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Alternating-Direction Implicit Finite-Difference Method for Transient 2D Heat Transfer in a Metal Bar using Finite Difference Method

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Abstract— Different analytical and numerical methods are commonly used to solve transient heat conduction problems. In this problem, the use of Alternating Direct Implicit scheme (ADI) was adopted to solve temperature variation within an infinitesimal long bar of a square cross-section. The bottom right quadrant of the square cross-section of the bar was selected. The surface of the bar was maintained at constant temperature and temperature variation within the bar was evaluated within a time frame. The Laplace equation governing the 2-dimesional heat conduction was solved by iterative schemes as a result of the time variation. The modelled problem using COMSOL-MULTIPHYSICS software validated the result of the ADI analysis. On comparing the Modelled results from **COMSOL MULTIPHYSICS** and the results from ADI iterative scheme graphically, there was an high level of agreement between both results.

Index Terms— ADI, Iteration, Metal Bar, Transient Heat Transfer

1 INTRODUCTION

Analytical solutions are difficult to arrive at, due to the increasing complexities encountered in the development of technology. For these problems, numerical solutions are very useful, most notably when the geometry of the object is irregular and the boundary conditions are non-linear.

The number of numerical methods and versions of each, available for use in tackling a given heat-flow problem, has increased rapidly; however, the comparative advantages of the different techniques with respect to accuracy, stability, and cost remain unclear [1].

Numerical methods can be used to solve many practical problems in heat conduction that involve – complex 2D and 3D geometries and complex boundary conditions.

Alternating Direction implicit (ADI) scheme is a finite difference method in numerical analysis, used for solving parabolic, hyperbolic and elliptic differential equations. ADI is mostly used to solve the problem of heat conduction. The equations that have to be solved with ADI in each step, have a similar structure and can be solved efficiently with theTridiagonal Matrix Algorithm.

1.2 HISTORICAL BACKGROUND

A lot of trends have occurred in the application of ADI method.The Alternating Direction Implicit scheme was first developed and employed by Peaceman and Rachford in 1955 **[3]** for the computation of two dimensional parabolic and elliptic Partial differential equations.

Thomas et al [1] determined the ADI scheme as a cost effective technique with stability and accuracy, as compared with other standard Finite-element method for the analytical solutions for two problems approximating different stages in steel ingot processing.

Afsheen [2] used ADI two step equations to solve an Heat-transfer Laplace 2D problem for a square metallic plate and used a Fortran90 code to validate the results. Finally, the results show the effect of Neumann boundary conditions and Dirichlet boundary conditions on the scheme.

ADI has found application in diffusion, Adérito et al [3] employed ADI to solve a two-dimensional hyperbolic diffusion problem, where it is assumed that both convection and diffusion are responsible for flow motion. They established the stability of the method using discrete energy method. Their result showcased the accuracy of the Alternating direction implicit method.

Dehghan [4] used ADI scheme as the basis to solve the two dimensional time dependent diffusion equation with non-local boundary conditions.

In this work, we used an Alternating direction implicit scheme to solve a transient conduction heat problem within an infinitesimal long bar of a square cross-section. We also modelled the problem using COMSOL multiphysics and compared its result with that of the ADI scheme numerical result.

2.0 ALTERNATING DIRECTION IMPLICIT METHOD FOR 2D TRANSIENT HEAT TRANSFER

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2.1 PROBLEM FORMULATION

An infinitely long bar of thermal diffusivity α has a cross section of side 2a. It is initially at a uniform temperature $\theta_\circ$ and then suddenly has its surface maintained at a temperature θ_1. The subsequent temperatures $\theta(x, y, t)$ inside the bar are to be solved and computed at various time-steps.

Dimensionless distances, time, and temperature are defined by

$$X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad \tau = \frac{\alpha t}{a^2}, \quad and\ T = \frac{\theta - \theta_\circ}{\theta_1 - \theta_\circ}$$

Unsteady state conduction is governed by $\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = \frac{\partial T}{\partial \tau}$

2.2 BOUNDARY CONDITIONS

2.2.1 Initial Boundary Condition

$\tau = 0$: $T = 0$ Throughout the region

2.2.2 Final Boundary Condition

$\tau > 0$: : $T = 1$ Along the sides $X = 1$ $and\ Y = 1$,

$\frac{\partial T}{\partial x} = 0$ And $\frac{\partial T}{\partial Y} = 0$

Along the sides $X = 0$ $and\ Y = 0$ respectively.

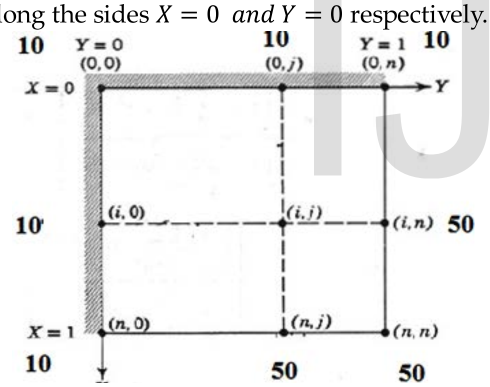

Figure 1.0: Initial temperature distributions of the sectioned bar

Where $\theta_1 = 50^\circ C$

$\theta_\circ = 10^\circ C$

2.3 Elliptic equation

$$T_{xx} = \frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta X^2} \quad (2.3.1)$$

$$T_{yy} = \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta Y^2} \quad (2.3.2)$$

$$T_{xx} + T_{yy} = 0 \quad (2.3.3)$$

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta X^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta Y^2} = 0 \quad (2.3.4)$$

$$\Delta X^2 = \Delta Y^2 = \Delta Z^2$$

$$T_{m,n}^{i+1} - T_{m,n}^{i} = \frac{\propto \Delta t}{\Delta^2}[T_{m+1,n}^{i} - 2T_{m,n}^{i} + T_{m-1,n}^{i} + T_{m,n+1}^{i} - 2T_{m,n}^{i} + T_{m,n-1}^{i}] \quad (2.3.5)$$

But $\tau = \frac{\propto \Delta t}{\Delta^2}$

3.0 COMPUTATION OF MESH FUNCTION ALONG COLUMNS

$$T_{m,n}^{i+1} - T_{m,n}^{i} = \tau[T_{m+1,n}^{i} - 2T_{m,n}^{i} + T_{m-1,n}^{i}] + [T_{m,n+1}^{i} - 2T_{m,n}^{i} + T_{m,n-1}^{i}] \quad (3.0.1)$$

3.1 COMPUTATION OF MESH FUNCTION ALONG ROWS

$$T_{m,n}^{i+2} - T_{m,n}^{i+1} = \tau[T_{m+1,n}^{i+2} - 2T_{m,n}^{i+2} + T_{m-1,n}^{i+2}] + [T_{m,n+1}^{i+1} - 2T_{m,n}^{i+1} + T_{m,n-1}^{i+1}] \quad (3.1.1)$$

For i = 1,2,3…..n-1 and j = 1,2,3…..n-1 , both equations yields a tridiagonal system of equations.

At **When $i = 0$ and $m = 1$ and $n = 1$**

$$-\tau T_{1,2}^{(1)} + [1 + 2\tau]T_{1,1}^{(1)} - \tau T_{1,0}^{(1)} = \tau T_{0,1}^{(1)} + [1 - 2\tau]T_{1,1}^{(0)} + \tau T_{2,1}^{(0)} \quad (3.1.2)$$

3.1.1 ITERATION ONE ($\tau = 1$)

$$-\tau 50 + [1 + 2\tau]T_{1,1}^{(1)} - 1\ 0 = 0 + [1 - 2\tau]0 + 0$$

$$T_{1,1}^{(1)} = 20$$

3.1.2 ITERATION TWO

Equation (a) was used in computing $T_{1,1}^{(1)}$.

It's direction was alternated and used in computing the function value for $T_{1,1}^{(2)}$ on the row, using equation (b)

$$-T_{m-1,n}^{i+2} + 3T_{m,n}^{i+2} - T_{m+1,n}^{i+2} = T_{m,n+1}^{i+1} - T_{m,n}^{i+1} + T_{m,n-1}^{i+1} \quad (3.1.3)$$

When i = 0 , n = 1 and m = 1

$$-T_{0,1}^{(2)} + 3T_{1,1}^{(2)} - T_{2,1}^{(2)} = T_{1,2}^{(1)} - T_{1,1}^{(1)} + T_{1,0}^{(1)}$$

$$T_{1,1}^{(2)} = 33.3$$

3.1.3 ITERATION THREE

equation (a) and (c) yields

$$-\tau T_{m,n+1}^{i+1} + [1 + 2\tau]T_{m,n}^{i+1} - \tau T_{m,n-1}^{k+1} = \tau T_{m-1,n}^{i} + [1 - 2\tau]T_{m,n}^{i} + \tau T_{m+1,n}^{i} \quad (3.1.4)$$

When i=2, m=1 and j=1

$$[1 + 2\tau]T_{1,1}^{(3)} = \tau 10 + \tau 10 + \tau 50 + \tau 50 + [1 - 2\tau]33.3$$

$$(\boldsymbol{\tau} = \mathbf{1})$$

$$3T_{1,1}^{(3)} = 10 + 10 + 50 + 50 + [1 - 2]33.3$$

$$T_{1,1}^{(3)} = 28.9$$

3.1.4 ITERATION FOUR

equation (b) and (f) are defined as

$$-T_{m-1,n}^{i+2} + 3T_{m,n}^{i+2} - T_{m+1,n}^{i+2} = T_{m,n+1}^{i+1} - T_{m,n}^{i+1} + T_{m,n-1}^{i+1} \quad (3.1.5)$$

when $i = 2$, $n = 1$ and $m = 1$

$$-T_{0,1}^{(4)} + 3T_{1,1}^{(4)} - T_{2,1}^{(4)} = T_{1,2}^{(3)} - T_{1,1}^{(3)} + T_{1,0}^{(3)}$$

$$T_{1,1}^{(4)} = 30.4$$

3.2 COMPUTATIONS OF THE VARIOUS TIME STEPS (ΔT) FOR EACH ITERATION

$$\tau = \frac{\propto \Delta t}{\Delta^2} \qquad (3.1.6)$$

Where $\propto = 0.1516$, $\Delta = 4$, $\tau = 1,2,3,4$, $\Delta t = \frac{\Delta^2}{\propto}$

3.2.1 ITERATION ONE

Time (Δt) = 26.38s

3.2.2 ITERATION TWO

Time (Δt) = 52.77s

3.2.3 ITERATION THREE

Time (Δt) = 79.15s

3.2.4 ITERATION FOUR

Time (Δt) = 105.54

4.0 RESULTS AND DISCUSSION

4.1 TEMPERATURE DISTRIBUTIONS

Table 1.0: Initial Temperature distribution.

10	**10**	**10**
10	**10**	**10**
10	**10**	**10**

After Iteration one, (time $= 26.38$) , values were

Table 2: Temperature distribution at 26.38s

10	**10**	**10**
10	**20**	**50**
10	**50**	**50**

After Iteration two (time=52.8) , values were

Table 3: Temperature distribution at 52.8s

10	10	10
10	**33.3**	**50**
10	**50**	**50**

After Iteration three (time=79.2) values were

Table 4: Temperataure distribution at 79.2s

10	**10**	**10**
10	**28.9**	**50**
10	**50**	**50**

After Iteration Four (time =105.54) values were

Table 5: Tempertaure distribution at 105.54s

10	**10**	**10**
10	**30.4**	**50**
10	**50**	**50**

4.2 VALIDATION OF RESULTS BY COMSOL MULTIPHYSICS

Using Comsol Multiphysics, The metal bar was modelled with, with the following parameters assumed, to achieve the temperature distribution within the metal bar.

Length=1.16m

Width=0.45m

Thermal conductivity (K) = 1W/(m.K)

Density(ϱ) = 1kg/m^3

Heat capacity (C_ρ =1 J/kg.K)

The result which was in graphical user interface form is shown below as .

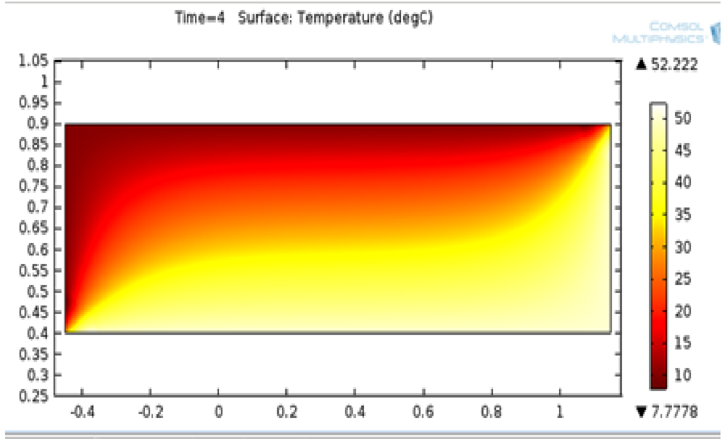

Figure 2: Temperature distribution within the metal bar

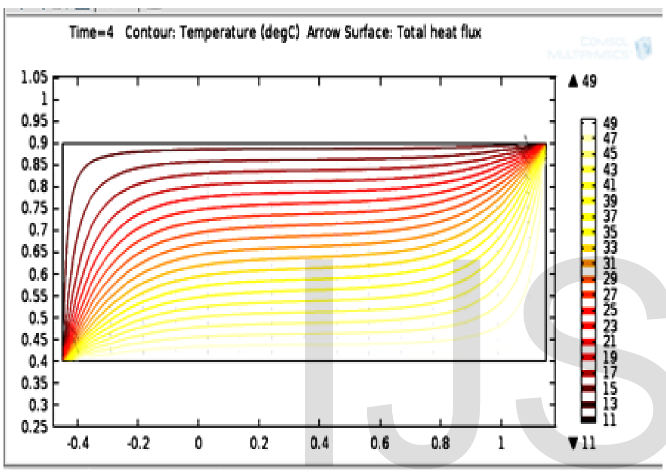

Figure 3: Isothermal contour showing temperature distribution within the metal bar

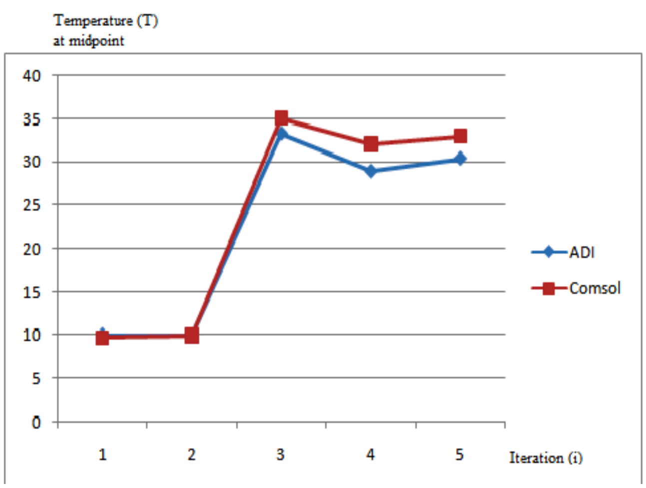

Figure 4: A comparison of ADI results with COMSOL results

CONCLUSIONS

The two Dimensional Heat problem was modelled using **COMSOL MULTIPHYSICS** which gave a graphical distribution of temperature within the metal and a graph showing the convergence of the finite difference iterative scheme.

The ADI iterative scheme was highly effective in determining the nodal temperatures within the sectioned metal bar.

On comparing the Modelled results from **COMSOL MULTIPHYSICS** and the results from ADI iterative scheme, there was an high level of agreement between both results , notably if one observe closely the results for node $T^1_{1,1}$=20, $T^2_{1,1}$=33.3 , $T^3_{1,1}$=28.9 , $T^4_{1,1}$=30.4 with the temperature distribution of the Mid-section of the Metal Bar with COMSOL MULTIPHYSICS, a large level of conformity exists.

For problems with a simple geometry, the ADI finite difference method

is cost-effective with stability and accuracy similar to the finite-element methods.

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