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The sequential price of anarchy for affine congestion games with few players

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ABSTRACT

This paper determines the sequential price of anarchy for Rosenthal congestion games with affine cost functions and few players. We show that for two players, the sequential price of anarchy equals 1.5, and for three players it equals $1039/488 \approx 2.13$. While the case with two players is analyzed analytically, the tight bound for three players is based on the explicit computation of a worst-case instance using linear programming. The basis for both results are combinatorial arguments to show that finite worst-case instances exist.

1. Introduction

Congestion games, introduced by Rosenthal [25], are among the most studied classes of games. In a congestion game, we have n players $N = \{1, \dots, n\}$ that compete for certain subsets of a set $R = \{1, \dots, m\}$ of m resources, while the congestion (or cost) that each player experiences per resource, depends on the set of players using it. Subsets of resources that are feasible for a given player $i \in N$ are given by player specific collections $\mathcal{A}_i \subseteq 2^R$. Applications of this class of games include many sorts of allocation problems with scarce resources, and specifically atomic network routing problems, where the resources are network links and each of the n players is interested in a least congested path from her origin to destination. Rosenthal congestion games are also referred to as *atomic* congestion games sometimes. This is mainly to distinguish from other types of congestion games such as e.g. the Wardrop traffic models which are also known as non-atomic network routing games, as studied e.g. by Roughgarden and Tardos [26].

Rosenthal proved the existence of pure Nash equilibria in any congestion game, by defining what is known as a (Rosenthal) potential function [25]. Assuming that one is interested in some objective function on the set of equilibrium outcomes of a game, the price of anarchy, introduced in 1999 by Koutsoupias and Papadimitriou [20,21], relates the cost of a worst-case Nash equilibrium to the cost of a globally optimal solution. Shortly after the price of anarchy had been determined for non-atomic network routing games by Roughgarden and Tardos [26], the price of anarchy for atomic congestion games was analyzed as well. Christodoulou and Koutsoupias [9], and independently Awerbuch et al. [3,4] show that the price of anarchy for congestion games with affine cost functions equals 2 for two players, and $5/2$ for three or more players.

However, worst-case Nash equilibria can be overly pessimistic in some situations, and specifically in situations where players take their decisions sequentially, worst-case Nash equilibria can often not be realized as equilibrium outcomes. In 2012, Paes Leme et al. [24] have therefore introduced the idea to study the *sequential* price of anarchy. The sequential price of anarchy relates the total cost of the worst-case outcome of a subgame perfect equilibrium of a corresponding extensive form game to the total cost of an optimal solution. This paper establishes tight bounds for the sequential price of anarchy for congestion games with affine cost functions for the case with $n = 2$ and $n = 3$ players.

2. Related work and contribution

Also prior to Paes Leme et al. [24], some authors have addressed the quality of outcomes of network routing games in which players act sequentially. This includes Olver [23], Harks, Heinz and Pfetsch [14], Harks and Vegh [15], as well as Farzad, Olver and Vetta [11]. Without going into details, in these papers the cost of a player depends only on preceding players on the same resource, like in traffic situations where players enter resources in sequence, and a player's cost only depends on the players that entered the resource before. Consequently, players experience different costs on one and the same resource.

We here follow a simpler model where all players experience the same cost per resource. Hence a rational player, if aiming to be farsighted, needs to take into account all future players. This indeed leads to the definition of the sequential price of anarchy as defined by Paes Leme et al. [24]. Subsequent to [24], various researchers

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studied the sequential price of anarchy for several classes of games, with mixed results. For a handful of problems it was shown that the sequential price of anarchy is strictly smaller than the price of anarchy [16,18,24], while for others this turns out to be exactly opposite [1,2,6,7,10]. Specifically, Correa et al. [10] show that in symmetric, atomic network routing games, the sequential price of anarchy can become as large as $\Omega(\sqrt{n})$ for large n, n being the number of players, while the price of anarchy is known to be at most 5/2 for all $n \geq 3$ [3,4]. This of course implies the same lower bound for general congestion games with affine cost functions. Subsequently to [10], work of Groenland and Schäfer [13] suggests that the presence of ties is pivotal for the existence of such worst-case instances. This paper shows that when the number of players is 2 or 3, sequential play and subgame perfect equilibria lead to outcomes that are strictly better than the worst-case Nash equilibria. More precisely, we show that the sequential price of anarchy equals 1.5 for two players, and $1039/488 \approx 2.13$ for three players. Both bounds are shown to be tight. The case with more than three players will be briefly discussed in Section 6.

We believe that, although considering the cases of two and three players only, our results are interesting for at least two reasons. First, from an application's point of view it seems not unreasonable to consider (sequential) congestion games with a small number of players, such as the competition for scarce resources by a small number of corporate players, for example. Moreover, through the lens of computational tractability, sequential games with few players are also the limiting cases that still appear practically feasible. However, note that already for two players, computing the outcome of a subgame perfect equilibrium may be NP-hard [10]. For an arbitrary number of players, it is PSPACE-hard [24]. Second, we obtain our results by using a sequence of combinatorial arguments which yield that the worst-case must be attained on a finite instance of reasonable size. That leads to a linear programming (LP) formulation to actually compute an instance that attains the worst-case for the sequential price of anarchy. In that sense, our proof can be seen as a computerized proof, and the correctness of the result depends on the correctness of the underlying LP; even though we also verify the resulting worst-case instance. We believe that both the sequence of combinatorial arguments as well as the LP formulation itself are of independent interest, also because it is a first example of doing that for sequential games and subgame perfection.

Inspired by our work, Correa et al. have used another, LP-based proof to show that for two players, the sequential price of anarchy for atomic *network routing* games is even smaller than that of general two-player congestion games; it equals 7/5 [10]. That said, it should be noted that linear programming techniques, and specifically linear programming duality has been used before, e.g. by Nadav and Roughgarden [22] as well as Bilò [5] in order to obtain new or improved bounds for the price of anarchy in several settings; see also [8].

We end this section by mentioning that -naturally- also more general than affine cost functions have been considered in the context of the price of anarchy for congestion games. A detailed account of all these results is beyond the scope of this paper, and we refer, e.g. to the work by Fotakis [12] for price of anarchy results with general cost functions.

3. Notation and preliminaries

An instance I of a *congestion game* is defined as follows: There is a finite set R of resources, a finite set $N = \{1, \dots, n\}$ of players, and the action space $\mathcal{A}_i \subseteq 2^R$ of any player $i \in N$ is a set of subsets of R. We say a player i *chooses* a resource $r \in R$, if player i chooses an action $A_i \ni r$. Each resource $r \in R$ has a cost function $c_r : \mathbb{N} \to \mathbb{R}^+$ that is nonnegative and nondecreasing. Given an action profile $A = (A_1, \dots, A_n)$, each resource r has cost $c_r(x_r)$, where x_r denotes the number of players choosing r, $x_r = |\{A_i \mid r \in A_i\}|$. Each player $i \in N$ pays the cost of each of the resources in her chosen action A_i, which we denote by $C_i(A) = C_i(A_1, \dots, A_n) = \sum_{r \in A_i} c_r(x_r)$. The total cost of all players in profile A is denoted $C(A) = \sum_{i \in N} C_i(A)$.

Affine congestion games are congestion games where the cost functions of all resources $r \in R$ are of the form $c_r(x_r) = \alpha_r + \beta_r x_r$ for $\alpha_r \geq 0$, $\beta_r \geq 0$. Here, α_r can be thought of as an activation cost for resource r. We refer to α_r as the *constant* cost of resource r, and to β_r as the *weight* of resource r. We also denote by $\alpha(R') = \sum_{r \in R'} \alpha_r$ the total constant cost of resources in $R' \subseteq R$ and by $\beta(R') = \sum_{r \in R'} \beta_r$ the total weight of resources in $R' \subseteq R$.

In a sequential congestion game, let us assume that players act in the order $1, \dots, n$. The strategy of player 1 is then one of the available actions from $\mathcal{A}_1$. The strategy of a player $k > 1$, however, is more complex, as it has to prescribe one action $A_k \in \mathcal{A}_k$ for each of the possible states that the game can be in, depending on the actions of all players $i = 1, \dots, k-1$. That means that the strategy of player 2 can be written as a vector of actions of length $|\mathcal{A}_1|$, for player 3 a vector of actions of length $|\mathcal{A}_1| \cdot |\mathcal{A}_2|$, etc. Of course other, more compact representations of players' strategies might be possible, however this is not relevant for what follows. We will use $S = (S_1, \dots, S_n)$ to denote the profile of strategies of the resulting extensive form game (for a fixed order of players), and S^{SPE} to denote a subgame perfect equilibrium in that game.

If we denote by A^{OPT} the profile of actions of the n players in an allocation that minimizes the total cost $C(A)$, and by A^{SPE} the profile of actions in the outcome of a subgame perfect equilibrium S^{SPE}, the *sequential price of anarchy* [24] is defined as

$$\mathrm{SPoA} := \sup \frac{C(A^{\mathrm{SPE}})}{C(A^{\mathrm{OPT}})},$$

where the supremum is taken over all possible finite congestion games, all possible permutations of the players, and all possible subgame perfect equilibria.

4. Warmup: The case of two players

We start with the sequential price of anarchy for two players.

Theorem 1. *The sequential price of anarchy* SPoA *equals 1.5 for linear and affine atomic congestion games with two players.*

The lower bound is based on the following simple example.

Example 2. There are two players 1,2 and three resources 1, 2, 3, with linear cost functions with weights $\beta_1 = \beta_2 = 1$, $\beta_3 = 2$. Player 1 can choose either resource 1 or resource 2. Player 2 can choose either resource 2 or resource 3. This example is illustrated in Fig. 1. ◁

For Example 2 it is clearly optimal when player 1 chooses resource 1 and player 2 chooses resource 2, giving a total cost of $1 + 1 = 2$. In the outcome of a worst-case subgame perfect equilibrium, however, first player 1 chooses resource 2 and then player 2 chooses resource 3, with a total cost of $1 + 2 = 3$. The subgame perfect strategy of player 2 is to choose resource 2 if player 1 chooses resource 1, and resource 3 if player 1 chooses resource 2. In fact, player 1 is then indifferent, and the worst-case happens if she chooses resource 2. Given that player 1 chooses resource 2, also player 2 is indifferent, and again, the worst-case happens if she chooses resource 3.

Note that this example rests on the existence of ties. However, this can be easily avoided, by decreasing the weight of resource 2 by some small constant ϵ and decreasing the weight of resource 3 by 3ϵ. Then the above described subgame perfect equilibrium is unique.

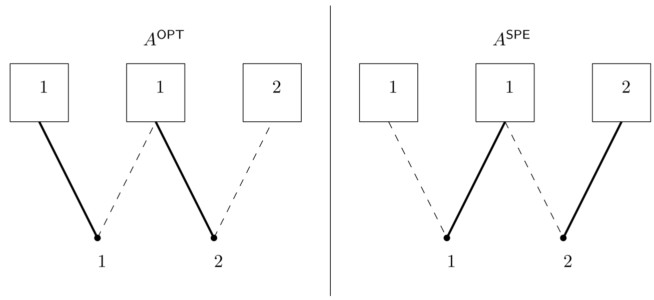

Fig. 1. The congestion game from Example 2. Dots represent players, squares represent resources, numbers in resources denote weights, and edges represent actions. Fat edges represent chosen actions.

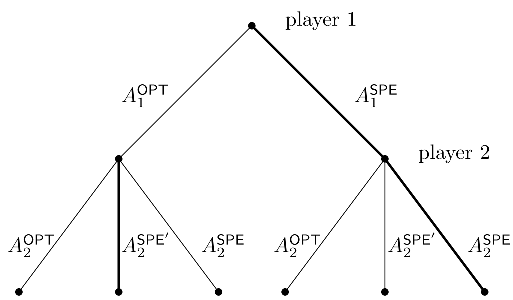

Fig. 2. All relevant actions in the game tree for 2 player congestion games. Fat lines represent subgame perfect actions.

We next prove a matching upper bound. The idea is to realize that only a small part of the game tree is relevant, namely the actions that are played in some optimal solution, and the actions that are played in equilibrium. That means, we only need to consider *two* actions of player 1, namely optimum and equilibrium, and *three* actions of player 2, namely optimum and the best responses to the two relevant actions of player 1; see also Fig. 2.

To formalize this into a proof of the upper bound on the sequential price of anarchy, we introduce the following notation. Denote by $A^{\mathsf{OPT}} = (A_1^{\mathsf{OPT}}, A_2^{\mathsf{OPT}})$ the pair of actions of the two players 1 and 2, respectively, in an optimal allocation. Denote by $A^{\mathsf{SPE}} = (A_1^{\mathsf{SPE}}, A_2^{\mathsf{SPE}})$ the actions in the outcome of a subgame perfect equilibrium. Finally, denote by $A_2^{\mathsf{SPE}'}$ the subgame perfect action of player 2, in the subgame induced by A_1^{OPT} i.e. $A_2^{\mathsf{SPE}'}$ is a best response of player 2 to A_1^{OPT}. Note that both players might have more actions at their disposal, however these are not relevant for the analysis. Also note that we do not exclude cases where any two sets from $A_1^{\mathsf{OPT}}, A_1^{\mathsf{SPE}}, A_2^{\mathsf{OPT}}, A_2^{\mathsf{SPE}'}$ or A_2^{SPE} overlap. It could even be that they are equal. The situation is shown in Fig. 2; it displays only the relevant part of the game tree.

Recall that $\alpha(R')$ and $\beta(R')$ respectively denote the total constant cost and weight of a set R' of resources. For brevity, we also use the following notation:

$$a = \alpha(A_1^{\mathsf{OPT}}), \qquad b = \alpha(A_2^{\mathsf{OPT}}),$$
$$c = \beta(A_1^{\mathsf{OPT}} \setminus A_2^{\mathsf{OPT}}), \qquad d = \beta(A_2^{\mathsf{OPT}} \setminus A_1^{\mathsf{OPT}}),$$
$$\gamma = \beta(A_1^{\mathsf{OPT}} \cap A_2^{\mathsf{OPT}}), \qquad \delta = \beta(A_1^{\mathsf{OPT}} \cap A_2^{\mathsf{SPE}'}) - \beta(A_1^{\mathsf{OPT}} \cap A_2^{\mathsf{OPT}}).$$

Note that δ denotes the difference in the total weight of shared resources, when player 1 chooses A_1^{OPT} and player 2 switches from A_2^{OPT} to $A_2^{\mathsf{SPE}'}$.

Observation 3. $C(A^{\mathsf{OPT}}) = C_1(A^{\mathsf{OPT}}) + C_2(A^{\mathsf{OPT}}) = (a + c) + (b + d) + 4\gamma$.

We prove the upper bound for Theorem 1 by deriving an upper bound on $C_1(A^{\mathsf{SPE}})$ (Lemma 4) and two upper bounds on $C_2(A^{\mathsf{SPE}})$ (Lemma 5 and Lemma 6).

Lemma 4. $C_1(A^{\mathsf{SPE}}) \leq a + c + 2\gamma + \delta$.

Proof.

$$\begin{aligned} C_1(A^{\mathsf{SPE}}) &\leq C_1(A_1^{\mathsf{OPT}}, A_2^{\mathsf{SPE}'}) \\ &= \alpha(A_1^{\mathsf{OPT}}) + \beta(A_1^{\mathsf{OPT}}) + \beta(A_1^{\mathsf{OPT}} \cap A_2^{\mathsf{SPE}'}) \\ &= a + (c + \gamma) + (\gamma + \delta) \\ &= a + c + 2\gamma + \delta, \end{aligned} \tag{1}$$

where the inequality follows from the Nash inequality (since a subgame perfect equilibrium induces a Nash equilibrium in every state). □

Lemma 5. $C_2(A^{\mathsf{SPE}}) \leq 2(b + d + \gamma - \delta)$.

Proof.

$$\begin{aligned} C_2(A^{\mathsf{SPE}}) &\leq C_2(A_1^{\mathsf{SPE}}, A_2^{\mathsf{SPE}'}) \\ &\leq \alpha(A_2^{\mathsf{SPE}'}) + 2\beta(A_2^{\mathsf{SPE}'}) \\ &\leq 2(\alpha(A_2^{\mathsf{SPE}'}) + \beta(A_2^{\mathsf{SPE}'})), \end{aligned}$$

where the first inequality follows from the Nash inequality, and the second inequality follows from the fact that each resource can be chosen by at most two players. We now show that $\alpha(A_2^{\mathsf{SPE}'}) + \beta(A_2^{\mathsf{SPE}'}) \leq b+d+\gamma-\delta$, proving the lemma. By the Nash inequality, we obtain

$$\begin{aligned} C_2(A_1^{\mathsf{OPT}}, A_2^{\mathsf{SPE}'}) &\leq C_2(A^{\mathsf{OPT}}) \\ &= b + d + 2\gamma \\ &= b + d + \gamma - \delta + (\gamma + \delta). \end{aligned} \tag{2}$$

Since $\beta(A_1^{\mathsf{OPT}} \cap A_2^{\mathsf{SPE}'}) = \gamma + \delta$, plugging

$$C_2(A_1^{\mathsf{OPT}}, A_2^{\mathsf{SPE}'}) = \alpha(A_2^{\mathsf{SPE}'}) + \beta(A_2^{\mathsf{SPE}'}) + \beta(A_1^{\mathsf{OPT}} \cap A_2^{\mathsf{SPE}'})$$

into (2), we obtain

$$\alpha(A_2^{\mathsf{SPE}'}) + \beta(A_2^{\mathsf{SPE}'}) \leq b + d + \gamma - \delta. \quad \square \tag{3}$$

Lemma 6. $C_2(A^{\mathsf{SPE}}) \leq a + c + b + d + 3\gamma$.

Proof.

$$\begin{aligned} \beta(A_1^{\mathsf{SPE}}) &\leq C_1(A^{\mathsf{SPE}}) \\ &\leq a + c + 2\gamma + \delta, \end{aligned} \tag{4}$$

where the second inequality follows from Lemma 4. Now,

$$\begin{aligned} C_2(A^{\mathsf{SPE}}) &\leq C_2(A_1^{\mathsf{SPE}}, A_2^{\mathsf{SPE}'}) \\ &\leq \alpha(A_2^{\mathsf{SPE}'}) + \beta(A_2^{\mathsf{SPE}'}) + \beta(A_1^{\mathsf{SPE}}) \\ &\leq b + d + \gamma - \delta + a + c + 2\gamma + \delta \\ &= a + c + b + d + 3\gamma. \end{aligned}$$

The first inequality follows from the Nash inequality. The second inequality follows from the fact that each resource that player 1 chooses adds at most the weight of that resource to the cost of player 2. The third inequality follows from (3) and (4). □

Lemma 7. SPoA ≤ 1.5 *for affine atomic congestion games with two players.*

Proof.

$$\begin{aligned}2(C(A^{\mathrm{SPE}})) &= 2(C_1(A^{\mathrm{SPE}}) + C_2(A^{\mathrm{SPE}}))\\ &\leq 2a + 2c + 2(2\gamma + \delta) + 2(b + d + \gamma - \delta)\\ &\quad + a + c + b + d + 3\gamma\\ &= 3a + 3c + 3b + 3d + 9\gamma\\ &\leq 3C(A^{\mathrm{OPT}}),\end{aligned}$$

where the first inequality follows from Lemmas 4–6, and the last inequality follows from Observation 3. □

Theorem 1 now follows from Lemma 7. While the case with two players can still be written down algebraically, for three players we turn the same proof idea into a linear programming formulation.

5. The case of three players

Along the lines of the proof for the case with two players, we also settle the case with three players.

Theorem 8. SPoA $= \frac{1039}{488} \approx 2.13$ *for affine and linear congestion games with three players.*

To prove the theorem, we use a linear programming (LP) approach. We first use a sequence of simple, combinatorial arguments to show that a worst-case instance is moderate in size. This is done in Lemmas 9–11. These lemmas apply to games with three players or more. We then compute a worst-case instance for the case with three players using a standard LP solver.

5.1. Worst-case instances of moderate size

We use the following notation. Define the series

$$z_1 := 2 \text{ and } z_i := 1 + \prod_{j<i} z_j \text{ for all } i \geq 2.$$

Note that $z_2 = 3$, $z_3 = 7$, $z_4 = 43$, and that z_i grows doubly-exponential.

Lemma 9. *For any instance I of a congestion game, there exists an instance I', such that $|\mathcal{A}_i| \leq z_i$ for all players $i = 1, \ldots, n$, and* $\mathrm{SPoA}(I') = \mathrm{SPoA}(I)$.

Proof. Denote by A^{OPT} an optimal outcome for instance I and by A^{SPE} the outcome of a worst-case subgame perfect equilibrium. The proof goes by successively eliminating all actions that are not chosen, neither in A^{OPT} nor in the subgame perfect equilibrium, in the order of the players $1, 2, \ldots, n$. For the first player, we thereby restrict $\mathcal{A}_1$ to only two actions: A_1^{OPT} and A_1^{SPE}. For the second player, we thereby restrict $\mathcal{A}_2$ to at most $z_2 = 3$ actions, namely the actions prescribed by the subgame perfect equilibrium for the two relevant actions of player 1, plus A_2^{OPT}. More generally, for the kth player, we restrict $\mathcal{A}_k$ to at most $1 + \prod_{i<k} z_i$ actions, namely the actions prescribed by the subgame perfect equilibrium in each of the at most $\prod_{i<k} z_i$ states, plus A_k^{OPT}. In the so reduced game I', S^{SPE} is also subgame perfect, as the actions that were removed are all actions with inferior or identical cost for the respective player. □

Lemma 10. *For any instance I of a congestion game, there exists an instance I', such that $|\mathcal{A}_i| \leq z_i$ for all players $i = 1, \ldots, n$, and $|R| \leq 2^{\sum_{i\in N}|\mathcal{A}_i|} - 1$, and* $\mathrm{SPoA}(I') = \mathrm{SPoA}(I)$.

Proof. By Lemma 9, we may restrict to instances I with $|\mathcal{A}_i| \leq z_i$ for all players i. Suppose the claim is false. Then choose among all instances that falsify the claim an instance I that minimizes the number of resources $|R|$. Then each resource $r \in R$ is chosen in at least one of the actions. As we have at most $\sum_{i\in N}|\mathcal{A}_i|$ different actions, there are at most $2^{\sum_{i\in N}|\mathcal{A}_i|} - 1$ different non-empty subsets of the set of all actions. Since $|R| > 2^{\sum_{i\in N}|\mathcal{A}_i|} - 1$, by the pigeonhole principle the "incidence matrix" of all resources and actions contains at least two identical rows. In other words, there must exist two resources $r, r' \in R$ such that every action either contains both r and r', or it contains neither r nor r'. Now we can construct an instance I' identical to I, except instead of r and r', it contains a new resource r'' for which $\beta_{r''} = \beta_r + \beta_{r'}$ and $\alpha_{r''} = \alpha_r + \alpha_{r'}$. Other than this replacement, instance I' has the same sets of actions as instance I. Now each outcome A in I' has the same costs as A in I. Therefore the same actions are subgame perfect and $\mathrm{SPoA}(I') = \mathrm{SPoA}(I)$. As I' has one resource less, we obtain a contradiction. □

Recall that A^{OPT} denotes an action profile that minimizes $C(A)$ for a given instance I. The final lemma bounds the constant cost and weights in a worst-case instance.

Lemma 11. *For any instance I of a congestion game with affine cost functions, there exists an instance I' such that $|\mathcal{A}_i| \leq z_i$ for all players $i = 1, \ldots, n$, $|R| \leq 2^{\sum_{i\in N}|\mathcal{A}_i|} - 1$, $\alpha_r + \beta_r \leq nC(A^{\mathrm{OPT}})\ \forall\, r \in R$, and* $\mathrm{SPoA}(I') = \mathrm{SPoA}(I)$.

Proof. We are only left to show the claim on the cost functions. However resources r with $\alpha_r + \beta_r > nC(A^{\mathrm{OPT}})$ can safely be eliminated, as it cannot be subgame perfect for any player i to choose resource r: choosing A_i^{OPT} instead, yields a cost at most $nC_i(A^{\mathrm{OPT}}) \leq nC(A^{\mathrm{OPT}}) < \alpha_r + \beta_r$. □

Specifically, for congestion games with three players, in order to find a worst-case instance we only need to consider games of moderate size. It suffices to let $|\mathcal{A}_1| = 2$, $|\mathcal{A}_2| = 3$, $|\mathcal{A}_3| = 7$, $|R| = 2^{2+3+7} - 1 = 4095$, and $\alpha_r + \beta_r \leq 3C(A^{\mathrm{OPT}})$ for all resources $r \in R$. The linear program now works as follows: It maximizes the sequential price of anarchy over all instances with the properties described above. We have $2^{12} - 1 = 4095$ resources, one for every (potential) nonempty intersection of actions. The LP decides the weight $\beta_r \geq 0$ and constant cost $\alpha_r \geq 0$ of each resource. We define some fixed outcome as the optimum solution with total cost normalized to 1, which yields that $\alpha_r + \beta_r \leq 3$ for all $r \in R$, and we maximize the sequential price of anarchy.

5.2. Details LP formulation

In violation to our earlier nomenclature, we here choose to denote actions by lowercase letters a, a', b, b', c, c', μ and ν, because they appear as index to the decision variables. We use binary parameters $\delta_{\mu r}$ to specify whether resource r is chosen in action μ. For each resource r, we have decision variables α_r and β_r, the constant cost and weight of r, respectively. We fix a subgame perfect equilibrium S^{SPE} with subgame perfect outcome A^{SPE} and we use binary parameters z_a^1, z_{ab}^2 and z_{abc}^3 to prescribe which actions are part of the subgame perfect equilibrium S^{SPE} for players 1, 2, 3 respectively. To clarify these, for example $z_{ab}^2 = 1$ means that action b is the subgame perfect action by player 2, given player 1 plays a. We denote by $i.x$ the xth action of player i. We define the action profile $(1.1, 2.1, 3.1)$ as the optimal outcome with total cost normalized to 1. The subgame perfect equilibrium is also fixed, namely for player 1 it is action 1.2, for player 2 it is action 2.2 (given 1.1) and action 2.3 (given 1.2). For player 3, in branch $a \geq 1$ of the game tree, we let action $3.(a + 1)$ be the subgame perfect action. This implies that action profile $(1.2, 2.3, 3.7)$ is the outcome of the worst-case subgame perfect equilibrium.

We denote by $v_\mu = \sum_{r\in R} \delta_{\mu r}(\beta_r + \alpha_r)$ the cost of a player who chooses action μ, assuming no other players would be there. Next, we denote by $o_{\mu\nu} = \sum_{r\in R} \delta_{\mu r}\delta_{\nu r}\beta_r$ the additional costs that two players with actions μ and ν incur due to overlap in resources. We will use these auxiliary variables to determine the total cost of player i when players 1, 2 and 3 choose actions a, b and c, respectively. This we denote by $C_i(abc)$.

Since we have to be able to describe a subgame perfect equilibrium, we need a bit more of notation: $C_1(a)$ and $C_2(ab)$ denote the cost of actions of players 1 and 2 respectively, when successors play subgame perfect. For instance, $C_2(ab)$ denotes the cost of action b for player 2, when player 1 chooses action a and player 3 plays subgame perfect. Finally $C(A^{\mathrm{SPE}})$ will denote the total costs for all players in the subgame perfect outcome.

For each of the following parameters, variables, and constraints, we assume the following conventions on nomenclature:

$$a, a' \in \mathcal{A}_1, b, b' \in \mathcal{A}_2, c, c' \in \mathcal{A}_3, \mu, \nu \in \cup_{i\in N}\mathcal{A}_i, r \in R, i \in N.$$

In writing, a and a' denote actions of player 1, b and b' denote actions of player 2, c and c' denote actions of player 3, μ and ν are arbitrary actions, r is a resource, and i is a player. Also, let us denote by S^{SPE} the worst-case subgame perfect equilibrium that we seek to compute, so that S_i^{SPE} is the strategy of player $i = 1, 2, 3$. Also recall that $A^{\mathrm{SPE}} = (1.2, 2.3, 3.7)$ is the resulting action profile. The linear program is now as follows.

Binary parameters

$$\delta_{\mu r} \quad \forall r, \mu \quad \begin{cases} 1 & \text{if } r \in \mu \\ 0 & \text{otherwise} \end{cases}$$

$$z_a^1 \quad \forall a \quad \begin{cases} 1 & \text{if } a \text{ is prescribed by } S_1^{\mathrm{SPE}} \\ 0 & \text{otherwise} \end{cases}$$

$$z_{ab}^2 \quad \forall a, b \quad \begin{cases} 1 & \text{if } b \text{ is prescribed by } S_2^{\mathrm{SPE}} \text{ in state } a \\ 0 & \text{otherwise} \end{cases}$$

$$z_{abc}^3 \quad \forall a, b, c \quad \begin{cases} 1 & \text{if } c \text{ is prescribed by } S_3^{\mathrm{SPE}} \text{ in state } ab \\ 0 & \text{otherwise} \end{cases}$$

Variables

α_r	$\forall r$	constant cost of r
β_r	$\forall r$	weight of r
υ_μ	$\forall \mu$	constant cost plus weight of resources in μ
$o_{\mu\nu}$	$\forall \mu, \nu \mid \mu \neq \nu$	weight of resources in $\mu \cap \nu$
$C_i(abc)$	$\forall a, b, c, i$	cost of player i when players 1,2,3 choose a, b, c respectively
$C(A^{\mathrm{SPE}})$		costs in subgame perfect outcome A^{SPE}
$C_1(a)$	$\forall a$	cost of player 1 when she chooses a and 2,3 choose as prescribed by S^{SPE}
$C_2(ab)$	$\forall a, b$	costs of player 2 when players 1,2 choose a, b and 3 chooses as prescribed by S^{SPE}

Constraints

$$0 \le \alpha_r + \beta_r \le 3 \qquad \forall r \tag{5}$$

$$\upsilon_\mu = \sum_{r\in R} \delta_{\mu r}(\beta_r + \alpha_r) \qquad \forall \mu \tag{6}$$

$$o_{\mu\nu} = \sum_{r\in R} \delta_{\mu r}\delta_{\nu r}\beta_r \qquad \forall \mu, \forall \nu \mid \mu \neq \nu \tag{7}$$

$$C_1(abc) = \upsilon_a + o_{ab} + o_{ac} \qquad \forall a, b, c \tag{8}$$

$$C_2(abc) = \upsilon_b + o_{ab} + o_{bc} \qquad \forall a, b, c \tag{9}$$

$$C_3(abc) = \upsilon_c + o_{ac} + o_{bc} \qquad \forall a, b, c \tag{10}$$

$$C_3(abc) \le C_3(abc') \qquad \forall a, b, c \mid z_{abc}^3 = 1, c' \tag{11}$$

$$C_2(ab) \le C_2(ab') \qquad \forall a, b \mid z_{ab}^2 = 1, b' \tag{12}$$

$$C_1(a) \le C_1(a') \qquad \forall a \mid z_a^1 = 1, a' \tag{13}$$

$$C_1(a) = C_1(abc) \qquad \forall a, b \mid z_{ab}^2 = 1, c \mid z_{abc}^3 = 1 \tag{14}$$

$$C_2(ab) = C_2(abc) \qquad \forall a, b, c \mid z_{abc}^3 = 1 \tag{15}$$

$$\sum_{i\in N} C_i(1.1, 2.1, 3.1) = 1 \tag{16}$$

$$C(A^{\mathrm{SPE}}) = \sum_{i\in N} C_i(1.2, 2.3, 3.7) \tag{17}$$

Constraints (6) and (7) define costs of actions υ_μ and intersection costs $o_{\mu\nu}$ for all actions μ, ν. Constraints (8), (9), and (10) define the costs in each outcome for each player. Constraints (11), (12), and (13) guarantee that no player can improve from the subgame perfect equilibrium S^{SPE}. Constraints (14) define $C_1(a)$, and constraints (15) define $C_2(ab)$, as explained earlier. The optimal solution is the action profile $(1.1, 2.1, 3.1)$ and has total cost equal to 1, encoded into constraint (16). Constraint (17) defines $C(A^{\mathrm{SPE}})$. Finally, constraints (5) are not necessary, but bound the variables α_r and β_r.

The objective is to maximize $C(A^{\mathrm{SPE}})$, since, due to the normalization, this value equals the sequential price of anarchy. This allows us to give the LP-based proof of the main theorem of this paper. This result is particularly interesting in comparison to the tight bound 2.5 for the price of anarchy for non-sequential three player congestion games [3,4,9].

Proof of Theorem 8. We have solved the above described linear program using the AIMMS modeling framework with CPLEX 12.5 as LP solver. We obtain an optimal solution with value $\frac{1039}{488} \approx 2.13$, which proves Theorem 8. □

5.3. Tight lower bound instance

We have used a mixed integer linear program (MIP) in order to model situations with smaller action spaces and fewer resources. Here, we use integer *variables* (instead of parameters) to decide which actions are subgame perfect for the third player. Naturally, such a MIP only yields lower bounds on the sequential price of anarchy, but no upper bounds. Nevertheless, after inspection of the solution of the linear program, using the MIP we have been able to compute a lower bound example that matches the upper bound of Theorem 8, so that player 3 only uses four different actions. We here give this instance, scaled to integers such that the cost in the optimum solution equals 488.

Example 12. There are three players. Player 1 has two actions 1.1, 1.2. Player 2 has three actions 2.1, 2.2, 2.3. Player 3 has 4 actions 3.1, 3.2, 3.3, 3.4. There are 13 resources $R = \{1, \dots, 13\}$ with constant costs $\alpha_r = 0$ for all $r \in R$. Table 1 shows for each resource, its weight and the actions that contain it. This instance is also illustrated in a more intuitive way in Fig. 3. ◁

Backward induction on the game tree yields a subgame perfect outcome where player 1 chooses 1.2, player 2 chooses 2.3, and player 3 chooses 3.4, yielding a total cost of $C(A^{\mathrm{SPE}}) = 1039$. The game tree is shown in Fig. 4.

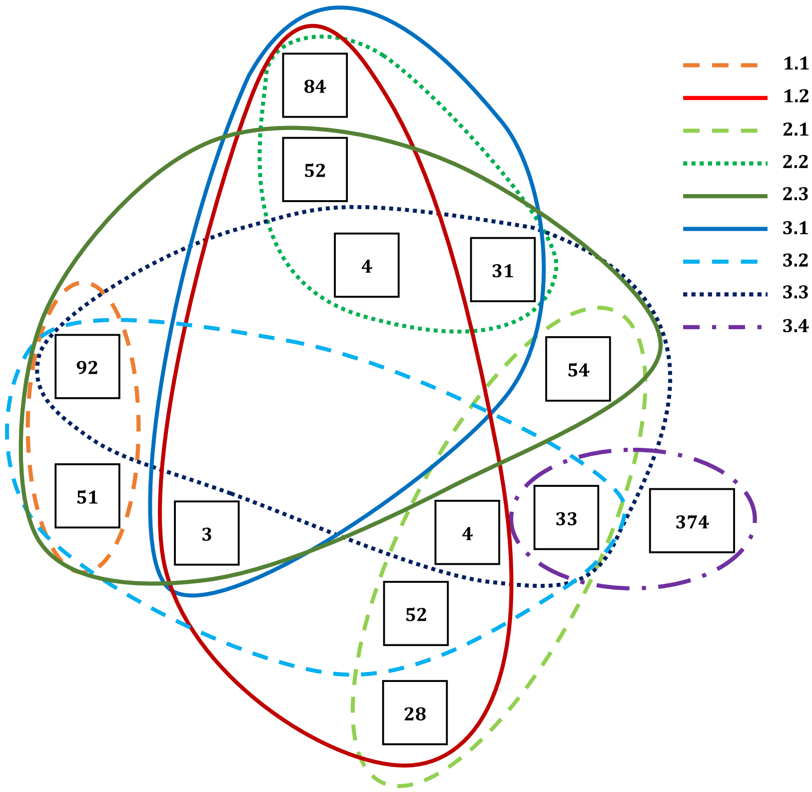

Fig. 3. Illustration of Example 12. Squares represent resources. The number in each resource denotes its weight. Encircled resources depict the actions.

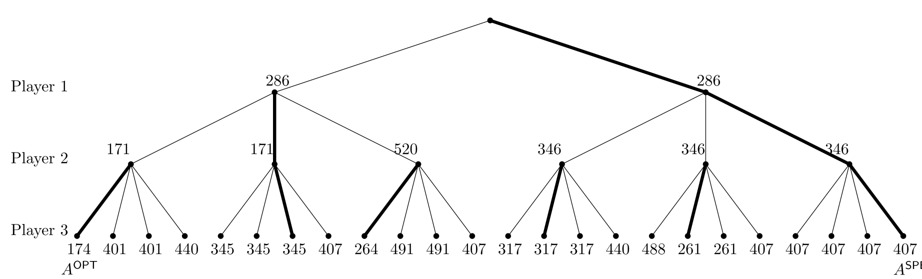

Fig. 4. The game tree of Example 12. The number at each action denotes the cost of the corresponding player when all successors play subgame perfect. Fat lines are actions of the subgame perfect equilibrium.

Table 1
The weights and actions of each resource. ✓denotes that the corresponding action contains the corresponding resource.

r	β_r	1.1	1.2	2.1	2.2	2.3	3.1	3.2	3.3	3.4
1	84		✓		✓		✓			
2	52		✓		✓	✓	✓			
3	3		✓			✓	✓		✓	
4	4		✓		✓	✓	✓	✓		
5	31				✓	✓	✓	✓		
6	52		✓	✓					✓	
7	54			✓		✓		✓		
8	92	✓				✓		✓	✓	
9	51	✓				✓			✓	
10	28		✓	✓						
11	4		✓	✓				✓	✓	
12	33			✓				✓		✓
13	374									✓

6. Conclusions

The linear program that we proposed here for the case of three players, is no longer practically feasible for the case with four players, as the number of actions of the fourth player would be 43, and the number of resources would become $2^{55} - 1$. Yet we have experimented with a reduced size MIP in order to compute lower bounds on the sequential price of anarchy for congestion games with four or more players. Even though the corresponding MIP was large and posed some computational challenges, we have been able to show that SPoA > 2.55 for four players [19], so specifically it is larger than 5/2 (which is the price of anarchy for Nash equilibria).

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