11. Method for the determination of Clebsch Gordan coefficients of finite magnetic groups

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Recently a new method for the determination of Clebsch Gordan (CG) coefficients of finite nonmagnetic groups has been developed (1, 2, 3). Here we generalise this method for magnetic groups. Because of lack of space we restrict our discussion to unitary-antiunitary representations of type I.

Let G be a finite group and G_o a subgroup of G of index 2. A unitary-antiunitary (UA) representation of G with respect to G_o is a mapping D from G into the unitary matrices of some dimension n such that

$$D(g)\ D^g(g') = D(gg') \qquad \forall\, g,g' \in G$$

where D^g is defined by

$$D^g = \begin{cases} D & \text{if } g \in G_o \\ D^* & \text{if } g \notin G_o \end{cases}$$

Let a complete set of irreducible inequivalent UA representations of G with respect to G_o be given by D^α, D^β,... and let their dimensions be d_α, d_β,...
The direct product $D^\alpha \otimes D^\beta$ is defined by

$$(D^\alpha \otimes D^\beta)(g)_{ij,k\ell} = D^\alpha(g)_{ik}\ D^\beta(g)_{j\ell} \qquad \forall\, g \in G$$

$D^\alpha \otimes D^\beta$ is a UA representation which is reducible in general; suppose it is equivalent with the direct sum $\sum_\gamma \oplus\, m_\gamma D^\gamma$. Then there exists a unitary matrix U with the property

$$(D^\alpha \otimes D^\beta)(g)\ U^g = U \sum_\gamma \oplus\ m_\gamma\ D^\gamma(g) \qquad \forall\, g \in G$$

The matrix elements of U are the CG coefficients. We label the rows of U by the pairs (i,j); i = 1,2,...,d_α; j = 1,2,...,d_β and the columns of U by the triples (γ,τ,k) with $m_\gamma \neq 0$; $\tau = 1,2,\dots,m_\gamma$ and $k = 1,2,\dots,d_\gamma$. It is convenient to consider columns of the matrix U separately; let therefore $c(\gamma,\tau,k)$ denote the column of dimension $d_\alpha d_\beta$ whose elements are given by

$$c(\gamma,\tau,k)_{ij} = U_{ij,\gamma\tau k}$$

These columns can be calculated separately for each γ. We will restrict ourselves here, due to lack of space, to those γ for which D^γ is of type I, which means that the restriction of D^γ to G_o is an irreducible representation of G_o. For those D^γ the columns $c(\gamma,\tau,k)$ can be calculated as follows. Fix upon one value of γ and let the matrices $A(\gamma,\ell,k)$ and $B(\gamma,\ell,k)$ of dimension $d_\alpha d_\beta$ be defined by

$$A(\gamma,\ell,k)_{mn,ij} = \frac{d_\gamma}{|G|} \sum_{g\in G_o} D^{\gamma*}_{\ell k}(g)\; D^\alpha_{mi}(g)\; D^\beta_{nj}(g)$$

$$B(\gamma,\ell,k)_{mn,ij} = \frac{d_\gamma}{|G|} \sum_{g\notin G_o} D^{\gamma*}_{\ell k}(g)\; D^\alpha_{mi}(g)\; D^\beta_{nj}(g)$$

Consider first the matrices $A(\gamma,k,k)$ and $B(\gamma,k,k)$ for some fixed value of k. Let $A(\gamma,k,k)_{mn,mn}$ be a diagonal element of $A(\gamma,k,k)$ which is not equal to zero. Then $A(\gamma,k,k)_{mn,mn}$ is real and positive. Let ψ and ϕ be the mn-th column of $A(\gamma,k,k)$ and $B(\gamma,k,k)$ respectively:

$$\psi_{ij} = A(\gamma,k,k)_{ij,mn}$$

$$\phi_{ij} = B(\gamma,k,k)_{ij,mn}$$

If $A(\gamma,k,k)_{mn,mn} \neq -B(\gamma,k,k)_{mn,mn}$ we can take

$$c(\gamma,1,k) = \frac{\psi+\phi}{\{A(\gamma,k,k)_{mn,mn} + \mathrm{Re}\, B(\gamma,k,k)_{mn,mn}\}}$$

If $A(\gamma,k,k) = -B(\gamma,k,k)$ we can take

$$c(\gamma,1,k) = \frac{i(\psi-\phi)}{\{2A(\gamma,k,k)_{mn,mn}\}^{\frac{1}{2}}}$$

If $m_\gamma \geq 2$ we define $A'(\gamma,k,k)$ and $B'(\gamma,k,k)$ by

$$A'(\gamma,k,k) = A(\gamma,k,k) - \tfrac{1}{2}\, c(\gamma,1,k)\, \tilde{c}^*(\gamma,1,k)$$

$$B'(\gamma,k,k) = B(\gamma,k,k) - \tfrac{1}{2}\, c(\gamma,1,k)\, \tilde{c}(\gamma,1,k)$$

Here $\tilde{c}$ means the transpose of c. From these matrices we obtain in the same way as above the column $c(\gamma,2,k)$. If $m_\gamma \geq 3$ we can proceed in this way by defining

$$A''(\gamma,k,k) = A'(\gamma,k,k) - c(\gamma,2,k)\, \tilde{c}^*(\gamma,2,k)$$

and so on. In this way we determine all m_γ columns $c(\gamma,\tau,k)$.
The columns $c(\gamma,\tau,\ell)$ for $\ell \neq k$ are now determined by

$$c(\gamma,\tau,\ell) = A(\gamma,\ell,k)\cdot c(\gamma,\tau,k).$$

This method provides us with a set of CG coefficients for those γ for which D^{γ} is of type I.

A proof of this method, together with similar but different methods for D^{γ} of types II and III, will be published elsewhere.

1 S. Schindler and R. Mirman, J. Math. Phys. 18, 1678 (1977)
2 P.M. van den Broek and J.F. Cornwell, Phys. St. Sol. b 90, 211 (1978)
3 R. Dirl, J. Math. Phys. (to be published).