PERIODIC LINEAR DIFFERENTIAL STOCHASTIC PROCESSES*

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Abstract. Periodic linear differential processes are defined and their properties are analyzed. Equivalent representations are discussed, and the solutions of related optimal estimation problems are given. An extension is presented of Kailath and Geesey's [1] results concerning the innovations representation of stochastic processes with a given covariance function.

1. Introduction. A periodic linear differential stochastic process X_t , $0 \leq t \leq T$, is defined as the solution of the stochastic differential equation

$$(1.1) \quad dX_t = AX_t dt + dW_t, \quad 0 \leq t \leq T,$$

with the periodicity condition

$$(1.2) \quad X_0 = X_T.$$

Here X_t is an n -dimensional real vector stochastic process, T a given real number, A a constant real $n \times n$ matrix, and W_t , $0 \leq t \leq T$, n -dimensional real Brownian motion with $E(dW_t dW_t') = V dt$, where the prime denotes the transpose and V is a given real symmetric nonnegative definite constant matrix. Processes of this kind can be used to model a variety of periodic random phenomena such as occur in communications and physics (t is not necessarily time but may also denote a space variable). In this paper the properties of the solution to (1.1)–(1.2) are studied, equivalent representations of the process X_t , $0 \leq t \leq T$, are given, and filtering, smoothing and prediction problems related to such processes are solved.

2. Properties of periodic differential processes. Solution of the differential equation (1.1) with the use of (1.2) yields the explicit representation

$$(2.1) \quad X_t = e^{At}(I - e^{AT})^{-1} \int_0^T e^{A(T-s)} dW_s + \int_0^t e^{A(t-s)} dW_s, \quad 0 \leq t \leq T,$$

where the integrals are stochastic integrals. The existence of this solution is guaranteed if A has no characteristic values that are an integral multiple of $2\pi i/T$, which henceforth will be assumed. It is easily verified that (2.1) constitutes the unique solution (in a mean square sense) to (1.1) and (1.2).

The expression (2.1) clearly defines a Gaussian process, which, therefore, is completely defined by its mean value function $E(X_t)$, $0 \leq t \leq T$, and its matrix covariance function $\text{cov}(X_t, X_s)$. Obviously,

$$(2.2) \quad E(X_t) = 0, \quad 0 \leq t \leq T.$$

To find the matrix covariance function, (2.1) is rewritten in the form

$$(2.3) \quad X_t = B_1 \int_0^t e^{A(t-s)} dW_s - B_2 \int_t^T e^{A(t-s)} dW_s, \quad 0 \leq t \leq T,$$

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where

$$(2.4) \quad B_1 = (I - e^{AT})^{-1}, \quad B_2 = (I - e^{-AT})^{-1}.$$

Note that $B_1 + B_2 = I$. It follows from (2.3) by direct computation that

$$(2.5) \quad \begin{aligned} R(t, s) = E(X_t X_s') &= B_1 \left[\int_0^s e^{A(t-\theta)} V e^{A'(s-\theta)} d\theta \right] B_1' \\ &\quad - B_1 \left[\int_s^t e^{A(t-\theta)} V e^{A'(s-\theta)} d\theta \right] B_2' \\ &\quad + B_2 \left[\int_t^T e^{A(t-\theta)} V e^{A'(s-\theta)} d\theta \right] B_2' \quad \text{for } s \leq t. \end{aligned}$$

It is seen that the variance matrix is given by

$$(2.6) \quad R(t, t) = B_1 \left[\int_0^t e^{A(t-\theta)} V e^{A'(t-\theta)} d\theta \right] B_1' + B_2 \left[\int_t^T e^{A(t-\theta)} V e^{A'(t-\theta)} d\theta \right] B_2',$$

$$0 \leq t \leq T.$$

By some simple substitutions it follows that

$$(2.7) \quad R(t, t) = B_1 \left[\int_0^T e^{A\theta} V e^{A'\theta} d\theta \right] B_1', \quad 0 \leq t \leq T,$$

which proves that the variance matrix is a constant matrix, which henceforth will be denoted by Q . By premultiplying (2.7) by A , postmultiplying by A' , and integrating by parts it is easily found that Q satisfies the linear matrix equation

$$(2.8) \quad AQ + QA' + B_1 V + V B_1' - V = 0.$$

If $\lambda_j + \lambda_k \neq 0$ for each pair of characteristic values λ_j and λ_k , $j, k = 0, 1, \dots, n$, of A , Q is the unique solution of this matrix equation.

Equation (2.5) shows that for fixed s the matrix covariance function R is a continuous and differentiable function of t . Partial differentiation with respect to t yields

$$(2.9) \quad \frac{\partial R(t, s)}{\partial t} = AR(t, s) - V e^{A'(s-t)} B_2', \quad s \leq t.$$

Integration of this differential equation with the initial condition $R(s, s) = Q$ gives

$$(2.10) \quad \begin{aligned} R(t, s) &= e^{A(t-s)} Q - \left[\int_s^t e^{A(t-\theta)} V e^{A'(s-\theta)} d\theta \right] B_2' \\ &= e^{A(t-s)} Q - e^{A(t-s)} \left[\int_0^{t-s} e^{-A\tau} V e^{-A'\tau} d\tau \right] B_2', \quad s \leq t. \end{aligned}$$

Since $R(t, s) = R'(s, t)$, one immediately obtains

$$(2.11) \quad R(t, s) = Q e^{-A'(t-s)} - B_2 \left[\int_0^{t-s} e^{-A\tau} V e^{-A'\tau} d\tau \right] e^{-A'(t-s)}, \quad t \leq s.$$

THEOREM 2.1 (Covariance function). *Suppose that A has no characteristic values that are integer multiples of $2\pi i/T$. Then the solution of (1.1) and (1.2) is a stationary Gaussian process on $[0, T]$, with zero mean and matrix covariance function*

$$(2.12) \quad R(t, s) = \begin{cases} (I, 0) e^{F(t-s)} \begin{pmatrix} Q \\ B'_2 \end{pmatrix}, & s \leq t, \\ (Q, B_2) e^{-F'(t-s)} \begin{pmatrix} I \\ 0 \end{pmatrix}, & t \leq s, \end{cases}$$

where the variance matrix $Q = \text{var}(X_t)$ is given by (2.7), and where

$$(2.13) \quad F = \begin{pmatrix} A & -V \\ 0 & -A' \end{pmatrix}.$$

Proof. The relations (2.10) and (2.11) show that $R(t, s)$ is a function of $t - s$ alone. Therefore, the process X_t , $0 \leq t \leq T$, is wide-sense stationary, and, hence, strictly stationary. That R may be represented as in (2.12) follows from (2.10) and (2.11) together with the fact that

$$(2.14) \quad e^{F(t-s)} = \begin{pmatrix} e^{A(t-s)} & -e^{A(t-s)} \int_0^{t-s} e^{-A\tau} V e^{-A'\tau} d\tau \\ 0 & e^{-A'(t-s)} \end{pmatrix}. \quad \square$$

It is illuminating to consider the spectral representation of the process X_t , $0 \leq t \leq T$. Let $R(t, s) = \tilde{R}(t - s)$. Then according to Bochner's theorem there exists a matrix spectral distribution function Ψ such that

$$(2.15) \quad \tilde{R}(\theta) = \int_{-\infty}^{\infty} e^{i2\pi\gamma\theta} d\Psi(v), \quad 0 \leq \theta \leq T.$$

Here

$$(2.16) \quad \Psi(v) = \sum_{k=-\infty}^{\infty} \Psi_k H(v - v_k), \quad -\infty \leq v \leq \infty,$$

where H is the Heaviside step function, $v_k = k/T$, and Ψ_k is the coefficient matrix

$$(2.17) \quad \Psi_k = \frac{1}{T} \int_0^T \tilde{R}(\theta) e^{-i2\pi\gamma_k\theta} d\theta, \quad k = 0, \pm 1, \pm 2, \dots$$

It follows from (2.9) that

$$(2.18) \quad \frac{d\tilde{R}(\theta)}{d\theta} = A\tilde{R}(\theta) - V e^{-A'\theta} B'_2, \quad 0 \leq \theta \leq T.$$

Multiplication of both sides of (2.18) by $(1/T) \exp(-i2\pi v_k)$ and integration over $[0, T]$ yields

$$(2.19) \quad \Psi_k = \frac{1}{T} \Phi(i2\pi v_k), \quad k \text{ integer},$$

where

$$(2.20) \quad \Phi(s) = (sI - A)^{-1}V(-sI - A')^{-1}, \quad s \text{ complex}.$$

This result is interesting, for the following reason. Suppose that A is stable, i.e., all its characteristic values have strictly negative real parts. Then the matrix equation

$$(2.21) \quad A\bar{Q} + \bar{Q}A' + V = 0$$

has a unique symmetric nonnegative definite solution $\bar{Q}$. Furthermore, the stochastic differential equation

$$(2.22) \quad d\bar{X}_t = A\bar{X}_t + dW_t, \quad t \geq 0,$$

with $W_t, t \geq 0$, Brownian motion with $E(dW_t dW_t') = V dt$, and $\bar{X}_0$ a Gaussian stochastic variable, independent of $W_t, t \geq 0$, with expectation zero and variance matrix $\bar{Q}$, generates a stationary Gaussian stochastic process $\bar{X}_t, t \geq 0$, with spectral density matrix $\Phi(i2\pi\nu)$. The process $\bar{X}_t, t \geq 0$, may be called the *long term version* of the solution of the stochastic differential equation (1.1), and the solution of (1.1) and (1.2) the *periodic version* of this solution. Thus, the spectral coefficients of the periodic version are directly related to the spectral density matrix of the long term version by (2.19). It furthermore follows from (2.19) that the covariance matrix function $\tilde{R}$ of the periodic version and the covariance matrix function $\bar{R}(t-s) = E(\bar{X}_t \bar{X}_s')$ of the long term version of the process are related by

$$(2.23) \quad \tilde{R}(\theta) = \sum_{n=-\infty}^{\infty} \bar{R}(\theta + nT), \quad 0 \leq \theta \leq T.$$

This suggests that the periodic and long term versions of the process are related by

$$X_t = \text{l.i.m.}_{N \rightarrow \infty} \frac{1}{\sqrt{N+1}} \sum_{n=0}^{\infty} \bar{X}_{t+nT}, \quad 0 \leq t \leq T,$$

which may be verified. These connections between the periodic and long term versions of the process of course make no sense if A is not stable, because in this case the long term version of the process is not defined.

3. Equivalent representations. This section is addressed to the question whether there exists another process $X_t^*, 0 \leq t \leq T$, defined by

$$(3.1) \quad \begin{aligned} dX_t^* &= A^* X_t^* dt + dW_t^*, \quad 0 \leq t \leq T, \\ X_0^* &= X_T^*, \end{aligned}$$

with $E(dW_t^* dW_t^{*'}) = V^* dt$, such that the processes $X_t^*, 0 \leq t \leq T$, and $X_t, 0 \leq t \leq T$, have the same matrix covariance functions. Let

$$(3.2) \quad \Phi^*(s) = (sI - A^*)^{-1}V^*(-sI - A^{*'})^{-1}.$$

Then, since $\Phi(s)$ and $\Phi^*(s)$ agree for all $s = i2\pi k/T, k$ integer,

$$(3.3) \quad \Phi(s) = \Phi^*(s), \quad \text{all complex } s.$$

Assume now that $V = BB'$, and that the pair (A, B) is completely controllable. For the system theoretic terminology employed in this section, reference is made, for example, to Kalman, Falb and Arbib [2]. Then if $\Phi(s)$ is considered as the transfer matrix of a time-invariant linear differential system, (F, G, H) forms a realization of this system, i.e., $\Phi(s) = H(sI - F)^{-1}G$, where

$$(3.4) \quad F = \begin{pmatrix} A & -V \\ 0 & -A' \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad H = (I, 0).$$

From the fact that (A, B) is completely controllable it may be shown by straightforward verification that (F, G) is completely controllable and (F, H) completely observable, so that (F, G, H) is a minimal realization of $\Phi(s)$, i.e., the matrices F , G and H have the smallest possible dimensions. Let

$$(3.5) \quad F^* = \begin{pmatrix} A^* & -V^* \\ 0 & -A^{*'} \end{pmatrix}.$$

Since $\Phi(s) \equiv \Phi^*(s)$, (F^*, G, H) is another realization of $\Phi(s)$. As this realization has the same dimension as (F, G, H) , it is also minimal. Therefore, there exists [2] a nonsingular $2n \times 2n$ transformation matrix M such that $M^{-1}FM = F^*$, $M^{-1}G = G$, and $HM = H$. It is easily found that the latter two equalities are satisfied if and only if

$$(3.6) \quad M = \begin{pmatrix} I & 0 \\ P & 0 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix},$$

where P is an arbitrary $n \times n$ matrix. With this it follows from $M^{-1}FM = F^*$ that

$$(3.7) \quad \begin{pmatrix} A - VP & -V \\ -A'P - PA + PVP & -A' + PV \end{pmatrix} = \begin{pmatrix} A^* & -V^* \\ 0 & -A^{*'} \end{pmatrix},$$

which leads to the following result.

THEOREM 3.1 (Equivalent representations). *Suppose that $V = BB'$, and that (A, B) is completely controllable. Then if the process X_t^* , $0 \leq t \leq T$, has the same matrix covariance function as X_t , $0 \leq t \leq T$,*

$$(3.8) \quad \begin{aligned} A^* &= A - VP, \\ V^* &= V, \end{aligned}$$

where P is a solution of the nonlinear matrix equation

$$(3.9) \quad 0 = A'P + PA - PVP.$$

Equation (3.9) is a special form of the algebraic matrix Riccati equation, about which a great deal is known (see e.g., [3]).

4. Innovations representation and estimation of stochastic processes with a given covariance function. In this section an extension is given of Kailath and Geesey's results [1] concerning the representation and estimation of stochastic processes with given covariance functions. The results given here differ from those of Kailath and Geesey in that vector-valued processes are considered,

nonzero means are accounted for, the existence of the solution of an essential matrix differential equation is proved, and expressions are given for the variance matrices of the reconstruction errors. The results of this section will be applied to periodic differential processes in § 5.

Consider a vector-valued, mean square continuous stochastic process Z_t , $t \in \Theta$, where $\Theta = [t_0, t_1]$, with the mean value function

$$(4.1) \quad E(Z_t) = \int_{t_0}^t m_Z(s) ds, \quad t \in \Theta,$$

and the matrix covariance function

$$(4.2) \quad \text{cov}(Z_t, Z_s) = \int_{t_0}^t dt' \int_{t_0}^s L_Z(t', s') ds' + \int_{t_0}^{\min(t, s)} V(t') dt', \quad t, s \in \Theta.$$

Here $V(t)$, $t \in \Theta$, is a given, continuous, symmetric matrix function, positive definite for every $t \in \Theta$. The function m_Z is given in the form

$$(4.3) \quad m_Z(t) = M(t)\Psi(t, t_0)m_0, \quad t \in \Theta,$$

while L_Z is given in the form

$$(4.4) \quad L_Z(t, s) = \begin{cases} M(t)\Psi(t, s)N(s), & s \leq t, \\ N'(t)\Psi'(s, t)M'(s), & t \leq s. \end{cases}$$

Here M and N are continuous matrix functions on Θ , m_0 is a constant vector, and Ψ is a fundamental matrix satisfying the matrix differential equation

$$(4.5) \quad \begin{aligned} \frac{\partial}{\partial t} \Psi(t, s) &= F(t)\Psi(t, s), & t, s \in \Theta, \\ \Psi(s, s) &= I, & s \in \Theta, \end{aligned}$$

with F a continuous matrix function on Θ . It will finally be assumed that the self-adjoint operator Q defined by

$$(4.6) \quad (Qv)(t) = V(t)v(t) + \int_{t_0}^{t_1} L_Z(t, s)v(s) ds, \quad t_0 \leq t \leq t_1,$$

where $v \in L_2[t_0, t_1]$, is positive definite.

The following result (Kailath and Geesey [1]) is essential.

THEOREM 4.1 (Innovations representation). *Suppose that the matrix differential equation*

$$(4.7) \quad \begin{aligned} \dot{S}(t) &= F(t)S(t) + S(t)F'(t) + K(t)V(t)K'(t), & t \in \Theta, \\ S(t_0) &= 0, \end{aligned}$$

where

$$(4.8) \quad K(t) = [N(t) - S(t)M'(t)]V^{-1}(t), \quad t \in \Theta,$$

has a solution on Θ . Then the process Z_t^* , $t \in \Theta$, defined by

$$(4.9) \quad \begin{aligned} dZ_t^* &= M(t)P_t dt + dI_t, & t \in \Theta, \\ Z_{t_0} &= 0, \end{aligned}$$

$$(4.10) \quad \begin{aligned} dP_t &= F(t)P_t dt + K(t) dI_t, & t \in \Theta, \\ P_{t_0} &= m_0, \end{aligned}$$

where I_t , $t \in \Theta$, is Brownian motion with $E(dI_t dI_t') = V(t) dt$, has the mean value function defined by (4.1) and (4.3), and the matrix covariance function defined by (4.2) and (4.4).

Proof. This theorem may be proved by direct calculation of the mean value function and matrix covariance function of the process Z_t^* , $t \in \Theta$, as defined by (4.7)–(4.10). This calculation is laborious but straightforward. It helps to note that $S(t) = \text{var}(P_t)$, $t \in \Theta$.

THEOREM 4.2 (Existence of the solution of the matrix differential equation). *The matrix differential equation (4.7)–(4.8) has a unique solution on every finite interval $[t_0, t_1]$.*

Proof. The proof of Kailath and Geesey of the existence of the solution of the matrix differential equation (4.7)–(4.8) is based on the assumption of the existence of some lumped Markov model for the process Z . This assumption is not required in the proof to follow. Define the matrix functions

$$(4.11) \quad \begin{aligned} S^*(t) &= S(t_0 + t_1 - t), & F^*(t) &= F(t_0 + t_1 - t), & N^*(t) &= N(t_0 + t_1 - t), \\ M^*(t) &= M(t_0 + t_1 - t), & V^*(t) &= V(t_0 + t_1 - t), \end{aligned}$$

for all $t \in [t_0, t_1]$. Then it easily follows that (4.7)–(4.8) can be rewritten in the form

$$(4.12) \quad \begin{aligned} -\dot{S}^*(t) &= [F^{*'}(t) - M^{*'}(t)V^{*-1}(t)N^{*'}(t)]S^*(t) \\ &\quad + S^*(t)[F^{*'}(t) - M^{*'}(t)V^{*-1}(t)N^{*'}(t)] \\ &\quad + S^*(t)M^{*'}(t)V^{*-1}(t)M^*(t)S^*(t) + N^*(t)V^{*-1}(t)N^{*'}(t), \end{aligned}$$

$$t_0 \leq t \leq t_1,$$

$$S^*(t_1) = 0.$$

This matrix differential equation, and hence (4.7)–(4.8), has a unique solution if and only if the following optimal control problem has a unique solution [4]. Maximize

$$(4.13) \quad \int_{t_0}^{t_1} [x'(t)N^*(t)V^{*-1}(t)N^{*'}(t)x(t) - u'(t)V^*(t)u(t)] dt$$

with respect to $u(t)$ and $x(t)$, $t_0 \leq t \leq t_1$ subject to

$$(4.14) \quad \begin{aligned} \dot{x}(t) &= [F^{*'}(t) - M^{*'}(t)V^{*-1}(t)N^{*'}(t)]x(t) + M^{*'}(t)u(t), & t_0 \leq t \leq t_1, \\ x(t_0) &= x_0, \end{aligned}$$

with x_0 a given vector. By substituting $u(t) - V^{*-1}(t)N^{*'}(t)x(t) = u^*(t)$, $t_0 \leq t \leq t_1$, it follows that this problem is equivalent to the problem of minimizing

$$(4.15) \quad J = \int_{t_0}^{t_1} [u^{*'}(t)V^*(t)u^*(t) + 2u^{*'}(t)N^{*'}(t)x(t)] dt$$

with respect to $u^*(t)$ and $x(t)$, $t_0 \leq t \leq t_1$, subject to

$$(4.16) \quad \begin{aligned} \dot{x}(t) &= F^{*'}(t)x(t) + M^{*'}(t)u^*(t), \quad t_0 \leq t \leq t_1, \\ x(t_0) &= x_0. \end{aligned}$$

Let $\Psi^*(t, s) = \Psi'(t_0 + t_1 - s, t_0 + t_1 - t)$ denote the fundamental matrix corresponding to $\dot{x}(t) = F^{*'}(t)x(t)$. Then substitution of

$$(4.17) \quad x(t) = \Psi^*(t, t_0)x_0 + \int_{t_0}^t \Psi^*(t, s)M^{*'}(s)u^*(s) ds, \quad t_0 \leq t \leq t_1,$$

into (4.15) yields, with changes of integration variables from s to $t_0 + t_1 - s$ and t to $t_0 + t_1 - t$, and with the notation $u^*(t_0 + t_1 - t) = v(t)$, $t_0 \leq t \leq t_1$,

$$(4.18) \quad \begin{aligned} J &= \int_{t_0}^{t_1} v'(t)V(t)v(t) dt + 2 \int_{t_0}^{t_1} v'(t)N'(t)\Psi'(t_1, t)x_0 dt \\ &\quad + 2 \int_{t_0}^{t_1} v'(t)N'(t) dt \int_t^{t_1} \Psi'(s, t)M'(s)v(s) ds \\ &= \int_{t_0}^{t_1} v'(t)V(t)v(t) dt + 2 \int_{t_0}^{t_1} v'(t)N'(t)\Psi'(t_1, t)x_0 dt \\ &\quad + \int_{t_0}^{t_1} dt \int_t^{t_1} v'(t)N'(t)\Psi'(s, t)M'(s)v(s) ds \\ &\quad + \int_{t_0}^{t_1} ds \int_{t_0}^s v'(s)M(s)\Psi(s, t)N(t)v(t) dt \\ &= \int_{t_0}^{t_1} v'(t)V(t)v(t) dt + 2 \int_{t_0}^{t_1} v'(t)N'(t)\Psi'(t_1, t)x_0 dt \\ &\quad + \int_{t_0}^{t_1} \int_{t_0}^{t_1} v'(t)L_Z(t, s)v(s) dt ds. \end{aligned}$$

In functional analytic form this may be rewritten as

$$(4.19) \quad J = \langle v, Qv \rangle + 2\langle v, f \rangle,$$

where Q is the operator defined by (4.6), f is the function $f(t) = N'(t)\Psi'(t_1, t)x_0$, $t_0 \leq t \leq t_1$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_2[t_0, t_1]$. Since by assumption Q is positive definite, there exists a unique $v \in L_2[t_0, t_1]$ that minimizes J . Hence, the matrix equation (4.7)–(4.8) has a unique solution on any interval $[t_0, t_1]$. $\square$

Once the innovations representation (4.9)–(4.10) has been obtained, it is easy to obtain the solution to various filtering problems (Kailath and Geesey [1]).

Let R_t , $t \in \Theta$, be another vector-valued, mean square continuous process, defined on the same probability space as the process Z_t , $t \in \Theta$, with the mean value function

$$(4.20) \quad E(R_t) = M_r(t)\Psi(t, t_0)m_0, \quad t \in \Theta.$$

Suppose that the matrix cross covariance function of the processes R and Z is given in the form

$$(4.21) \quad \text{cov}(R_t, Z_s) = \int_{t_0}^s L_{RZ}(t, s') ds', \quad t, s \in \Theta,$$

where

$$(4.22) \quad L_{RZ}(t, s) = \begin{cases} M_r(t)\Psi(t, s)N(s), & s \leq t, \\ N'_r(t)\Psi'(s, t)M'(s), & t \leq s. \end{cases}$$

Here N_r and M_r are given, continuous matrix functions on Θ .

THEOREM 4.3 (Filtering problem). *Let Z_t , $t \in \Theta$, and R_t , $t \in \Theta$, be mean square continuous, vector-valued stochastic processes, with mean value functions and (cross) covariance matrix functions defined by (4.1)–(4.4) and (4.20)–(4.22), respectively. Let $R_{t|t}$ denote the minimum variance unbiased linear estimator of R_t given Z_θ , $t_0 \leq \theta \leq t$. Then*

$$(4.23) \quad R_{t|t} = M_r(t)P_t, \quad t \in \Theta,$$

where P_t , $t \in \Theta$, is the solution of the stochastic differential equation

$$(4.24) \quad \begin{aligned} dP_t &= F(t)P_t dt + K(t)[dZ_t - M(t)P_t dt], \quad t \in \Theta, \\ P_{t_0} &= m_0. \end{aligned}$$

The variance matrix of the estimation error is given by

$$(4.25) \quad \text{var}(R_{t|t} - R_t) = \text{var}(R_t) - M_r(t)S(t)M'_r(t), \quad t \in \Theta.$$

Proof. $R_{t|s}$ is the minimum variance unbiased linear estimator of Z_t given Z_θ , $t_0 \leq \theta \leq s$, if and only if

$$(4.26) \quad E(R_{t|s} - R_t) = 0,$$

and

$$(4.27) \quad \text{cov}(R_{t|s} - R_t, Z_\theta) = 0, \quad t_0 \leq \theta \leq s.$$

It may be verified by straightforward but tedious calculations that the estimator $R_{t|t}$ as given by (4.23) satisfies these conditions for $s = t$. The variance matrix of the estimation error as given by (4.25) follows from the fact that

$$(4.28) \quad \text{var}(R_{t|s} - R_t) = \text{var}(R_t) - \text{var}(R_{t|s}). \quad \square$$

THEOREM 4.4 (Prediction problem). *Let the processes Z_t , $t \in \Theta$, and R_t , $t \in \Theta$, be defined as before. Let $R_{t|s}$, $t \geq s$, denote the minimum variance unbiased linear estimator of R_t , given Z_θ , $t_0 \leq \theta \leq s$. Then*

$$(4.29) \quad R_{t|s} = M_r(t)\Psi(t, s)P_s, \quad t \geq s,$$

where P_t , $t \in \Theta$, is the solution of (4.24). The prediction error variance matrix is given by

$$(4.30) \quad \text{var}(R_{t|s} - R_t) = \text{var}(R_t) - M_r(t)\Psi(t, s)S(s)\Psi'(t, s)M_r'(t), \quad t \geq s.$$

Proof. Again it may be verified by direct computation that the estimator as given by (4.29) satisfies the conditions (4.26) and (4.27). The variance matrix (4.30) follows from (4.28). $\square$

THEOREM 4.5 (Smoothing problem). *Let the processes Z_t , $t \in \Theta$, and R_t , $t \in \Theta$, be defined as before. Let $R_{t|s}$, $t \leq s$, denote the minimum variance unbiased linear estimator of R_t , given Z_θ , $t_0 \leq \theta \leq s$. Then*

$$(4.31) \quad R_{t|s} = R_{t|t} + [N_r'(t) - M_r(t)S(t)]\Lambda_{t|s}, \quad t \leq s,$$

where $\Lambda_{t|s}$, $t_0 \leq t \leq s$, can be solved from the stochastic differential equation

$$(4.32) \quad \begin{aligned} d\Lambda_{t|s} &= -[F(t) - K(t)M(t)]'\Lambda_{t|s}dt - M'(t)V_2^{-1}(t)[dZ_t - M(t)P_t dt], \quad t \leq s, \\ \Lambda_{s|s} &= 0. \end{aligned}$$

The smoothing error variance matrix is given by

$$(4.33) \quad \begin{aligned} \text{var}(R_{t|s} - R_t) &= \text{var}(R_t) - M_r(t)S(t)M_r'(t) \\ &\quad - [N_r'(t) - M_r(t)S(t)] \left[\int_t^s \Xi'(\theta, s)M'(\theta)V_2^{-1}(\theta)M(\theta)\Xi(\theta, s) d\theta \right] \\ &\quad [N_r(t) - S(t)M_r'(t)], \quad t \leq s, \end{aligned}$$

where the fundamental matrix Ξ is the solution of

$$(4.34) \quad \begin{aligned} \frac{\partial}{\partial t}\Xi(t, s) &= [F(t) - K(t)M(t)]\Xi(t, s), \quad t, s \in \Theta, \\ \Xi(s, s) &= I, \quad s \in \Theta. \end{aligned}$$

Proof. Again it may be verified by direct calculation that (4.31) satisfies (4.26) and (4.27). The variance matrix (4.33) follows from (4.28). $\square$

5. Filtering, prediction and smoothing for periodic linear differential processes.

In this section the results of the preceding section are applied to the periodic processes of §§ 1 and 2. Consider the periodic process defined by

$$(5.1) \quad \begin{aligned} dX_t &= AX_t dt + dW_{1,t}, \quad 0 \leq t \leq T, \\ X_0 &= X_T, \end{aligned}$$

with $E(dW_{1,t}dW_{1,t}') = V_1 dt$. Suppose furthermore that a process Z_t , $0 \leq t \leq T$, is observed, which is given by

$$(5.2) \quad \begin{aligned} dZ_t &= CX_t dt + dW_{2,t}, \quad 0 \leq t \leq T, \\ Z_0 &= 0, \end{aligned}$$

where C is a constant matrix, and $W_{2,t}$, $0 \leq t \leq T$, Brownian motion, independent of $W_{1,t}$, with $E(dW_{2,t}dW_{2,t}') = V_2 dt$. Using the results of § 2, it is easily seen that the mean value function of the process Z_t , $0 \leq t \leq T$, is identical to zero, while

its covariance matrix function may be expressed in the form

$$(5.3) \quad \text{cov}(Z_t, Z_s) = \int_0^t dt' \int_0^s L_Z(t', s') ds' + \int_0^{\min(t,s)} V_2 dt', \quad 0 \leq t, s \leq T,$$

with

$$(5.4) \quad L_Z(t, s) = \begin{cases} M\Psi(t, s)N, & s \leq t, \\ N'\Psi'(s, t)M', & t \leq s, \end{cases}$$

where

$$(5.5) \quad M = (C, 0), \quad \Psi(t, s) = e^{F(t-s)}, \quad N = \begin{pmatrix} QC' \\ B_2' C' \end{pmatrix}.$$

Here

$$(5.6) \quad F = \begin{pmatrix} A & -V_1 \\ 0 & -A' \end{pmatrix},$$

while Q is the variance matrix of the process X_t . Furthermore, it is easily established that the cross covariance matrix of the processes X_t and Z_t is given by

$$(5.7) \quad \text{cov}(X_t, Z_s) = \int_0^s L_{XZ}(t, s') ds', \quad 0 \leq t, s \leq T,$$

where

$$(5.8) \quad L_{XZ}(t, s) = \begin{cases} M_x \Psi(t, s)N, & s \leq t, \\ N'_x \Psi'(s, t)M', & t \leq s, \end{cases}$$

such that

$$(5.9) \quad M_x = (I, 0), \quad N_x = \begin{pmatrix} Q \\ B_2' \end{pmatrix}.$$

The results (5.3)–(5.4) and (5.7)–(5.8) conform to the assumptions of § 4. The matrix differential equation (4.7), which plays a central role, in the present case takes the form

$$(5.10) \quad \dot{S}(t) = \begin{pmatrix} A & -V_1 \\ 0 & -A' \end{pmatrix} S(t) + S(t) \begin{pmatrix} A' & 0 \\ -V_1 & -A \end{pmatrix} + K(t) V_2 K'(t), \quad 0 \leq t \leq T, \\ S(0) = 0,$$

where

$$(5.11) \quad K(t) = \left[\begin{pmatrix} QC' \\ B_2' C' \end{pmatrix} - S(t) \begin{pmatrix} C' \\ 0 \end{pmatrix} \right] V_2^{-1}, \quad 0 \leq t \leq T.$$

It is convenient to make the substitution

$$(5.12) \quad U(t) = \begin{pmatrix} Q & B_2' \\ B_2' & 0 \end{pmatrix} - S(t), \quad 0 \leq t \leq T.$$

This results in the matrix differential equation

$$(5.13) \quad \begin{aligned} \dot{U}(t) &= FU(t) + U(t)F' - U(t)H'V_2^{-1}HU(t) + GV_1G', \quad 0 \leq t \leq T, \\ U(0) &= \begin{pmatrix} Q & B_2 \\ B_2' & 0 \end{pmatrix}, \end{aligned}$$

while

$$(5.14) \quad K(t) = U(t)H'V_2^{-1}, \quad 0 \leq t \leq T.$$

Here, with a change from an earlier notation,

$$(5.15) \quad F = \begin{pmatrix} A & -V_1 \\ 0 & -A' \end{pmatrix}, \quad G = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad H = (C, 0).$$

The existence and uniqueness of the solutions of (5.10)–(5.11) and (5.13) are guaranteed by Theorem 4.2.

The solutions of the filtering, prediction and smoothing problems for the processes described by (5.1)–(5.2) are now easily solved using the results of § 4. It is advantageous to express these solutions in terms of the solution of the matrix differential equation (5.13).

THEOREM 5.1 (Filtering problem). *The minimum variance unbiased estimator $X_{t|t}$ of X_t , given Z_θ , $0 \leq \theta \leq t$, is given by*

$$(5.16) \quad X_{t|t} = (I, 0)P_t, \quad 0 \leq t \leq T,$$

where the $2n$ -dimensional process P_t , $0 \leq t \leq T$, is the solution of

$$(5.17) \quad \begin{aligned} dP_t &= \begin{pmatrix} A & -V_1 \\ 0 & -A' \end{pmatrix} P_t dt + K(t)[dZ_t - (C, 0)P_t dt], \quad 0 \leq t \leq T, \\ P_0 &= 0. \end{aligned}$$

The filtering error variance matrix is given by

$$(5.18) \quad \text{var}(X_{t|t} - X_t) = U_{11}(t), \quad 0 \leq t \leq T,$$

where U_{11} is obtained by partitioning $U(t)$ into four $n \times n$ matrices as

$$(5.19) \quad U(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U'_{12}(t) & U_{22}(t) \end{pmatrix}, \quad 0 \leq t \leq T.$$

THEOREM 5.2 (Prediction problem). *The minimum variance unbiased estimator $X_{t|s}$ of X_t , given Z_θ , $0 \leq \theta \leq s$, is given by*

$$(5.20) \quad X_{t|s} = (I, 0) e^{F(t-s)} P_s, \quad 0 \leq s \leq t \leq T,$$

with F given by (5.15) and where P_t , $0 \leq t \leq T$, is obtained from (5.17). The prediction error variance matrix is

$$(5.21) \quad \begin{aligned} \text{var}(X_{t|s} - X_t) &= (I, 0) \left[e^{F(t-s)} U(s) e^{F'(t-s)} \right. \\ &\quad \left. + \int_s^t e^{F(t-\theta)} G V_1 G' e^{F'(t-\theta)} d\theta \right] \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad 0 \leq s \leq t \leq T. \end{aligned}$$

THEOREM 5.3 (Smoothing problem). *The minimum variance unbiased estimator $X_{t|s}$ of X_t , given Z_θ , $0 \leq \theta \leq s$, is given by*

$$(5.22) \quad X_{t|s} = X_{t|t} + (I, 0)U(t)\Lambda_{t|s}, \quad 0 \leq t \leq s \leq T,$$

where $\Lambda_{t|s}$ is solved from

$$(5.23) \quad \begin{aligned} d\Lambda_{t|s} &= -[F - K(t)H]' \Lambda_{t|s} dt + H' V_2^{-1} [dZ_t - H P_t dt], \quad 0 \leq t \leq s, \\ \Lambda_{s|s} &= 0. \end{aligned}$$

The smoothing error variance matrix may be expressed as

$$(5.24) \quad \begin{aligned} \text{var}(X_{t|s} - X_t) &= U_{11}(t) - (I, 0)U(t) \left[\int_t^s \Xi'(\theta, s) H' V_2^{-1} H \Xi(\theta, s) d\theta \right] U(t) \begin{pmatrix} I \\ 0 \end{pmatrix}, \\ &0 \leq t \leq s \leq T, \end{aligned}$$

where Ξ is the fundamental matrix satisfying

$$(5.25) \quad \begin{aligned} \frac{\partial}{\partial t} \Xi(t, s) &= [F - K(t)H] \Xi(t, s), \quad 0 \leq t, s \leq T, \\ \Xi(s, s) &= I, \quad 0 \leq s \leq T. \end{aligned}$$

It is noted that the filtering, prediction and smoothing solutions as given would have been obtained if the problem had been considered of estimating the first n components of a $2n$ -dimensional process D_t , $0 \leq t \leq T$, which is the solution of

$$(5.26) \quad dD_t = F D_t dt + d\Sigma_{1,t}, \quad 0 \leq t \leq T,$$

with $\Sigma_{1,t}$, $0 \leq t \leq T$, Brownian motion with

$$(5.27) \quad E(d\Sigma_{1,t} d\Sigma'_{1,t}) = \begin{pmatrix} V_1 & 0 \\ 0 & 0 \end{pmatrix} dt,$$

together with the observation equation

$$(5.28) \quad dZ_t = (C, 0)D_t dt + d\Sigma_{2,t}, \quad 0 \leq t \leq T, \quad Z_0 = 0,$$

where $\Sigma_{2,t}$, $0 \leq t \leq T$, is Brownian motion independent of Σ_1 , with

$$E(d\Sigma_{2,t} d\Sigma'_{2,t}) = V_2 dt,$$

and where, finally, D_0 is a stochastic vector, independent of the Brownian motion processes, with expectation zero and variance matrix

$$(5.29) \quad \text{var}(D_0) = \begin{pmatrix} Q & B_2 \\ B_2' & 0 \end{pmatrix}.$$

Such a process D_t , $0 \leq t \leq T$, does not exist, however, since the right-hand side of (5.29) is an indefinite matrix, and hence is not a variance matrix.

A simplification of the solution of the smoothing problem is obtained for $s = T$.

THEOREM 5.4 (Simplification of the smoothing solution). *The minimum variance unbiased estimator $X_{t|T}$ of X_t , given Z_θ , $0 \leq \theta \leq T$, may be solved from*

$$(5.30) \quad dX_{t|T} = AX_{t|T} dt + V_1 \Gamma_t dt, \quad 0 \leq t \leq T,$$

where Γ_t , $0 \leq t \leq T$, satisfies

$$(5.31) \quad d\Gamma_t = -A'\Gamma_t dt - C'V_2^{-1}(dZ_t - CX_{t|T} dt), \quad 0 \leq t \leq T.$$

The boundary conditions are

$$(5.32) \quad X_{0|T} = X_{T|T}, \quad \Gamma_0 = \Gamma_T.$$

The reconstruction error variance matrix $\text{var}(X_{t|T} - X_t)$ is constant on $[0, T]$.

Proof. Equations (5.30)–(5.32) may be proved by differentiating $X_{t|T}$ as derived from (5.22). That $\text{var}(X_{t|T} - X_t)$ is a constant matrix follows by recognizing that $E_t = X_t - X_{t|T}$ and Γ_t satisfy

$$(5.33) \quad \begin{pmatrix} dE_t \\ d\Gamma_t \end{pmatrix} = \begin{pmatrix} A & -V_1 \\ -C'V_2^{-1}C & -A' \end{pmatrix} \begin{pmatrix} E_t \\ \Gamma_t \end{pmatrix} dt + \begin{pmatrix} dW_{1,t} \\ -C'V_2^{-1}dW_{2,t} \end{pmatrix}, \quad 0 \leq t \leq T,$$

$$\begin{pmatrix} E_0 \\ \Gamma_0 \end{pmatrix} = \begin{pmatrix} E_T \\ \Gamma_T \end{pmatrix}.$$

Equation (5.33) defines a periodic differential process, which proves that $\text{var}(E_t, \Gamma_t)$ and hence also $\text{var}(E_t) = \text{var}(X_{t|T} - X_t)$ is constant on $[0, T]$. $\square$

6. Conclusions. Periodic differential stochastic processes turn out to have several interesting properties. Kailath and Geesey's results provide a convenient method to solve filtering problems for such processes.

REFERENCES

- [1] T. KAILATH AND R. A. GEESEY, *An innovations approach to least squares estimation—Part IV: Recursive estimation given lumped covariance functions*, IEEE Trans. Automatic Control, AC-16 (1971), pp. 720–727.
- [2] R. E. KALMAN, P. L. FALB AND M. A. ARBIB, *Topics in Mathematical Systems Theory*, McGraw-Hill, New York, 1969.
- [3] H. KWAKERNAK AND R. SIVAN, *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.
- [4] R. E. KALMAN, *Contributions to the theory of optimal control*, Bol. Soc. Mat. Mexicana, 5 (1960), pp. 102–119.