Optimal Particle-Filter-Based Detector

Yvo Boers and Pranab K. Mandal

***Abstract*—In this letter, we propose and prove the asymptotic optimality of a particle-filter-based detection scheme. The detection method can be used in a general nonlinear/nonGaussian signal detection problem. The proposed detection mechanism is based on likelihood ratio (LR) and thus optimal in the Neyman-Pearson (NP) sense, but we approximate the LR based on a particle filter (PF). We show the asymptotic optimality by proving that the PF-based approximation of the LR converges to the true LR as the number of particles increases to infinity. We also discuss the practical and operational implications of the result, the main one being that it is optimal in the sense that no other processing and detection mechanism can have higher probability of detection while have same or lower false alarm rate.**

***Index Terms*—Particle Filters, Detection Problem, Hypothesis Testing, Neyman Pearson Optimality**

I. INTRODUCTION

We consider the detection problem for a possibly nonlinear and/or non-Gaussian state-space system. In a previously published letter [1], a mechanism has been proposed which combines a particle filter (PF) and a detection mechanism based on the likelihood ratio (LR) test, where the likelihoods were estimated/approximated from the running PF. The authors conclude with a "remaining question" about a possible convergence result for the approximate LR [1, Section III]. To the best of our knowledge, no explicit convergence results for the PF-based approximated LR can be found in the existing literature.

In this letter, we fill that gap by presenting a proof of the convergence of the PF-based LR to the true LR, for which a closed form expression is usually not available. In the proof, we make use of the existing convergence results of a particle filter, e.g., [2]. The proposed detection mechanism is thus asymptotically NP optimal in the detection part and when combined with the PF, it becomes Bayes optimal for the state estimation part. The NP optimality is appealing not only from a theoretical point of view, but also from the practical point of view, because the optimized quantities, namely the detection probability and the false alarm rate, are very important and relevant in real life situations. For example, in radar, sonar and infrared systems or combined sensing systems, a guaranteed false alarm rate is of paramount importance while minimizing the missed detection probability. This is precisely what an LR-test provides. Also, we verify the convergence result through a numerical simulation with a linear-Gaussian system, where the true likelihood is known and can be easily calculated. Numerical results show that the particle filter does provide a more accurate estimate of the signal as well as the likelihood ratio, as the number of used particles increases.

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The proposed detection mechanism can, in principle, be used in a (radar) track-before-detect (TBD) processing chain for a radar system that needs to be able to detect and track very small objects in adverse conditions. However, we do not pursue this as this is not the main purpose of this correspondence.

Contributions of this letter are, thus:

1) A proof that the particle-filter-based LR converges to the true, possibly unknown, LR, as the number of particles tends to infinity.
2) A numerical illustration of the above statement by means of a simulation example.
3) A discussion on the practical relevance of the result and how the proposed detector differs from some of the current state-of-the-art.

The organisation of the remainder of this letter is as follows: first, in Section II, we define precisely the system that we consider and we present some preliminary results that will be needed later on. We present our main result in Section III. In Section IV, we present the simulation results that support the results from Section III. Finally, in Section V, we draw some conclusions. Relevant remarks and discussions can be found throughout the letter.

II. SYSTEM AND PRELIMINARIES

Let us consider the following general nonlinear system: for $k = 0, 1, 2 \ldots$,

$$\text{(state:)} \qquad s_{k+1} = f(s_k, w_k) \tag{1}$$

$$\text{(measurement:)} \qquad z_k = h(s_k, v_k) \tag{2}$$

with

$$\text{(initial condition:)} \qquad s_0 \sim p(s_0), \tag{3}$$

where the states $s_k \in \mathcal{S} \subset \mathbb{R}^n$, the measurements $z_k \in \mathbb{R}^p$, the state (or process) noises $w_k \in \mathbb{R}^m$, with $w_k \sim p_w$, the measurement noises $v_k \in \mathbb{R}^q$ with

$$v_k \sim p_v \tag{4}$$

and f and h are mappings of appropriate dimensions.

We denote the a posteriori density corresponding to the system (1) – (4), by:

$$p(s_k \mid Z_k), \tag{5}$$

where Z_k contains the measurement history up to and including time k. Moreover, we denote the (cumulative probability) distribution associated with $p(s_k \mid Z_k)$ by $P(s_k \mid Z_k)$.

In our analysis, we assume that the detection scheme uses a Sequential Importance Resampling (SIR) particle filter with the importance density to be the state predictive density and resampling performed at each time step k; see, e.g., [3].

Adopting the notation of [1] we denote the output of the particle filter by:

$$\{s_k^i, \tilde{s}_k^i, q_k^i, \tilde{q}_k^i\}_{i=1,\dots,N} \quad \text{with} \quad \tilde{q}_k^i := p(z_k \mid s_k^i), \tag{6}$$

where $s_k^i \in S$ represents a particle before resampling, $\tilde{q}_k^i \in \mathbb{R}^+$ and $q_k^i \in [0,1]$ are the corresponding unnormalized and normalized weights, respectively, and $\tilde{s}_k^i \in S$ denotes a particle after resampling. The weights after resampling are, of course, all equal to $\frac{1}{N}$. Then the empirical a posteriori distribution is given by (see, e.g., [2] and [1]):

$$\hat{P}_N(s_k) := \frac{1}{N}\sum_{i=1}^{N} u(s_k - \tilde{s}_k^i), \tag{7}$$

where $u(.)$ is the Heaviside step function [4]. Let us also define the empirical one-step-ahead prediction distribution (i.e., corresponding to $p(s_k \mid Z_{k-1})$) as:

$$\hat{P}_N(s_k \mid Z_{k-1}) := \frac{1}{N}\sum_{i=1}^{N} u(s_k - s_k^i). \tag{8}$$

Remark 1. *It is worth noting that the empirical a posteriori distributions defined by (7) and (8) are stochastic, in the sense that they depend on the (stochastic/random) outcome of the (stochastically simulated) particles ($\tilde{s}_k^i$ or s_k^i). This is, e.g., also stated in [2, section V]. Furthermore, the particles are not independent, whereas in standard statistics literature, when one considers an empirical distribution of the form (7), the samples (particles) are, usually, considered to be so.*

The following theorem provides convergence results for the empirical distributions (posterior and one-step-ahead prediction) obtained from a SIR particle filter.

Theorem 1. *Consider the dynamical system (1) – (4). Suppose that the likelihood $p(z_k \mid s_k)$ is bounded and continuous as a function of s_k and the transition density $p(s_k \mid s_{k-1})$ satisfies the so-called Feller property (see, e.g., [2]). Let $\hat{P}_N(s_k)$ and $\hat{P}_N(s_k \mid Z_{k-1})$ be as given in (7) and (8), obtained from a SIR-PF applied to the dynamical system. Then, as $N \to \infty$,*
(a) $\hat{P}_N(s_k)$ converges to $P(s_k \mid Z_k)$ almost surely, and
(b) $\hat{P}_N(s_k \mid Z_{k-1})$ converges to $P(s_k \mid Z_{k-1})$ almost surely. Here, almost surely refers to the randomness in the simulated particles.

Proof. Part (a) coincides with Theorem 1 of [2].
For Part (b), note that $\hat{P}_N(s_k \mid Z_{k-1}) = c^N \circ b_k(\pi^N_{k|k-1})$ and $P(s_k \mid Z_{k-1}) = b_k(\pi_{k|k-1})$, where the notations $c^N, b_k, \pi_{k|k-1}$ are as in [2]. The result then follows from [2, Eqn. (14)]. ■

We note furthermore (see, e.g., [2, IV-B.1]) that the statement in Theorem 1(b), for example, should be interpreted as follows. For any continuous, bounded function ϕ on S

$$E_{\hat{P}_N(s_k|Z_{k-1})}\phi \overset{N\to\infty}{\longrightarrow} E_{P(s_k|Z_{k-1})}\phi. \tag{9}$$

For additional background information on the convergence of a particle filter, we refer the readers to [5], [6] and [7].

III. Detection Problem and the LR-Detector

The problem of determining whether or not an object/signal is present in an observed situation/data amounts to a detection problem, see, e.g., [8]. This can also be formulated as a statistical hypothesis testing problem. Suppose, we want to perform the detection over a fixed finite horizon $M \in \mathbb{N}^+$. In other words, the null hypothesis is that there is no object/signal present in the last M time points and the alternative is that there is one.

Note that the measurement model (2) holds if there is an object present. Without any object, the measurements will be only noise. Thus, at each time step k, we want to test:

$\mathcal{H}_0$: No signal present, i.e.,
$z_j = v_j, \; j = k-M+1,\dots,k$
versus
$\mathcal{H}_1$: There is a signal present, i.e.,
$z_j = h(s_j, v_j), \; j = k-M+1,\dots,k.$

A. Main Result / Convergence of LR

Under the Neyman-Pearson (NP) paradigm, an optimal test procedure is based on the likelihood ration (LR) given by

$$L(z_{k-M+1},\dots,z_k) = \frac{p(z_{k-M+1},\dots,z_k \mid \mathcal{H}_1)}{p(z_{k-M+1},\dots,z_k \mid \mathcal{H}_0)}. \tag{10}$$

The *likelihood ratio test* rejects the null hypothesis if $L > \tau$, where τ is chosen in such a way that the probability of false alarm is below a certain acceptable threshold. The Neyman-Pearson Lemma (see, e.g., [8]) states that this procedure has the highest detection probability among all the procedures with false alarm probability below the threshold.

The authors in [1] have exploited the running particle filter to approximate the LR as

$$L(z_{k-M+1},\dots,z_k) \approx \hat{L}_N := \frac{\Pi_{j=k-M+1}^{k}(\sum_{i=1}^{N}\tilde{q}_j^i)}{N^M\,\Pi_{j=k-M+1}^{k} p_v(z_j)} \tag{11}$$

where $\tilde{q}_j^i$ and $p_v(\cdot)$ are as given in (6) and (4), respectively. The authors have subsequently used the approximated LR to perform the hypothesis test. However, as pointed out in the introduction, an explicit asymptotic property of $\hat{L}_N$ is missing in the literature on this topic.

The following theorem provides such an asymptotic result.

Theorem 2. *Consider the dynamical system (1) – (4) and suppose the conditions of Theorem 1 hold. Let the PF output be given by (6) and $\hat{L}_N \equiv \hat{L}_N(z_{k-M+1},\dots,z_k)$ be as given in (11). Then*

$$\lim_{N\to\infty}\hat{L}_N(z_{k-M+1},\dots,z_k) = L(z_{k-M+1},\dots,z_k). \tag{12}$$

Proof. Observe that under the null hypothesis $\mathcal{H}_0$

$$p(z_{k-M+1},\dots,z_k \mid \mathcal{H}_0) = \Pi_{j=k-M+1}^{k} p_v(z_j), \tag{13}$$

and under the alternative hypothesis

$$p(z_{k-M+1},\dots,z_k \mid \mathcal{H}_1) = \Pi_{j=k-M+1}^{k} p(z_j \mid Z_{j-1}), \tag{14}$$

where $p(z_j \mid Z_{j-1})$ can be determined from the measurement equation (2) as follows:

$$\begin{aligned} p(z_j \mid Z_{j-1}) &= \int_{\mathcal{S}} p(z_j \mid s_j)\, p(s_j \mid Z_{j-1})\, ds_j \\ &= E_{P(s_j|Z_{j-1})}\, p(z_j \mid \cdot) \\ &= \lim_{N\to\infty} E_{\hat{P}_N(s_j|Z_{j-1})}\, p(z_j \mid \cdot) \qquad \text{[using (9)]} \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} \int_{\mathcal{S}} \frac{1}{N} \delta(s_j - s_j^i)\, p(z_j \mid s_j)\, ds_j \quad \text{[by (8)]} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} p(z_j \mid s_j^i) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \tilde{q}_j^i \quad \text{[by (6)].} \end{aligned}$$

The result then follows from the fact that the limit of the product of (finite number of) sequences equals the (finite) product of the limits. ■

We have thus proven that the particle-filter-based likelihood ratio is asymptotically equal to the true likelihood ratio.

B. *Optimality*

The particle-filter-based LR-detectors are especially useful for a (radar) Track-Before-Detect application; see, e.g., [9]. Note that in our results/model, it is (implicitly) assumed that an object/signal is either present or absent during the entire M-sized observation window. This is fundamentally different from, e.g., the assumptions made for a Bernoulli filter (see [10]) where objects/signals can appear/disappear at any time instant. The Bernoulli filter is Bayes optimal for the *joint estimation* of object position and object existence. On the other hand, the combination of a running particle filter together with the PF-based LR-detector, as described above, will be Bayes optimal[1] for the position estimation part and NP optimal for the detection part. We stress that the NP optimality is highly desired in a (radar) sensor application, because it keeps the false alarm probability fixed at a specific (low) level and maximizes the probability of detection. The latter quantities are used, e.g., in (commercial) radar systems, see [12] and are well known and well understood by the designers and end-users alike. We can thus conclude that it is not possible to construct another detection mechanism that performs strictly better than the proposed LR-detector in both aspects.

It is worthwhile to remark here that one could design a detector from the Bernoulli filter, by thresholding the probability of existence of an object. See, for example, [13]. However, as mentioned in [13, 1st paragraph, Section IV.D], the false alarm probability generated by the detector cannot be guaranteed to be fixed (at a low level). This could be considered as a drawback of the Bernoulli-filter-based detector, if a guaranteed (low) level of false probability is required. A comprehensive comparison study between the two detectors remains as an intersting future work.

[1]Bayes optimality here refers [11] to the fact that the method provides the *posterior* distribution of the object “positions”, and also, in a sequential manner so that the posterior at the previous time instant can be used as the *prior* in calculating the posterior for the current time. In the case of a PF, the *a posteriori* density is reconstructed approximately, but in a sequential manner.

IV. Simulation Results

In this section, we validate the results through a simple numerical example, namely, a linear-Gaussian system. The reason is that for such a system we can determine the likelihood ratio exactly/analytically. In particular, the likelihoods $p(z_k \mid Z_{k-1})$ can be calculated/evaluated by using the expressions for the innovations in a running Kalman Filter (KF), see e.g. [14]. This *ground truth* can then be compared to the PF-based approximation. In particular, we take the absolute difference between the true $p(z_k \mid Z_{k-1})$, as obtained from the KF, and its PF-based approximation, as given in the proof of Theorem 2. We call this difference $\Delta p(z_k \mid Z_{k-1})$, see also below. We are fully aware that our numerical simulation study is not representative for a true and realistic target tracking situation. Applying our technique to such an example and comparing it to alternative methods, remains a future task for now.

The linear-Gaussian system we consider is:

$$s_{k+1} = s_k + w_k \tag{15}$$

$$z_k = s_k + v_k \tag{16}$$

$$s_0 \sim N(\mu, \sigma_0^2) \tag{17}$$

where the process and measurement noises are assumed to be independent and zero mean Gaussian random variables with standard deviations: $\sigma_{w_k} = 0.5$ and $\sigma_{v_k} = 0.5$, respectively. Once a set of synthetic data set is generated, we run a Kalman filter and a few particle filters with varying number of particles on the (same) measurement data.

The simulation results are presented below. Table I provides the approximation error of the particle-based likelihood as a function of the number of particles used. The likelihoods – the true (from KF) and the PF-based approximations with $N = 100$ and $N = 5000$, for the first 10 time steps – are presented in Fig. 2 and Fig. 4, respectively. Fig. 1 and Fig. 3 show the corresponding PF-based state estimates, together with the KF estimates (the true posterior mean).

Clearly, the figures and the table support the main convergence result of Section III. Indeed, as stated in Theorem 2, the PF-based LR converges to the true LR, as the number of particles tends to infinity.

approximation error of $p(z_k \mid Z_{k-1})$	
N	$\Delta p(z_k \mid Z_{k-1})$
100	0.012
1000	0.0047
5000	0.0014
20k	0.0010
50k	0.00047
100k	0.00028
500k	0.00013

TABLE I
ERROR OF THE PARTICLE-BASED LIKELIHOOD AS A FUNCTION OF THE NUMBER OF PARTICLES

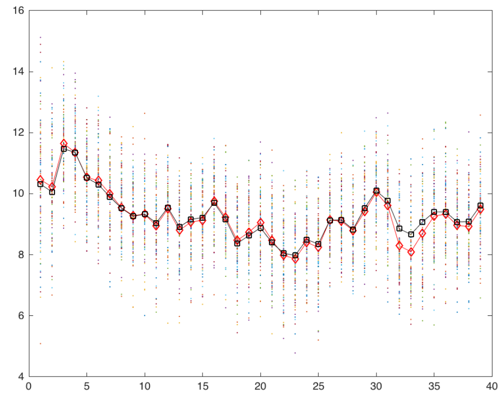

Fig. 1. Particle filter output – individual particles ($N = 100$) and PF-means (black squares) – and Kalman filter estimates (red diamonds). The horizontal axis denotes time-step k and vertical axis: (simulated) state.

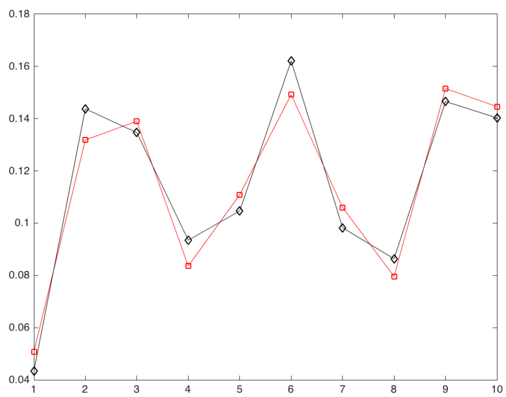

Fig. 2. True, Kalman innovation based, likelihood $p(z_k \mid Z_{k-1})$ (red squares) and particle-based approximation (black diamonds) with $N = 100$. The horizontal axis denotes time-step k.

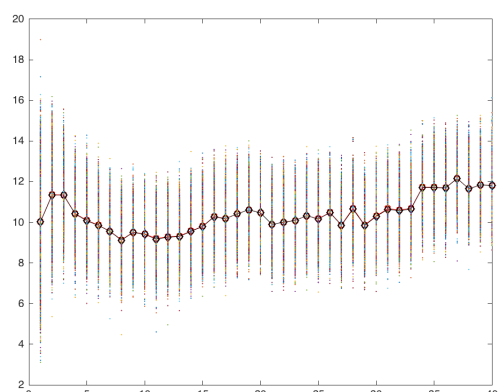

Fig. 3. Particle filter output – individual particles ($N = 5000$) and PF-means (black squares) – and Kalman filter estimates (red diamonds). The horizontal axis denotes time-step k and vertical axis: (simulated) state.

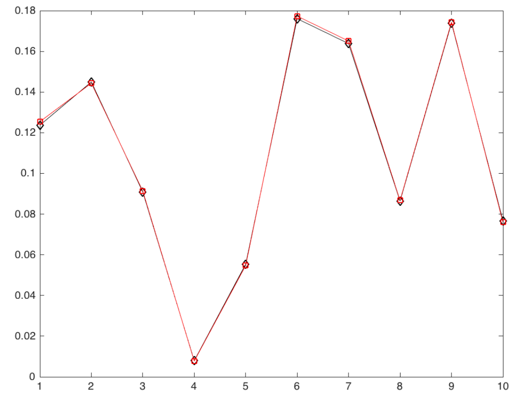

Fig. 4. True, Kalman innovation based, likelihood $p(z_k \mid Z_{k-1})$ (red squares) and particle-based approximation (black diamonds) with $N = 5000$. The horizontal axis denotes time-step k.

V. Conclusion

This letter deals with the detection problem in possibly nonlinear and/or non-Gaussian systems. A particle filter combined with a likelihood ratio detector is considered. This is very appealing, e.g., in radar tracking of stealth/dim objects, since the NP optimal detector directly relates to operationally relevant quantities such as the probability of detection and the probability of false alarm. The most important contributions and conclusions of this letter are the following.

- It provides a proof of the convergence of the particle-filter-based detector to the true NP-optimal LR detector for nonlinear and/or nonGaussian systems/signals.
- It provides a numerical illustration/verification of the main result by showing, via a simulation example, that the PF-based likelihoods converge to the true likelihoods.

An interesting topic for further research is to compare the proposed method, which is based on an NP-optimal detector, to one which is based on a Bernoulli filter [10], [13]. This, however, would require some adjustments in the theory and/or the application to allow for the appearing/disappearing targets. Furthermore, one would also require a carefully constructed criterion to be used for comparison, because, as mentioned in Section III-B, the two mechanisms optimize different quantities/aspects of the dynamical system under consideration.

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