ON THE DETERMINATION OF FACTOR SYSTEMS OF PUA - REPRESENTATIONS

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Abstract: A method is developed to obtain a complete set of inequivalent factor systems of PUA - representations of a group with a subgroup of index two from the factor systems of this subgroup.

1. Introduction

Let G be a group which has a subgroup H of index two. A projective unitary-antiunitary (PUA-) representation of G is a mapping D from G into the operators on some Hilbert space $\mathcal{H}$ such that

i) the operator D(g) is unitary if $g \in H$ and antiunitary if $g \notin H$

ii) $D(g)\,D(g') = \sigma(g,g')\,D(gg')$ $\forall g,g' \in G$ for some mapping

$$\sigma: G \times G \to U(1).$$

It is customary to choose D(e) = I, where e is the identity of G and I is the identity operator on $\mathcal{H}$. Then σ satisfies

$$\sigma(g,e) = \sigma(e,g) = 1 \qquad \forall g \in G \tag{1.1}$$

and

$$\sigma(g_1,g_2)\,\sigma(g_1g_2,g_3) = \sigma(g_1,g_2g_3)\,\sigma^{g_1}(g_2,g_3)\ \forall g_1,g_2,g_3 \in G \tag{1.2}$$

where λ^g is defined by

$$\lambda^g = \begin{cases} \lambda & \text{if } g \in H \\ \lambda^* & \text{if } g \notin H \end{cases},$$

the asterisk denoting complex conjugation.

A mapping $\sigma: G \times G \to U(1)$ which satisfies (1.1) and (1.2) is called a factor system of G with respect to H. In the following a factor system of G shall always mean a factor system of G with respect to H. If D is a PUA-representation with factor system σ and c is a mapping from G into U(1) with c(e) = 1 then $D'(g) = c(g)\,D(g)$ is a PUA-representation of G with factor system

$$\sigma'(g_1,g_2) = \frac{c(g_1)\,c^{g_1}(g_2)}{c(g_1g_2)}\,\sigma(g_1,g_2) \tag{1.3}$$

Two factor systems σ and σ' are called equivalent if a mapping $c: G \to U(1)$ with c(e) = 1 exists such that (1.3) holds. Factor systems of H are defined in an analogous way, the only difference being the absence of the complex conjugation in

(1.2) and (1.3). The theory of PUA-representations and its use in physics is described by Murthy [6] , Parthasarathy [7] , Janssen [4] and Shaw & Lever [9] , [10] . Factor systems of PU-representations are studied quite extensively [1], [2] ,[5] ,[8] . This however is not the case for factor systems of PUA-representations.

It is the aim of this paper to determine a complete set of inequivalent factor systems of G when the factor systems of H are known. This problem has already been attacked by Bradley & Wallis [3] , but they have not obtained the general solution.

Janssen [4] has given a method to obtain the factor systems of G in the case where G is finite, without using the factor systems of the subgroup.

2. Reduction of the problem

First we choose an element a_o from $G \setminus H$ which remains fixed during the following. We can write all elements of $G \setminus H$ as $a_o h$ or $h' a_o$ for some $h, h' \in H$. Suppose σ is a factor system of G.

The restriction σ_H of σ to H x H is then a factor system of H. If σ_H is a factor system of H we may ask the question whether or not there exists a factor system σ of G such that its restriction to H x H is σ_H. If such σ exists it is called an extension of σ_H.

Lemma 1 An extension of a factor system σ_H of H to a factor system σ of G is completely determined by the elements $\sigma(a_o,a_o)$, $\sigma(h,a_o)$ and $\sigma(a_o,h)$ for all $h \in H$.

Proof The following relations follow immediately from (1.2):

$$\sigma(a_o h,h') = \frac{\sigma(a_o,hh')\ \sigma^*(h,h')}{\sigma(a_o,h)} \tag{2.1}$$

$$\sigma(h,h'a_o) = \frac{\sigma(h,h')\ \sigma(hh',a_o)}{\sigma(h',a_o)} \tag{2.2}$$

$$\sigma(ha_o,a_oh') = \frac{\sigma(h,a_o^2)\,\sigma(a_o,a_o)\,\sigma(ha_o^2,h')}{\sigma(h,a_o)\sigma^*(a_o,h')} \qquad (2.3)$$

This proves the lemma.

If these relations are substituted in (1.2) we obtain after laborious manipulation the following three equations for $\sigma(a_o,a_o)$, $\sigma(a_o,h)$ and $\sigma(h,a_o)$:

$$\sigma^*(h,a_o)\sigma^*(a_o,a_o^{-1}hh'a_o)\,\sigma(a_o^{-1}ha_o,a_o^{-1}h'a_o)\sigma^*(h',a_o)\,\sigma(a_o,a_o^{-1}ha_o)$$

$$\sigma(hh',a_o)\ \ \sigma(h,h')\,\sigma(a_o,a_o^{-1}h'a_o) = 1 \quad \forall h,h' \in H \qquad (2.4)$$

$$\sigma^*(a_oha_o^{-1},a_o^2)\,\sigma(a_o^2,a_o^{-1}\,ha_o)\,\sigma^*(a_o,h)\,\sigma(a_o,a_o^{-1}\,ha_o)$$

$$\sigma(a_oha_o^{-1},a_o)\,\sigma^*(h,a_o) = 1 \qquad \forall\, h \in H \qquad (2.5)$$

$$\sigma(a_o,a_o)\ \ \sigma(a_o,a_o)\ \ \sigma(a_o^2,a_o)\,\sigma^*(a_o,a_o^2) = 1 \qquad (2.6)$$

Note that for each solution $\sigma(h,a_o)$ and $\sigma(a_o,h)$ of (2.4) and (2.5) we obtain two values of $\sigma(a_o,a_o)$ from (2.6).

The following theorem has now been derived:

<u>Theorem 1</u> All extensions of a given factor system σ of H to factor systems of G are obtained from the solutions $\sigma(h,a_o)$ and $\sigma(a_o,h)$ of the equations (2.4) and (2.5).

For each solution of (2.4) and (2.5) there are two extensions which are given by the equations (2.6), (2.1), (2.2) and (2.3).

To obtain a complete set of inequivalent factor systems of G it is only necessary to consider extensions of inequivalent factor systems of H. This follows from the fact that if σ_H and σ'_H are two equivalent factor systems of H and σ is an extension of σ_H then σ'_H has an extension σ' which is equivalent with σ. On the other hand inequivalent factor systems of H have inequivalent extensions. So in order to obtain a complete set of inequivalent factor systems of G we have to find all inequivalent extensions of one representative of each class of equivalent factor systems of H.

3. Solution of the problem

In this section we present without proof a method to obtain a complete set of inequivalent extensions of a factor system of H. First we give a criterion to decide whether there exist extensions or not. Then we give for the case where extensions do exist a set of extensions which contains a complete set of inequivalent ones. Finally we obtain this complete set.

Define an equivalence relation in H: h and h' are called equivalent if there is a $n \in \mathbb{Z}$ such that $a_o^n h a_o^{-n} = h'$. In this way H is divided into classes. Let H_o be a set of elements of H which contains exactly one element from each class with an even number of elements and none from each other class. Let σ be a factor system of H and define the mapping D_σ from H into the complex numbers of modulus unity by

$$D_\sigma(h) \begin{cases} \prod_{\substack{n=0 \\ n \text{ even}}}^{p-2} \left[\sigma^*(a_o^{n+1}\, h\, a_o^{-(n+1)},\ a_o^2)\ \sigma(a_o^2,\ a_o^{n-1} h\, a_o^{-(n-1)})\right] & \text{if } h \in H_o \\ 1 & \text{if } h \notin H_o \end{cases}$$

where p is the number of elements in the class containing h. If σ' is a factor system of H' which is equivalent with σ then $D_\sigma = D_{\sigma'}$.

<u>Theorem 2</u> σ can be extended to a factor system of G if and only if there is a factor system σ' of H which is equivalent with σ and obeys

$$\sigma'(h,h')\ \sigma'(a_o^{-1} h a_o,\ a_o^{-1} h' a_o) = D_\sigma(h)\ D_\sigma(h')\ D_\sigma^*(hh') \qquad (3.1)$$

and

$$\sigma'^*(a_o h a_o^{-1},\ a_o^2)\sigma'(a_o^2,\ a_o^{-1} h a_o) = D_\sigma^*(a_o h a_o^{-1})\ D_\sigma(h) \quad . \qquad (3.2)$$

Let R(H) be the set of all unitary one-dimensional representations Δ of H with the property $\Delta(h) = \Delta(a_o h a_o^{-1})$

<u>Theorem 3</u> If σ satisfies (3.1) and (3.2) then a complete set of inequivalent extensions of σ is contained in the set extensions given by $\sigma(a_o,h) = 1$ and $\sigma(h,a_o) = \Delta(h)\, D_\sigma(h)$ where $\Delta \in R(H)$.

The only thing we still need is a criterion to decide when two extensions of this set are equivalent.

<u>Theorem 4</u> Let σ_1 and σ_2 be two extensions of the set defined above. Then there exists a $\Delta \in R(H)$ with $\frac{\sigma_1(h,a_o)}{\sigma_2(h,a_o)} = \Delta(h)$.

σ_1 and σ_2 are equivalent if and only if there exists a one-dimensional unitary representation Δ_o of H with the properties
$\Delta_o(ha_o^{-1}ha_o) = \Delta(h) \quad \forall\, h \in H$ and $\Delta_o^*(a_o^2) = \frac{\sigma_1(a_o,a_o)}{\sigma_2(a_o,a_o)}$.

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