ON THE DETERMINATION OF FACTOR SYSTEMS OF PUA - REPRESENTATIONS

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Abstract: A method is developed to obtain a complete set of inequivalent factor systems of PUA - representations of a group with a subgroup of index two from the factor systems of this subgroup.
1. Introduction

Let $G$ be a group which has a subgroup $H$ of index two. A projective unitary-antiunitary (PUA-) representation of $G$ is a mapping $D$ from $G$ into the operators on some Hilbert space $\mathcal{H}$ such that

i) the operator $D(g)$ is unitary if $g \in H$ and antiunitary if $g \notin H$

ii) $D(g) D(g') = \sigma(g,g') D(gg') \forall g, g' \in G$ for some mapping

$$\sigma: G \times G \to U(1).$$

It is customary to choose $D(e) = I$, where $e$ is the identity of $G$ and $I$ is the identity operator on $\mathcal{H}$. Then $\sigma$ satisfies

$$\sigma(g,e) = \sigma(e,g) = 1 \quad \forall g \in G \quad (1.1)$$

and

$$\sigma(g_1,g_2) \sigma(g_1,g_2,g_3) = \sigma(g_1,g_2,g_3) \sigma(g_1,g_2,g_3) \forall g_1,g_2,g_3 \in G \quad (1.2)$$

where $\lambda^g$ is defined by

$$\lambda^g = \begin{cases} 
\lambda & \text{if } g \in H \\
\lambda^* & \text{if } g \notin H
\end{cases}$$

the asterisk denoting complex conjugation.

A mapping $\sigma: G \times G \to U(1)$ which satisfies (1.1) and (1.2) is called a factor system of $G$ with respect to $H$. In the following a factor system of $G$ shall always mean a factor system of $G$ with respect to $H$. If $D$ is a PUA-representation with factor system $\sigma$ and $c$ is a mapping from $G$ into $U(1)$ with $c(e) = 1$ then

$$D'(g) = c(g) D(g)$$

is a PUA-representation of $G$ with factor system

$$\sigma'(g_1,g_2) = \frac{c(g_1)c^*(g_2)}{c(g_1)g_2)} \sigma(g_1,g_2) \quad (1.3)$$

Two factor systems $\sigma$ and $\sigma'$ are called equivalent if a mapping $c: G \to U(1)$ with $c(e) = 1$ exists such that (1.3) holds. Factor systems of $H$ are defined in an analogous way, the only difference being the absence of the complex conjugation in
(1.2) and (1.3). The theory of PUA-representations and its use in physics is described by Murthy [6], Parthasarathy [7], Janssen [4] and Shaw & Lever [9], [10]. Factor systems of PU-representations are studied quite extensively [1], [2],[5],[8]. This however is not the case for factor systems of PUA-representations.

It is the aim of this paper to determine a complete set of inequivalent factor systems of G when the factor systems of H are known. This problem has already been attacked by Bradley & Wallis [3], but they have not obtained the general solution.

Janssen [4] has given a method to obtain the factor systems of G in the case where G is finite, without using the factor systems of the subgroup.

2. Reduction of the problem

First we choose an element \( a_0 \) from \( G \setminus H \) which remains fixed during the following. We can write all elements of \( G \setminus H \) as \( a_0 h \) or \( h'a_0 \) for some \( h, h' \in H \). Suppose \( \sigma \) is a factor system of G.

The restriction \( \sigma_H \) of \( \sigma \) to \( H \times H \) is then a factor system of H. If \( \sigma_H \) is a factor system of H we may ask the question whether or not there exists a factor system \( \sigma \) of G such that its restriction to \( H \times H \) is \( \sigma_H \). If such \( \sigma \) exists it is called an extension of \( \sigma_H \).

**Lemma 1** An extension of a factor system \( \sigma_H \) of H to a factor system \( \sigma \) of G is completely determined by the elements \( \sigma(a_0, a_0) \), \( \sigma(h, a_0) \) and \( \sigma(a_0, h) \) for all \( h \in H \).

**Proof** The following relations follow immediately from (1.2):

\[
\sigma(a_0 h, h') = \frac{\sigma(a_0, hh') \sigma(h, h')}{\sigma(a_0, h)} \quad (2.1)
\]

\[
\sigma(h, h'a_0) = \frac{\sigma(h, h') \sigma(hh', a_0)}{\sigma(h', a_0)} \quad (2.2)
\]
\[ \sigma(ha_o, a_o h') = \frac{\sigma(h, a_o^2) \sigma(a_o, a_o) \sigma(ha_o, h')}{\sigma(h, a_o) \sigma(a_o, h')} \] (2.3)

This proves the lemma.

If these relations are substituted in (1.2) we obtain after laborious manipulation the following three equations for \( \sigma(a_o, a_o) \), \( \sigma(a_o, h) \) and \( \sigma(h, a_o) \):

\[ \sigma'(h, a_o) \sigma'(a_o, a_o^{-1} hh'a_o) \sigma(a_o^{-1} ha_o, a_o^{-1} h'a_o) \sigma'(h'o, a_o) \sigma(a_o, a_o^{-1} ha_o) \]

\[ \sigma(hh', a_o) \sigma(h, h') \sigma(a_o, a_o^{-1} h'a_o) = 1 \quad \forall h, h' \in H \] (2.4)

\[ \sigma'(a_o ha_o^{-1}, a_o^2) \sigma(a_o^2, a_o^{-1} ha_o) \sigma'(h, a_o) \sigma(a_o, a_o^{-1} ha_o) \]

\[ \sigma(a_o ha_o^{-1}, a_o) \sigma'(h, a_o) = 1 \quad \forall h \in H \] (2.5)

\[ \sigma(a_o, a_o) \sigma(a_o, a_o) \sigma(a_o^2, a_o) \sigma'(a_o, a_o^2) = 1 \] (2.6)

Note that for each solution \( \sigma(h, a_o) \) and \( \sigma(a_o, h) \) of (2.4) and (2.5) we obtain two values of \( \sigma(a_o, a_o) \) from (2.6).

The following theorem has now been derived:

**Theorem 1** All extensions of a given factor system \( \sigma \) of \( H \) to factor systems of \( G \) are obtained from the solutions \( \sigma(h, a_o) \) and \( \sigma(a_o, h) \) of the equations (2.4) and (2.5).

For each solution of (2.4) and (2.5) there are two extensions which are given by the equations (2.6), (2.1), (2.2) and (2.3).

To obtain a complete set of inequivalent factor systems of \( G \) it is only necessary to consider extensions of inequivalent factor systems of \( H \). This follows from the fact that if \( \sigma'_H \) and \( \sigma''_H \) are two equivalent factor systems of \( H \) and \( \sigma \) is an extension of \( \sigma'_H \) then \( \sigma''_H \) has an extension \( \sigma' \) which is equivalent with \( \sigma \).

On the other hand inequivalent factor systems of \( H \) have inequivalent extensions. So in order to obtain a complete set of inequivalent factor systems of \( G \) we have
to find all inequivalent extensions of one representative of each class of equivalent factor systems of \( H \).

3. Solution of the problem

In this section we present without proof a method to obtain a complete set of inequivalent extensions of a factor system of \( H \). First we give a criterion to decide whether there exist extensions or not. Then we give for the case where extensions do exist a set of extensions which contains a complete set of inequivalent ones. Finally we obtain this complete set.

Define an equivalence relation in \( H \): \( h \) and \( h' \) are called equivalent if there is a \( n \in \mathbb{Z} \) such that \( a_o^n h a_o^{-n} = h' \). In this way \( H \) is divided into classes. Let \( H_0 \) be a set of elements of \( H \) which contains exactly one element from each class with an even number of elements and none from each other class. Let \( \sigma \) be a factor system of \( H \) and define the mapping \( D_\sigma \) from \( H \) into the complex numbers of modulus unity by

\[
D_\sigma(h) = \begin{cases} 
\prod_{n=0}^{p-2} \left[ \sigma(a_o^{n+1} h a_o^{-n+1}, a_o^{2n} \sigma(a_o, a_o h a_o^{-n}) \right] & \text{if } h \in H_0 \\
1 & \text{if } h \notin H_0 
\end{cases}
\]

where \( p \) is the number of elements in the class containing \( h \). If \( \sigma' \) is a factor system of \( H' \) which is equivalent with \( \sigma \) then \( D_\sigma = D_{\sigma'} \).

Theorem 2. \( \sigma \) can be extended to a factor system of \( G \) if and only if there is a factor system \( \sigma' \) of \( H \) which is equivalent with \( \sigma \) and obeys

\[
\sigma'(h, h') \sigma'(a_o^{-1} h a_o, a_o^{-1} h' a_o) = D_\sigma(h) D_\sigma(h') D_\sigma(h h')
\]

and

\[
\sigma'(a_o h a_o^{-1}, a_o^2 \sigma(a_o, a_o h a_o^{-1})) = D_\sigma(a_o h a_o^{-1}) D_\sigma(h)
\]

Let \( R(H) \) be the set of all unitary one-dimensional representations \( \Delta \) of \( H \) with the property \( \Delta(h) = \Delta(a_o h a_o^{-1}) \).
Theorem 3 If $\sigma$ satisfies (3.1) and (3.2) then a complete set of inequivalent extensions of $\sigma$ is contained in the set extensions given by $\sigma(a_0, h) = 1$ and $\sigma(h, a_0) = \Delta(h) D_0(h)$ where $\Delta \in \mathbb{R}(H)$.

The only thing we still need is a criterion to decide when two extensions of this set are equivalent.

Theorem 4 Let $\sigma_1$ and $\sigma_2$ be two extensions of the set defined above. Then there exists a $\Delta \in \mathbb{R}(H)$ with
\[
\frac{\sigma_1(h, a_0)}{\sigma_2(h, a_0)} = \Delta(h).
\]

$\sigma_1$ and $\sigma_2$ are equivalent if and only if there exists a one-dimensional unitary representation $\Delta_0$ of $H$ with the properties
\[
\Delta_0(h a_0^{-1} h a_0) = \Delta(h) \quad \forall h \in H \quad \text{and} \quad \Delta_0^*(a_0^2) = \frac{\sigma_1(a_0, a_0)}{\sigma_2(a_0, a_0)}.
\]

References