EXACT GROUND STATES FOR A CLASS OF LINEAR QUANTUM SPIN SYSTEMS

W J CASPERS

Twente University of Technology, Enschede, The Netherlands

and

W MAGNUS*

Instituut voor Theoretische Fysica, University of Leuven, B-3030 Leuven, Belgium

Received 31 August 1982

The singlet pair state (S P S) is shown to be the exact ground state for a class of linear quantum spin systems with anisotropic interactions

1. Hamiltonian and notation

We consider a linear quantum spin system, described by the following Hamiltonian

$$H = 4 \sum_{r=x,y,z} J_r \sum_{i=1}^{2N} \sum_{n=1}^{N-1} \gamma(n) S_i^r S_{i+n}^r, \quad (1)$$

$2N$ is the number of spins ($S_i = 1/2$, $S_{2N+i} = S_i$ for every lattice site i), $\gamma(n) = 0$ for $n > N - 1$.

The singlet pair state (S P.S.) may be written: $|\Phi_0\rangle = \prod_{k=1}^N |0\rangle_k$; $|0\rangle_k$ denotes the singlet state for the spin pair $(2k - 1, 2k)$, $k = 1, 2, \dots, N$. Recently¹⁾, $|\Phi_0\rangle$ has been shown to be the ground state of H in the case $J_x = J_y = J_z$, if $\gamma(n)$ satisfies the conditions

$$\begin{aligned} \gamma(n) &= \gamma(1)(2s - n + 1)/2s, \quad \gamma(1) > 0, \quad 1 \leq n \leq 2s, \quad 2 \leq 2s \leq N - 1, \\ \gamma(n) &= 0, \quad \text{if } n > 2s \quad (2s \text{ is the interaction range}). \end{aligned} \quad (2)$$

This set of conditions is a generalization of the conditions derived by Van den Broek et al.²⁾.

From now on, we consider a parameter set $\{\gamma(n)\}$ satisfying following, more general conditions

$$\begin{aligned} \gamma(n) &= \sum_{l=l_n}^{[(N-1)/2]} (2l - n + 1) \delta^{(l)}, \quad l_n = \frac{1}{4}(2l + 1 - (-1)^n), \\ \delta^{(l)} &\geq 0 \text{ for } l < l_M, \quad \delta^{(l_M)} > 0, \quad \delta^{(l)} = 0 \text{ for } l > l_M, \end{aligned} \quad (3)$$

l_M being a measure for the interaction range.

* Onderzoeker I I K W, Belgium

The Hamiltonian can then be written as

$$H = 4 \sum_{r=x,y,z} J_r \sum_{l=1}^{2N} \sum_{l'=1}^{[(N-1)/2]} \sum_{n=1}^{2l} (2l-n+1) \delta^{(l)} S'_l S'_{l+n}.$$

It can be seen easily¹) that conditions (3) are sufficient for the S.P.S. to remain an exact eigenstate of H , even if J_x, J_y, J_z are chosen somewhat more freely. We use following notation

ϵ : ground-state energy per spin,

$$\tilde{H}_i^{(l)} = \sum_{r=x,y,z} \tilde{H}_{ir}^{(l)},$$

$$\tilde{H}_{ir}^{(l)} = 4J_r \delta^{(l)} \sum_{n=1}^{2l} \sum_{j=0}^{2l-n} S'_{i+j} S'_{i+j+n}, \quad l = 1, 2, \dots, [(N-1)/2],$$

$\alpha^{(l)}(\alpha_r^{(l)})$: lowest eigenvalue of $\tilde{H}_i^{(l)}(\tilde{H}_{ir}^{(l)})$,

$$H_0 = \sum_{k=1}^N H_{0k}, \quad H_{0k} = 4\gamma(1) \sum_{r=x,y,z} J_r S'_{2k-1} S'_{2k} \text{ describes the interaction within the } k\text{th neighbour pair.}$$

H can then be subdivided according to

$$H = \sum_{l=1}^{2N} \sum_{l'=1}^{[(N-1)/2]} \tilde{H}_i^{(l)}.$$

From now on J_x, J_y, J_z are restricted to the ranges defined by

$$J_x + J_y > 0, \quad J_y + J_z > 0, \quad J_z + J_x > 0. \quad (4)$$

Then $|0\rangle_k$ is the eigenstate of H_{0k} with the lowest eigenvalue

$$-\gamma(1)(J_x + J_y + J_z)$$

2. Lower and upper bounds for ϵ

For a set $\{\gamma(n)\}$ satisfying (3), we construct upper and lower bounds for ϵ , and look for cases in which the two bounds are equal. The eigenvalue equation for the lowest lying state $|\Phi_0\rangle$ of H_0 reads

$$H_0|\Phi_0\rangle = -N\gamma(1)(J_x + J_y + J_z)|\Phi_0\rangle.$$

As was remarked in section 1, $|\Phi_0\rangle$ is also an eigenstate of H , the corresponding eigenvalue being the same. Then it follows

$$2N\epsilon \leq -N\gamma(1)(J_x + J_y + J_z),$$

or

$$\epsilon \leq -\frac{1}{2}\gamma(1)(J_x + J_y + J_z), \quad (5)$$

from which an upper bound is obtained.

2.1. The case $J_x, J_y, J_z \geq 0$

$$\tilde{H}_{ir}^{(l)} = 2J_r \delta^{(l)} \left[(S'_{i,l})^2 - \frac{2l+1}{4} \right],$$

where $S'_{i,l} = \sum_{j=0}^{2l} S'_{i+j}$ is the r th component of the sum of $2l+1$ consecutive spins.

Therefore

$$\alpha_r^{(l)} = 2J_r \delta^{(l)} \left[\frac{1}{4} - \frac{1}{4}(2l+1) \right] = -J_r l \delta^{(l)}.$$

From (4) we have:

$$\epsilon \geq \sum_{r=x,y,z} \sum_{l=1}^{[(N-1)/2]} \alpha_r^{(l)} = - \sum_{r=x,y,z} J_r \sum_{l=1}^{(N-1)/2} l \delta^{(l)},$$

or, remembering (3),

$$\epsilon \geq -\frac{1}{2}\gamma(1)(J_x + J_y + J_z).$$

2.2. The case $J_x = J_y > 0, J_z < 0, |J_z| \leq J_x/(1+l_m)$.

J_x, J_z are still restricted to (4), whence, putting $J_y = J_x, y = |J_z|/J_x$, the upper bound for this case is

$$-\frac{1}{2}\gamma(1)(2J_x + J_z) = -\frac{1}{2}\gamma(1)J_x(2-y).$$

Now,

$$\tilde{H}_i^{(l)} = 2\delta^{(l)} J_x [S_i^2 - (1+y)(S_i^z)^2 - \frac{1}{4}(1+2l)(2-y)].$$

Putting $S_i^2 = S(S+1), S_i^z = S, S = \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{1}{2}(1+2l)$, we find the lowest eigenvalue of $\tilde{H}_i^{(l)}$ for fixed S

$$F(S, l) = 2\delta^{(l)} J_x [S - yS^2 - \frac{1}{4}(1+2l)(2-y)].$$

Because of the finite range of discrete values of S , $F(S, l)$ has two minima (l being fixed)

$$F(\frac{1}{2}, l) = -\delta^{(l)} J_x l(2-y) \text{ and } F(\frac{1}{2}(1+2l), l) = -\delta^{(l)} J_x l(2l+1)y.$$

It can be verified easily that, for every $l \leq l_m$,

$$F(\frac{1}{2}, l) \leq F(\frac{1}{2}(1+2l), l) \text{ if } y \leq 1/(1+l_m).$$

In that case $\alpha^{(l)} = F(\frac{1}{2}, l)$, and $\sum_{i=1}^{(N-1)/2} F(\frac{1}{2}, l)$ plays the role of a lower bound, which equals the upper bound (5).

3. Concluding remarks

If the parameter set $\{\gamma(n)\}$ satisfies the conditions (3), the S.P.S. remains the exact ground state for all anisotropic interaction strengths J_x, J_y, J_z , corresponding with the cases 2.1 and 2.2. Recently³, a two dimensional antiferromagnetic Hamiltonian with anisotropic interactions in the presence of a magnetic field, was analyzed by B.S. Shastry and B. Sutherland. Sufficient conditions for the S.P.S. to be the exact ground state of a two-dimensional lattice, have been derived. We also want to draw attention to the recent work of Klein⁴) on singlet-pair states. This author formulated independently the sufficient conditions

$$2\gamma(2t) = \gamma(2t-1) + \gamma(2t+1)$$

for a S.P.S. to exist in a linear chain with isotropic interactions¹).

References

- 1 W J Caspers, *Physica* **115A** (1982) 275
- 2 P M Van den Broek, *Phys Lett* **77A** (1980) 261, C K Majumdar and D K Ghosh, *J Math Phys* **10** (1969) 1399, C K Majumdar, *J Phys C* **3** (1970) 911
- 3 B S Shastry and B Sutherland, *Physica* **108B** (1981) 1069
- 4 D J Klein, *J Phys A, Math Gen* **15** (1982) 661