# Roots, symmetry and contour integrals in queueing systems

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**Abstract**

Many queueing systems are analysed using the probability-generating-function (pgf) technique. This approach often leads to expressions in terms of the (complex) roots of a certain equation. In this paper, we show that it is not necessary to compute the roots in order to evaluate these expressions. We focus on a certain class of pgfs with a rational form and represent it explicitly using symmetric functions of the roots. These functions can be computed using contour integrals.

We also study when the mean of the random variable corresponding to the considered pgf is an additive function of the roots. In this case, it may be found using one contour integral, which is more reliable than the root-finding approach. We give a necessary and sufficient condition for an additive mean. For example, the mean is an additive function when the numerator of the pgf has a polynomial-like structure of a certain degree, which means that the pgf can be represented in a special product form. We also give a necessary and sufficient condition for the mean to be independent of the roots.

## 1. Introduction

The analysis of many queueing systems involves finding roots of a characteristic equation. This often occurs in discrete-time models such as the $G_D/G_D/1$ queue, [13], bulk-service queue, [1], and multi-server queue $M/D/s$, [7], but also in continuous models, see, for example, [5], where authors consider general inter-arrival times and a Markovian service process, or [12], where two interrelating queues are analysed. In some special cases, there are formulas to compute the roots (see, e.g, [7] for the case of Poisson and Binomial arrivals), but there is no explicit formula in general. Moreover, the derived solution can be very sensitive to the precision of the roots, which, in turn, can be poor even due to a small mistake in the coefficients, see, e.g., the study of the so-called Wilkinson's polynomial [15]. For an example in queueing theory, we refer to [11], where the authors consider the bulk-service queue and show that there are instances, where the root-finding approach leads to a negative queue length.

In this paper, we consider a particular class of queueing systems, where the probability generating function (pgf) of the queue length (or another important value) has a rational form with several unknown variables $q_j$, $j = 0, \dots, n$, in the numerator:

$$X(z) = \frac{\sum_{j=0}^{n} q_j f_j(z)}{D(z)}(z-1), \tag{1}$$

An example of such systems is the bulk-service queue, see [1]. The classical approach to find the unknowns in the numerator is to consider the analyticity of the pgf in the unit disk and compute the zeroes of the denominator $\bar{z}_0 = 1, \bar{z}_1, \ldots, \bar{z}_n$. Due to the analyticity of the pgf, the zeros of the denominator are also zeros of the numerator. This yields $n$ linear equations for the unknowns:

$$\sum_{j=0}^{n} q_j f_j(\bar{z}_i) = 0, \qquad i = 1, \ldots, n. \tag{2}$$

One more equation follows from the normalisation equation $X(1) = 1$:

$$\sum_{j=0}^{n} q_j f_j(1) = D'(1). \tag{3}$$

This system of equations can be rewritten in a matrix form:

$$M(1, \bar{z}_1, \ldots, \bar{z}_n)(q_0, \ldots, q_n)^T = (D'(1), 0, \ldots, 0)^T, \tag{4}$$

where

$$M(z, z_1, \ldots, z_n) = \begin{pmatrix} f_0(z) & f_1(z) & \ldots & f_n(z) \\ f_0(z_1) & f_1(z_1) & \ldots & f_n(z_1) \\ \ldots & \ldots & \ldots & \ldots \\ f_0(z_n) & f_1(z_n) & \ldots & f_n(z_n) \end{pmatrix}. \tag{5}$$

In this paper, we use the properties of the matrix $M(z, z_1, \ldots, z_n)$ and of symmetric polynomials to find the pgf without computing the roots. We represent the pgf using a determinant of a certain matrix, where each entry is a symmetric function of the roots, which can be computed using contour integrals. The advantages of using contour integrals are that the results are generally more reliable compared to the root-finding approach, see [11], and can be used as an intermediate step for further results, see, e.g, [6].

We also study when the considered class of pgf can be represented in a special product form, where each term of the product depends on not more than one root. In this case, the mean of the corresponding random variable, e.g., the queue length, is an additive function of the roots and, under additional conditions, can be found using one contour integral. We give a sufficient and necessary condition for these properties to hold. The systems with such pgf include the bulk-service queue, see [1], the multi-server $M/D/s$ and $Geo/D/s$ queues, which are in some sense equivalent to the bulk-service queue with Poisson and Binomial arrivals, see [7] and [8], and the fixed-cycle traffic-light queue, see [11].

The paper is structured as follows. In Section 2, we give the definitions and required properties of symmetric polynomials and functions. In Section 3, we obtain the pgf in terms of symmetric functions for a general case of numerator. Then, we analyse a special sub-class of the pgfs in Section 4. Finally, we conclude the paper in Section 5.

## 2. Preliminaries

In this section, we give the required definitions and the preliminary results. First, in Subsection 2.1, we define symmetric, skew-symmetric and additive functions and alternant matrices. Then, we describe two types of symmetric polynomials and their properties, see Subsection 2.2. In Subsection 2.3, we analyse the determinant of an alternant matrix, which will be used later in Section 3 to obtain the pgf as a symmetric function of roots. Afterwards, we analyse linear dependency of functions in terms of singular matrices, see Subsection 2.4. Finally, in Subsection 2.5, we relate values of symmetric functions and contour integrals.

### 2.1. Definitions

Consider a function $f(z_1, \dots, z_n)$ of $n$ complex variables. We focus on two types of functions: symmetric and skew-symmetric.

**Definition 1.** Function $f(z_1, \dots, z_n)$ is called *symmetric* if

$$f(z_1, \dots, z_n) = f(z_{s(1)}, \dots, z_{s(n)}) \tag{6}$$

for any permutation $s \in \mathbf{S}_n$, where $\mathbf{S}_n$ is the set of all permutations of set $\{1, \dots, n\}$.

**Definition 2.** Function $f(z_1, \dots, z_n)$ is called *skew-symmetric* if

$$f(z_1, \dots, z_n) = \operatorname{sgn}(s) f(z_{s(1)}, \dots, z_{s(n)}) \tag{7}$$

for any permutation $s \in \mathbf{S}_n$. Here, $\operatorname{sgn}(s)$ is the sign of permutation $s$ and is equal to $(-1)^{m_s}$, where $m_s$ denotes the number of transpositions, i.e., permutations that interchange two elements, needed to construct $s$. The sign is independent of the representation of $s$ as a product of transpositions.

Skew-symmetric functions are sometimes called anti-symmetric. In the analysis of the following section, we also use a subtype of symmetric functions, namely additive functions.

**Definition 3.** Function $f(z_1, \dots, z_n)$ is called *additive* if

$$f(z_1, \dots, z_n) = \sum_{k=1}^{n} g(z_k), \tag{8}$$

for some function $g(z)$.

In the analysis of Sections 3 and 4, we mainly work with alternant matrices.

**Definition 4.** Consider functions $f_1(z), \dots, f_n(z)$ and points $z_1, \dots, z_n$. Matrix

$$\Lambda(z_1, \dots, z_n) = \begin{pmatrix} f_1(z_1) & \dots & f_n(z_1) \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots} \\ f_1(z_n) & \dots & f_n(z_n) \end{pmatrix} \tag{9}$$

is called an *alternant* matrix.

An example of an alternant matrix is a Vandermonde matrix, where $f_k(z) = z^{k-1}$. The determinant of the Vandermonde matrix is denoted by $V(z_1, \ldots, z_n) = \prod_{1\leqslant i<j\leqslant n}(z_j - z_i)$. Note that the determinant of an alternant matrix is a skew-symmetric function of $z_1, \ldots, z_n$. This follows immediately from the fact that if one interchanges two rows (or columns) in a square matrix, such an operation changes the sign of the determinant.

**Remark 1.** In what follows, we will work with rational functions of several variables, i.e., $f(z_1, \ldots, z_n)/g(z_1, \ldots, z_n)$. Suppose that both the numerator and the denominator are analytic functions and $g(z_1, \ldots, z_n)$ is not identically equal to 0. For the case of one variable, i.e., $n = 1$, there are not more than a finite number of points where this rational function is not defined, namely where $g(z_1) = 0$. However, for $n > 1$ this is not true. For example, function $1/(z_1+z_2)$ is not defined on a line $z_1 = -z_2$. Suppose that functions $f(z_1, \ldots, z_n)$ and $g(z_1, \ldots, z_n)$ are defined on the set $D_1^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon |z_i| < 1, i = 1, \ldots, n\}$, and there is at least one point in $D_1^n$ where $g(z_1, \ldots, z_n) \neq 0$. Then, one can prove that the function $f(z_1, \ldots, z_n)/g(z_1, \ldots, z_n)$ is defined on a dense open subset of $D_1^n$. In what follows, when we say that two rational functions are equal, we mean that they are equal on an open dense subset of $D_1^n$, where both of them are defined.

## 2.2. Symmetrical polynomials and their properties

In this subsection, we introduce two types of symmetric polynomials and their properties. The *elementary symmetric polynomials* are given by

$$\sigma_m = \sigma_m(z_1, \ldots, z_n) = \sum_{1\leqslant i_1<\ldots<i_m\leqslant n} z_{i_1} \cdots z_{i_m}. \tag{10}$$

The above formula is used for $m = 1, \ldots, n$. For convenience, $\sigma_0 = 1$, and $\sigma_m = 0$ if either $m > n$ or $m < 0$. The elementary symmetric polynomials naturally arise in Vieta's formulas that relate the coefficients of the polynomial with its roots. Consider a polynomial $\sum_{j=0}^{n} a_j z^j$ with roots $z_1, \ldots, z_n$, then it can be written as

$$a_n \prod_{i=1}^{n}(z - z_i) = a_n \sum_{j=0}^{n}(-1)^j \sigma_j \, z^{n-j}. \tag{11}$$

The proof of (11) requires the expansion of the left-hand side of (11), see [14].

We mainly use the *complete homogeneous symmetric polynomials* defined as

$$\zeta_m = \zeta_m(z_1, \ldots, z_n) = \sum_{1\leqslant i_1\leqslant\ldots\leqslant i_m\leqslant n} z_{i_1} \cdots z_{i_m}. \tag{12}$$

Note the difference in the definitions of the elementary and complete homogeneous symmetric polynomials: the indexes $i_k$ and $i_{k+1}$ for the latter case may coincide for some or all $k$. This allows us to use the above formula for $m > n$. For $m = 0$, we define $\zeta_0 = 1$, and, for $m < 0$, $\zeta_m = 0$.

The following property can be used to recursively find all complete homogeneous polynomials from the elementary symmetric polynomials. The proof is given in Appendix A.

**Property 1.** *For $m > 0$, the following equality holds*

$$\zeta_m = \sum_{j=1}^{n} (-1)^{j-1} \sigma_j \zeta_{m-j}. \tag{13}$$

Note that for $j > n$, $\sigma_j = 0$ and for $j > m$, $\zeta_{m-j} = 0$. Thus, the upper limit of summation in (13) can be changed to any number that is at least $\min\{n, m\}$.

In the case where $z_1, \dots, z_n$ are roots of an equation, the value of complete homogeneous and elementary symmetric polynomials can be found without actually knowing these roots, see Subsection 2.5. In the following Subsection 2.3, for an analytic function of one variable, we construct a symmetric function of $n$ variables using complete homogeneous symmetric polynomials. Such functions on $D_1^n$ are later used to rewrite the determinant of an alternant matrix as a product of a skew-symmetric Vandermonde determinant and a symmetric function.

## 2.3. Symmetric functions and alternant matrices

In this subsection, we write the determinant of the alternant matrix in terms of the Vandermonde determinant and symmetric functions. We use such determinants in Section 3 to give an alternative representation of the considered type of pgfs. First, we define a transformation rule of an analytic function in $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ to a symmetric function defined in $D_1^k \subset \mathbb{C}^k$. Then, we give several properties of this transformation. The main result of this subsection is presented in Lemma 1.

Consider an analytical function $f(z)$, with Taylor expansion at 0 given by $f(z) = \sum_{i=0}^{\infty} \alpha_i z^i$. Let

$$F_k^m = F_k^m(z_1, \dots, z_m) = \sum_{i=k}^{\infty} \alpha_i \zeta_{i-k}(z_1, \dots, z_m), \tag{14}$$

where $m$ corresponds to the number of variables. We call $F_k^m$ the $(k, m)$-transformation of function $f(z)$. The $(k, m)$-transformation of function $f_j(z)$ is denoted by $F_{j,k}^m$. Note that function $F_k^m(z_1, \dots, z_m)$ is a symmetric function of $z_1, \dots, z_m$.

In the analysis of the following sections, we will use the following properties.

**Property 2.** *Consider $m \leqslant n$. Then,*

$$F_k^n(z_1, \dots, z_m, 0, \dots, 0) = F_k^m(z_1, \dots, z_m). \tag{15}$$

The property follows from the definition of complete homogeneous symmetric polynomials and only requires the observation that $\zeta_j(z_1, \dots, z_m, 0, \dots, 0) = \zeta_j(z_1, \dots, z_m)$.

**Property 3.** *Consider the function $f(z) = \sum_{i=0}^{\infty} \alpha_i z^i$ and its $(l, n)$-transformations $F_l^n$, for $l = k, \dots, k + n$. Then,*

$$F_k^n + \sum_{j=1}^{n} (-1)^j \sigma_j F_{k+j}^n = \alpha_k. \tag{16}$$

This property follows from Property 1 and equality $\zeta_0 = 1$. In the following lemma, we show that the determinant of an alternant matrix can be written as the product of a Vandermonde determinant and a matrix composed from $(k, n)$-transformations of the functions $f_1(z), \dots, f_n(z)$ that are used in the alternant matrix. The proof is given in Appendix B.

**Lemma 1.** *Suppose functions $f_1(z), \dots, f_n(z)$ are analytical in a neighbourhood of 0. Then,*

$$\det \begin{pmatrix} f_1(z_1) & f_2(z_1) & \dots & f_n(z_1) \\ f_1(z_2) & f_2(z_2) & \dots & f_n(z_2) \\ \dots & \dots & \dots & \dots \\ f_1(z_n) & f_2(z_n) & \dots & f_n(z_n) \end{pmatrix} =$$

$$= V(z_1, \dots, z_n) \det \begin{pmatrix} F^n_{1,0} & F^n_{2,0} & \dots & F^n_{n,0} \\ F^n_{1,1} & F^n_{2,1} & \dots & F^n_{n,1} \\ \dots & \dots & \dots & \dots \\ F^n_{1,n-1} & F^n_{2,n-1} & \dots & F^n_{n,n-1} \end{pmatrix}, \quad (17)$$

*where $V(z_1, \dots, z_n)$ is the Vandermonde determinant.*

This lemma is an important result for our analysis in Sections 3 and 4. The matrix on the right-hand side of (17) consists of symmetric functions of $z_1, \dots, z_n$ and, for example, can be non-singular for $z_i = z_j$, which will allow us in Section 4 to give proofs using induction in the number of variables, $n$.

## 2.4. Singular matrices and linear independence

In this subsection, we give a sufficient condition for linear dependency between (not necessarily analytic) functions. The proof is given in Appendix C.

**Property 4.** *Consider numbers $a_i \in \mathbb{R}$, $i = 0, \dots, n$. Suppose matrix*

$$\Lambda_a = \begin{pmatrix} a_0 & \dots & a_n \\ f_0(z_1) & \dots & f_n(z_1) \\ \dots & \dots & \dots \\ f_0(z_n) & \dots & f_n(z_n) \end{pmatrix} \quad (18)$$

*is singular for all $z_1, \dots, z_n$. If $a_i \neq 0$ for some $i$, then the functions $f_0(z), \dots, f_n(z)$ are linearly dependent.*

From this property it follows that if matrix $\Lambda_a$ is singular for all $z_1, \dots, z_n$ and functions $f_0(z), \dots, f_n(z)$ are linearly independent, then $a_i = 0$ for $i = 0, \dots, n$.

## 2.5. Roots and contour integrals

In this subsection, we provide a way of computing the values of symmetric polynomials at special points. Consider analytic function $D(z)$. Suppose $1, \bar{z}_1, \dots, \bar{z}_n$ are the only roots of equation

$$D(z) = 0 \quad (19)$$

in the closed unit disk $\bar{D}_1 = \{z \in \mathbb{C}\colon |z| \leqslant 1\}$. Then, it is possible to compute $\zeta_k(\bar{z}_1, \ldots, \bar{z}_n)$ without finding the roots. The first step is to represent the complete homogeneous symmetric polynomials in terms of the elementary symmetric polynomials, see Property 1. Then, we recursively use Newton's formula, see [9],

$$k\sigma_k = \sum_{j=1}^{k} (-1)^{j-1} \sigma_{k-j} \eta_j, \quad k = 1, \ldots, n, \tag{20}$$

to find the elementary symmetric polynomials in terms of the power sums $\eta_j = \eta_j(z_1, \ldots, z_n) = \sum_{l=1}^{n} z_l^j$, $j = 1, \ldots, n$. The power sums, in turn, are found using Cauchy's residue theorem. Namely,

$$\eta_j(\bar{z}_1, \ldots, \bar{z}_n) + 1 = \frac{1}{2\pi i} \oint_{S_{1+\varepsilon}} \frac{D'(z)}{D(z)} z^j \, dz, \tag{21}$$

where $\varepsilon > 0$ is defined so that there are no roots of equation (19) with $1 < |z| < 1 + \varepsilon$, and $S_{1+\varepsilon} = \{z \in \mathbb{C}\colon |z| = 1 + \varepsilon\}$, for more details see [11].

**Remark 2.** If the equation (19) does not have a zero at 1, one needs to change the left-hand side of (21) to just $\eta_j(\bar{z}_1, \ldots, \bar{z}_n)$. We explicitly consider the case that 1 is a root since the denominator of the pgfs analysed in Section 3 has a zero at 1.

**Remark 3.** If the function $f(z)$ is a polynomial, then $F_k^n(\bar{z}_1, \ldots, \bar{z}_n)$ is a finite sum of the complete homogeneous symmetric polynomials and can be found using equations (21), (20) and (13). The application of the Cauchy residue theorem, see (21), is a crucial step for going from the root-finding approach to the contour-integral approach. If function $f(z)$ is not a polynomial, one can truncate the infinite summation in $F_k^n(\bar{z}_1, \ldots, \bar{z}_n)$ using the following bound:

$$\left| F_k^n(\bar{z}_1, \ldots, \bar{z}_n) - \sum_{k=0}^{M} \alpha_k \zeta_k(\bar{z}_1, \ldots, \bar{z}_n) \right| \leqslant C \binom{M+n}{n-1} \frac{(qr)^{M+1}}{(1-qr)^n}, \tag{22}$$

where $|\bar{z}_k| \leqslant q$ for $k = 1, \ldots, n$, $|\alpha_l| \leqslant Cr^l$ for $l = 0, 1 \ldots$, $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$; and $qr < 1$, see proof in Appendix D. Therefore, the determinant of the matrix in the right-hand side of (17) can be found without computing the roots.

**Remark 4.** For the sake of completeness, we also give a different way of finding $F_k^n$ at $(\bar{z}_1, \ldots, \bar{z}_n)$. Let $Q(z) = \prod_{i=1}^{n} (z - \bar{z}_i)$ be the minimal polynomial, i.e., the polynomial of the lowest degree, with roots $\bar{z}_1, \ldots, \bar{z}_n$. Due to (11), $Q(z) = \sum_{j=0}^{n} (-1)^j \sigma_j(\bar{z}_1, \ldots, \bar{z}_n) z^{n-j}$, which means that this polynomial can be found without computing the roots, see equations (21) and (20). Then, see [2],

$$\zeta_m(\bar{z}_1, \ldots, \bar{z}_n) = \sum_{l=1}^{n} \frac{\bar{z}_l^{n+m-1}}{Q'(z_l)}. \tag{23}$$

By applying the Cauchy residue theorem, we get

$$\zeta_m(\bar{z}_1, \ldots, \bar{z}_n) = \frac{1}{2\pi i} \oint_{S_1} \frac{z^{n+m-1}}{Q(z)} dz, \tag{24}$$

where $S_1$ is the unit circle, i.e., $S_1 = \{z \in \mathbb{C}\colon |z| = 1\}$. Hence, we get

$$F_k^n(\bar{z}_1, \dots, \bar{z}_n) = \frac{1}{2\pi i} \oint_{S_1} \frac{f(z) - \sum_{l=0}^{k-1} \alpha_l z^l}{Q(z)} z^{n-k-1} dz. \tag{25}$$

Using this formula for each entry of the matrix in the right-hand side of (17), i.e., $n^2$ times, can be computationally demanding. Still, it can be useful in some special cases.

**Remark 5.** Suppose function $f(z_1, \dots, z_n)$ is an additive function, i.e., $f(z_1, \dots, z_n) = \sum_{k=1}^n g(z_k)$. If function $g(z)$ is analytic in $D_{1+\epsilon} \setminus \{1\}$, then the value of $f(\bar{z}_1, \dots, \bar{z}_n)$ is given by one contour integral:

$$f(\bar{z}_1, \dots, \bar{z}_n) = \frac{1}{2\pi i} \oint_{S_{1+\varepsilon}} \frac{D'(z)}{D(z)} g(z) dz - r_1, \tag{26}$$

where $\varepsilon < \epsilon$ is defined as in (21), and $r_1$ is the residue of the function $D'(z)g(z)/D(z)$ at 1.

## 3. Pgf as a symmetric function of roots

In this section, we consider pgfs of form (1), which occur, for example, in the bulk-service queue [1], in the multi-server queue, see [7], and in traffic-light queues, see [10] and [11]. Recall that such pgf has a rational form:

$$X(z) = \frac{\sum_{j=0}^n q_j f_j(z)}{D(z)} (z - 1), \tag{27}$$

where $q_j$ are unknown numbers, $f_j(z)$ are analytical functions, and $D(z)$ is an analytic function with $n + 1$ zeroes inside the unit disk including 1, which we denote by $\bar{z}_0 = 1, \dots, \bar{z}_n$. The goal of this section is to represent the pgf $X(z)$ as a symmetrical function of roots $\bar{z}_1, \dots, \bar{z}_n$, see Theorem 1 below. Such a representation together with Remark 3 allows us to find the pgf $X(z)$ without finding the roots. This is a computationally stable and reliable approach, see [11].

**Remark 6.** In the representation of the pgf, the term $(z-1)$ is usually included in functions $f_j(z)$. One can also rewrite (1) as

$$X(z) = \frac{\sum_{j=0}^n q_j f_j(z)}{\tilde{D}(z)}, \tag{28}$$

where $\tilde{D}(z) = D(z)/(z - 1)$ is a function with $n$ zeroes, $\bar{z}_1, \dots, \bar{z}_n$, inside the unit disk. If function $D(z)$ is analytical in some $D_r$ with $r > 1$, then so is $\tilde{D}(z)$.

As we showed in the introduction, the unknowns in the numerator can be found using an alternant matrix

$$M(z, z_1, \dots, z_n) = \begin{pmatrix} f_0(z) & f_1(z) & \dots & f_n(z) \\ f_0(z_1) & f_1(z_1) & \dots & f_n(z_1) \\ \dots & \dots & \dots & \dots \\ f_0(z_n) & f_1(z_n) & \dots & f_n(z_n) \end{pmatrix}. \tag{29}$$

In the following lemma, we represent the numerator of the pgf in terms of matrix $M(z, z_1, \dots, z_n)$. As a by-product of this representation, we get the following interesting result: it is not necessary to know $q_j$, $j = 0, \dots, n$ to know the numerator of (1). The solution (31) is given only for the proof.

**Lemma 2.** *Suppose matrix $M(1, \bar{z}_1, \dots, \bar{z}_n)$ is non-singular, and $(q_0, \dots, q_n)$ is a solution of (4), then*

$$\sum_{j=0}^{n} q_j f_j(z) = D'(1) \frac{\det M(z, \bar{z}_1, \dots, \bar{z}_n)}{\det M(1, \bar{z}_1, \dots, \bar{z}_n)}, \tag{30}$$

*and*

$$q_j = (-1)^j D'(1) \frac{\det M_j(\bar{z}_1, \dots, \bar{z}_n)}{\det M(1, \bar{z}_1, \dots, \bar{z}_n)}, \tag{31}$$

*where $M_j(z_1, \dots, z_n)$ is the matrix $M(z, z_1, \dots, z_n)$ without the first row and the $(j+1)^{st}$ column.*

*Proof.* Since matrix $M(1, \bar{z}_1, \dots, \bar{z}_n)$ is non-singular there is a unique solution for (4). Using the Laplace expansion, it is easy to check that (31) gives (30). Plugging (30) in the left-hand side of (2) and (3) gives identical equalities. □

**Remark 7.** The proof of Lemma 2 does not require the precise form of the matrix $M(z, z_1, \dots, z_n)$. We used the facts that only the first row depends on $z$ and the right-hand side is non-zero only in the first entry. Thus, matrix $M(z, z_1, \dots, z_n)$ can depend on $z_1, \dots, z_n$ in a different way. For example, instead of a row, a column may depend on one root. Such kind of systems also occur in queueing theory, see, e.g., [5].

**Remark 8.** The determinant $\det M(z, z_1, \dots, z_n)$ is a skew-symmetrical function of roots. Hence, the right-hand side of (30), which is equal to the numerator of (1), is a symmetrical function. However, form (30) does not show how to find the value of the numerator without finding the roots. Therefore, we need an equivalent representation, see Remark 1.

Consider the matrix

$$\bar{M}(z, z_1, \dots, z_n) = \begin{pmatrix} f_0(z) & f_1(z) & \dots & f_n(z) \\ F^n_{0,0} & F^n_{1,0} & \dots & F^n_{n,0} \\ \dots & \dots & \dots & \dots \\ F^n_{0,n-1} & F^n_{1,n-1} & \dots & F^n_{n,n-1} \end{pmatrix}. \tag{32}$$

From Lemma 1, we get that

$$\det M(z, z_1, \dots, z_n) = V(z_1, \dots, z_n) \det \bar{M}(z, z_1, \dots, z_n). \tag{33}$$

In particular,

$$\frac{\det M(z, z_1, \dots, z_n)}{\det M(1, z_1, \dots, z_n)} = \frac{h(z)}{h(1)}, \tag{34}$$

where

$$h(z) = h(z, z_1, \dots, z_n) = \det \bar{M}(z, z_1, \dots, z_n) \tag{35}$$

is a symmetric function of the roots. Therefore, from Lemma 2, we get the following theorem.

**Theorem 1.** *Suppose matrix* $M(1, \bar{z}_1, \dots, \bar{z}_n)$ *is non-singular, and vector* $(q_0, \dots, q_n)$ *is a solution of* (4)*, then*

$$\sum_{k=0}^{n} q_k f_k(z) = D'(1) \frac{h(z, \bar{z}_1, \dots, \bar{z}_n)}{h(1, \bar{z}_1, \dots, \bar{z}_n)}, \tag{36}$$

*and*

$$X(z) = \frac{D'(1) h(z, \bar{z}_1, \dots, \bar{z}_n)}{D(z) h(1, \bar{z}_1, \dots, \bar{z}_n)} (z - 1). \tag{37}$$

Using Remark 3, we can find $h(z, \bar{z}_1, \dots, \bar{z}_n)$ without knowing the roots by computing $n$ contour integrals. In Section 4, we consider a special case, in which the mean, $X'(1)$, may be found using only one contour integral.

## 4. Factorisation of the pgf

In this section, we give sufficient and necessary conditions for the pgf $X(z)$, given by (1), to have some special properties. The first property is the *factorisation property*, i.e., that the numerator of the pgf can be represented as a product, where each of the terms depends on not more than one root:

$$X(z) = D'(1) \frac{g(z) \prod_{k=1}^{n} g(z, \bar{z}_k)}{D(z)} (z - 1) \tag{38}$$

for some functions $g(z)$ and $g(z, w)$. This representation of the sum as a product is analogous to Vieta's formulas, see (11). For an example of the numerator that cannot be represented as such product, consider three functions $f_0(z) = 1$, $f_1(z) = z$ and $f_2(z) = z^3$. Then, the numerator has three roots: $z_1$, $z_2$ and $w = w(z_1, z_2)$. The third root (symmetrically) depends on both roots of the denominator, thus making the product form impossible.

The second property is the *additive-mean property*, i.e., $X'(1)$, which represents the mean of the corresponding random variable, is an additive function of the roots. One can easily see that the factorisation property implies the additive-mean property. Indeed, from (38), L'Hospital's rule gives us

$$1 = X(1) = g(1) \prod_{k=1}^{n} g(1, \bar{z}_k), \tag{39}$$

and the mean of the corresponding random variable is given by a symmetric additive function of roots:

$$X'(1) = -\frac{D''(1)}{2D'(1)} + \frac{g'(1)}{g(1)} + \sum_{k=1}^{n} \frac{\partial}{\partial z} \frac{g(z, \bar{z}_k)}{g(1, \bar{z}_k)} \bigg|_{z=1}. \tag{40}$$

To find the above equation, we took the derivative of (38) and used equality (39). An additive form, as in (40), implies that under some conditions, the mean value can be found using one contour integral, see Remark 5. Note that for certain functions $g(z, w)$, it is possible to represent the pgf as an exponent of a contour integral, see [4].

The following theorem gives sufficient and necessary conditions for the pgf to have the factorisation or additive-mean property.

**Theorem 2.** *Suppose the functions $f_k(z)$ are analytic in $D_1$, the matrix $M(z, z_1, \ldots, z_n)$ is defined by (5), and the function $h(z)$ is given by (35). Consider the following conditions:*

*(a) There exist a non-singular matrix*

$$A = \begin{pmatrix} a_{0,0} & \ldots & a_{0,n} \\ \ldots & \ldots & \ldots \\ a_{n,0} & \ldots & a_{n,n} \end{pmatrix}, \tag{41}$$

*an analytic function $B(z)$ in $D_1$ and a non-constant meromorphic function $C(z)$ in $D_1$ such that $f_j(z) = \sum_{k=0}^{n} a_{j,k} \tilde{f}_k(z)$, where*

$$\tilde{f}_k(z) = B(z)C(z)^k. \tag{42}$$

*(b) There exist meromorphic functions $g(z)$ and $g(z, w)$ such that*

$$\frac{h(z)}{h(1)} = \frac{h(z, z_1, \ldots, z_n)}{h(1, z_1, \ldots, z_n)} = g(z) \prod_{k=1}^{n} g(z, z_k). \tag{43}$$

*(c) There exist a constant $c$ and a meromorphic function $f(z)$ such that*

$$\frac{h'(1)}{h(1)} = \frac{\partial}{\partial z} \frac{h(z, z_1, \ldots, z_n)}{h(1, z_1, \ldots, z_n)}\bigg|_{z=1} = c + \sum_{k=1}^{n} f(z_k). \tag{44}$$

*If*

$(*)$ *the matrix $M(1, \hat{z}_1, \ldots, \hat{z}_n)$ is non-singular for some $\hat{z}_1, \ldots, \hat{z}_n$,*

*then conditions $(a)$ and $(b)$ are equivalent and $(c)$ follows from them. If, moreover,*

$(**)$ *there are $i$ and $j$ such that $\frac{f_i'(1)}{f_i(1)} \neq \frac{f_j'(1)}{f_j(1)}$,*

*then all three conditions are equivalent. Also, if the quotients in $(**)$ are equal for all $i$ and $j$, then $(c)$ is satisfied for a constant function $f(z)$. Furthermore, if conditions $(a)$ and $(*)$ hold, then one can choose*

$$g(z) = \frac{B(z)}{B(1)}, \qquad g(z, w) = \frac{C(z) - C(w)}{C(1) - C(w)}, \tag{45}$$

$$c = \frac{B'(1)}{B(1)}, \qquad f(w) = \frac{C'(1)}{C(1) - C(w)}. \tag{46}$$

In the following subsections, we will prove implications $(a) \Rightarrow (b)$ (under $(*)$), $(b) \Rightarrow (c)$, and $(c) \Rightarrow (a)$ (under conditions $(*)$ and $(**)$). Each part of the proof is in a separate subsection with several concluding remarks in Subsection 4.4. The proof of $(b) \Rightarrow (a)$ under $(*)$ is almost identical to the proof of $(c) \Rightarrow (a)$, and, therefore, we omit it.

### 4.1. From polynomials to factorised form

In this subsection, we prove that $(a) \Rightarrow (b)$. Consider the matrix $M(z, z_1, \dots, z_n)$ at some point such that the matrix is non-singular. Given $(a)$, we get that $M(z, z_1, \dots, z_n)$ is a linear transformation of an alternant matrix with functions $\tilde{f}_k(z)$:

$$M(z, z_1, \dots, z_n)(A^T)^{-1} = \begin{pmatrix} B(z) & \dots & B(z)C(z)^n \\ \dots & \dots & \dots \\ B(z_n) & \dots & B(z_n)C(z_n)^n \end{pmatrix}, \tag{47}$$

which is almost a Vandermonde matrix. Hence, its determinant is equal to

$$\det(M(z, z_1, \dots, z_n)(A^T)^{-1}) = B(z) \prod_{k=1}^{n} B(z_k) V_C(z, \dots, z_n), \tag{48}$$

where $V_C(z, \dots, z_n) = V(C(z), C(z_1), \dots, C(z_n))$ is the Vandermonde determinant for variables $C(z), C(z_1), \dots, C(z_n)$. Therefore, given the fact that matrices $A$ and $M(1, z_1, \dots, z_n)$ are non-singular, we get

$$\frac{h(z)}{h(1)} = \frac{\det(M(z, z_1, \dots, z_n)(A^T)^{-1})}{\det(M(1, z_1, \dots, z_n)(A^T)^{-1})} = \frac{B(z)}{B(1)} \prod_{k=1}^{n} \frac{C(z) - C(z_k)}{C(1) - C(z_k)}. \tag{49}$$

This is exactly equation (43) with functions $g(z)$ and $g(z, w)$ defined as in (45). Note that due to the continuity of the functions on the left-hand side and the right-hand side of (49) in their support, the equality holds also when the matrix $M(z, z_1, \dots, z_n)$ is singular.

### 4.2. From factorised form to an additive function

In this subsection, we prove that $(b) \Rightarrow (c)$. This implication does not require $(*)$ or $(**)$. Note that from (43) it follows that $g(1) \prod_{k=1}^{n} g(1, z_k) = 1$. Hence,

$$\frac{h'(1)}{h(1)} = \frac{\partial}{\partial z} \left( \frac{g(z)}{g(1)} \prod_{k=1}^{n} \frac{g(z, z_k)}{g(1, z_k)} \right)\Bigg|_{z=1} = \frac{g'(1)}{g(1)} + \sum_{k=1}^{n} \frac{\partial}{\partial z} \frac{g(z, z_k)}{g(1, z_k)}\Bigg|_{z=1}. \tag{50}$$

Thus, one can choose $c = g'(1)/g(1)$ and $f(w) = \partial/\partial z\, g(z, w)/g(1, w)|_{z=1}$. If functions $g(z)$ and $g(z, w)$ are defined as in (45), then $c$ and $f(w)$ are defined as in (46).

### 4.3. From an additive mean to a polynomial numerator

In this subsection, we prove that $(c) \Rightarrow (a)$. This is the most difficult part. First, we consider a linear transformation of functions $f_k(z)$, given by the following lemma, see proof in Appendix E.

**Lemma 3.** *If functions $f_k(z)$ are analytic in $D_1$ and linearly independent, then there exists point $z^* \in D_1$ and functions $\tilde{f}_j(z)$ such that $f_j(z) = \sum_{i=0}^{n} \tilde{a}_{j,i} \tilde{f}_i(z)$ and*

$$\tilde{f}_i(z) = (z - z^*)^i + o((z - z^*)^i) \text{ as } z \to z^*. \tag{51}$$

*Moreover, $z^*$ can be any point in $D_1$ except a finite set of points.1*

From the proof of the lemma, it follows that, given $(*)$, the function $\tilde{f}_{n-1}(z)$ can be chosen such that $\tilde{f}_{n-1}(1) \neq 0$ for all possible $z^*$ except a finite set of points, see Remark 9 in Appendix E. In our case, we can apply Lemma 3, because matrix $M(1, \hat{z}_1, \ldots, \hat{z}_n)$ is non-singular, and, therefore, functions $f_0(z), \ldots, f_n(z)$ are linearly independent. Since $z^*$ can be any point in the unit disk except a finite number of points, we can assume, without loss of generality, that $z^* = 0$, $a_{i,j} = \delta_{ij}$ and $\tilde{f}_{n-1}(1) \neq 0$, where $\delta_{ij}$ is Kronecker delta.

Now, suppose that (44) holds. We will focus on determining $f_{k+1}(z)/f_k(z)$. For this, we will consider the cases where $z_{k+1} = \ldots = z_n = 0$ for $k = 1, \ldots, n$. The following lemma gives the value of $h(z)/h(1)$ for each $k$. The proof is given in Appendix F.

**Lemma 4.** *For $k > 0$,*

$$\frac{h(z, z_1, \ldots, z_k, 0, \ldots, 0)}{h(1, z_1, \ldots, z_k, 0, \ldots, 0)} = \frac{\det \Lambda_k(z, z_1, \ldots, z_k)}{\det \Lambda_k(1, z_1, \ldots, z_k)}, \tag{52}$$

*where*

$$\Lambda_k(z, z_1, \ldots, z_k) = \begin{pmatrix} f_{n-k}(z) & \ldots & f_n(z) \\ f_{n-k}(z_1) & \ldots & f_n(z_1) \\ \ldots & \ldots & \ldots \\ f_{n-k}(z_k) & \ldots & f_n(z_k) \end{pmatrix}. \tag{53}$$

To find function $f(z)$ in (44), we consider $k = 1$. Using Lemma 4 gives us

$$\frac{h(z, z_1, 0, \ldots, 0)}{h(1, z_1, 0, \ldots, 0)} = \frac{f_n(z_1)f_{n-1}(z) - f_n(z)f_{n-1}(z_1)}{f_n(z_1)f_{n-1}(1) - f_n(1)f_{n-1}(z_1)}. \tag{54}$$

Let $C(z) = f_n(z)/f_{n-1}(z)$. Then, we can rewrite (54) as

$$\frac{h(z, z_1, 0, \ldots, 0)}{h(1, z_1, 0, \ldots, 0)} = \frac{f_{n-1}(z)}{f_{n-1}(1)} \frac{C(z) - C(z_1)}{C(1) - C(z_1)}. \tag{55}$$

Note that $C(1)$ and the right-hand side of (55) are well-defined since $f_{n-1}(1) = \tilde{f}_{n-1}(1) \neq 0$, see the choice of $z^*$. Taking the derivative gives us

$$\frac{\partial}{\partial z} \frac{h(z, z_1, 0, \ldots, 0)}{h(1, z_1, 0, \ldots, 0)}\bigg|_{z=1} = \frac{f'_{n-1}(1)}{f_{n-1}(1)} + \frac{C'(1)}{C(1) - C(z_1)}. \tag{56}$$

Equation (44) defines the constant $c$ and the function $f(w)$ up to a constant, i.e., constant $c$ can be arbitrarily chosen. Therefore, we redefine $f(w)$ as $C'(1)/(C(1) - C(w))$. This leads to $c = f'_{n-1}(1)/f_{n-1}(1) - (n-1)C'(1)/C(1)$ since $f(0) = C'(1)/C(1)$. Here, we used the fact that $C(0) = \lim_{z \to 0} f_n(z)/f_{n-1}(z) = \lim_{z \to 0} z^n/z^{n-1} = 0$.

Now, it is left to prove that condition $(a)$ follows from equation

$$\frac{\partial}{\partial z} \frac{h(z, z_1, \ldots, z_n)}{h(1, z_1, \ldots, z_n)}\bigg|_{z=1} = \frac{f'_{n-1}(1)}{f_{n-1}(1)} - (n-1)\frac{C'(1)}{C(1)} + \sum_{k=1}^{n} \frac{C'(1)}{C(1) - C(z_k)} \tag{57}$$

with $B(z) = f_n(z)/C(z)^n$.

Note that function $C(z)$ is not a constant since functions $f_n(z)$ and $f_{n-1}(z)$ are linearly independent. Now, it would be sufficient to prove that $f_{k+1}(z) = C(z)f_k(z)$ for $k = 0, \ldots, n-2$. However, it may be not true. For example, functions $1+z$, $z$ and $z^2$ satisfy conditions $(a)$ - $(c)$ and (51), but $z/(1+z) \neq z = z^2/z$. Thus, we use the following lemma, recursive application of which together with a linear transformation of functions $f_k(z)$ concludes the proof of Theorem 2. The proof of Lemma 5 is given in Appendix G.

**Lemma 5.** *Consider $k \geqslant 2$. Suppose that equation (57) holds for $z_{k+1} = \ldots = z_n = 0$. Suppose also that $f_{j+1}(z)/f_j(z) = C(z)$ for $j = n-k+1, \ldots, n-1$. Then, there exist coefficients $\beta_j$, $j = 0, \ldots, k$, such that*

$$f_{n-k}(z) = \beta_0 f_{n-k+1}(z)/C(z) + \sum_{j=1}^{k} \beta_j f_{n-k+j}(z). \tag{58}$$

### 4.4. Some remarks

Theorem 2 was partially proven for a specific function $C(z)$ in [11]. There the focus was on proving (44) for certain systems such as the bulk-service queue and the fixed-cycle traffic-light queue. This result allows to use contour integrals for finding the average queue length. Theorem 2 generalises the result of [11] and describes all systems for which (44) applies. However, it does not mean that these are the only systems for which the mean value can be found using one contour integral. In this paper, we have considered a special class of the pgf, see (1). If one is able to find the pgf in a factorisation form, e.g., as in (38), then the mean will be an additive function of the roots. For example, this result can be applied to the $G_D/G_D/1$ queue considered in [13].

Concerning implication $(b) \Rightarrow (a)$ under $(*)$, one can proceed from equation (55) and define $g(z,w)$ as in (45). Afterwards, one needs to prove an alternative of Lemma 5, more precisely equation (58). In the proof of Lemma 5, we expand an alternant matrix over the row that depends on $z_1$. In this case, the result requires an additional condition $C'(1) \neq 0$, which follows from $(**)$. If we want to prove implication $(b) \Rightarrow (a)$, we can expand the similar matrix over the row that depends on $z$ (in implication $(c) \Rightarrow (a)$ this is a constant row and we cannot use such expansion). Then, we do not need an extra condition $(**)$.

Note that from definition (35) of the function $h(z)$, it follows that if condition $(**)$ does not hold, than (44) holds for $f(z) = 0$ and $c = f_0'(1)/f_0(1)$. To conclude this section, we give an example of a queueing system without condition $(**)$. Consider a special bulk-service queue with vacations depending on the queue size. The arrivals are Poisson with rate 1. The size of the batch is 3 and the service time is deterministic and equal to some $d \in (2,3)$. If the server visits the queue and finds at least three customers, it immediately starts serving the first three customers. If upon a visit the server finds the queue with $j$ customers, $j < 3$, it takes a vacation of deterministic time $v_j$ with

$$v_0 = -1 + \sqrt{d^2 - 4d + 7}, \qquad v_1 = -1 + \sqrt{d^2 - 3d + 3}, \tag{59}$$

$$v_2 = d - 2. \tag{60}$$

Note that for $d > 2$, time $v_j$ is positive, $j = 0, 1, 2$. It is possible to find the pgf $X(z)$ of the queue length at the times when the server visits the queue, i.e., after a service or a vacation,

$$X(z) = \frac{\sum_{j=0}^{2} \pi_j \hat{f}_j(z)}{z^3 - e^{d(z-1)}}, \tag{61}$$

where $\pi_j$ is the probability of finding $j$ customers in the queue upon a visit, and functions $\hat{f}_j(z) = (z-1)f_j(z)$ are defined as follows:

$$\hat{f}_j(z) = e^{v_j(z-1)} z^3 - z^j e^{d(z-1)}. \tag{62}$$

One can check that $f_j'(1) = \hat{f}_j''(1)/2 = 2f_j(1) = \hat{f}_j'(1)$ for all $j = 0, 1, 2$, which means that (∗∗) does not hold, and, therefore, that $(c)$ holds for $f(z) = 0$. Thus, the mean queue length upon the server arrival is independent of the roots of the characteristic equation $z^3 = e^{d(z-1)}$:

$$X'(1) = \left(\frac{z-1}{z^3 - e^{d(z-1)}}\right)'\Bigg|_{z=1} (3-d) + \frac{\sum_{j=0}^{2} \pi_j f_j'(1)}{\sum_{j=0}^{2} \pi_j f_j(1)} = \tag{63}$$

$$= -\frac{6-d^2}{6-2d} + 2 = \frac{d^2 - 4d + 6}{6-2d}. \tag{64}$$

## 5. Conclusions

In this paper, we analysed the dependency of a certain type of pgfs on the roots of the characteristic equation. We showed that such a pgf depends on the roots in a symmetric way and gave an explicit matrix representation of the pgf. Our representation allows one to use the roots that are close to each other without encountering the corresponding sensitivity problems. Moreover, it is possible to find the pgf without computing the roots, which can further improve the accuracy.

We studied the cases where the pgf has a product form, and where the mean value is an additive function of the roots. We showed that these properties are equivalent under a non-degeneracy condition and gave a sufficient-and-necessary condition for them. For systems with these properties, both the pgf at a point and the mean may be found using one contour integral. If the non-degeneracy condition does not hold, the mean is independent of the roots.

## Appendix A Proof of Property 1

In this section, we prove Property 1. Recall that we need to prove that

$$\sum_{j=0}^{n} (-1)^j \sigma_j \zeta_{m-j} = 0 \tag{65}$$

for any $m > 0$. The proof will be done using generating functions. First, we want to note that

$$\sum_{m=0}^{\infty} \zeta_m z^m = \prod_{i=1}^{n-1} \sum_{k=0}^{\infty} (z_i z)^k = \prod_{i=1}^{n-1} \frac{1}{1 - z_i z}. \tag{66}$$

The above equality holds for sufficiently small $z$, i.e., for $|z| < \min_{i=1,\dots,n-1} 1/|z_i|$. Second, from (11), we get

$$\prod_{i=1}^{n-1} (1 - z_i z) = z^{n-1} \prod_{i=1}^{n-1} (z^{-1} - z_i) = \sum_{j=0}^{n-1} (-1)^j \sigma_j \, z^j. \tag{67}$$

Hence,

$$1 = \prod_{i=1}^{n-1} \frac{1 - z_i z}{1 - z_i z} = \sum_{m=0}^{\infty} \zeta_m z^m \sum_{j=0}^{n-1} (-1)^j \sigma_j \, z^j. \tag{68}$$

Note that the last equation is an equality of two analytical functions. Thus, the coefficients at powers of $z$ should coincide. Result (13) follows from considering the coefficient at $z^m$ for $m > 0$.

# Appendix B Proof of Lemma 1

In this section, we prove Lemma 1. We use the first Jacobi-Trudi formula, see [2], which can be written as

$$\det \begin{pmatrix} z_1^{m_1} & \dots & z_1^{m_n} \\ \dots & \dots & \dots \\ z_n^{m_1} & \dots & z_n^{m_n} \end{pmatrix} = V(z_1, \dots, z_n) \det \begin{pmatrix} \zeta_{m_1} & \dots & \zeta_{m_n} \\ \dots & \dots & \dots \\ \zeta_{m_1 - n + 1} & \dots & \zeta_{m_n - n + 1} \end{pmatrix}. \tag{69}$$

It is used for the Schur polynomials, for which $m_1 > \ldots > m_n$. However, the result is general. In particular, a permutation of rows gives the result for any $m_1, \ldots, m_n$ such that $m_i \neq m_j$ for any $i \neq j$. Note also that if $m_i = m_j$ for $i \neq j$, then both sides of (69) are equal to 0.

Lemma 1 follows from equation (69) by summing it for all possible combinations $(m_1, \ldots, m_n)$ with coefficients $\prod_{k=1}^{n} \alpha_{k,m_k}$. Note also that (69) is a special case of Lemma 1.

# Appendix C Proof of Property 4

In this section, we prove Property 4. We use the induction by $n$. Consider $n = 1$. Then the matrix

$$\begin{pmatrix} a_0 & a_1 \\ f_0(z_1) & f_1(z_1) \end{pmatrix} \tag{70}$$

is singular for all $z_1$, which means $a_0 f_1(z) = a_1 f_0(z)$ for all $z$ and either $a_0$ or $a_1$ is non-zero, i.e., the functions are linearly dependent. Suppose we proved the statement for $n - 1$. Using Laplace expansion, we find $0 = \det \Lambda_a = \sum_{k=0}^{n} \det \Lambda_{a,k} f_k(z_1)$, where $\Lambda_{a,k}$ is matrix $\Lambda_a$ without second row and $(k+1)^{\text{st}}$ column. If $\det \Lambda_{a,k}$ is not identically 0, then the functions are linearly dependent. Now suppose that $\det \Lambda_{a,k}$ is identically 0 for all $k = 0, \ldots, n$. Without loss of generality, we can assume $a_1 \neq 0$. Applying property for matrix $\Lambda_{a,0}$, we find that functions $f_1(z), \ldots, f_n(z)$ are linearly dependent, and so are functions $f_0(z), \ldots, f_n(z)$.

# Appendix D Proof of bound (22)

In this section, we prove bound (22). Suppose $|\bar{z}_k| \leqslant q$ for $k = 1, \ldots, n$, and $|\alpha_l| \leqslant Cr^l$ for $l = 0, 1, \ldots$, where $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$. The latter condition holds if function $f(z)$ is analytic in a disk with radius more than $1/r$. Suppose also that $qr < 1$. Then, we can give the following bound

$$\left| F_k^n(\bar{z}_1, \ldots, \bar{z}_n) - \sum_{k=0}^{M} \alpha_k \zeta_k(\bar{z}_1, \ldots, \bar{z}_n) \right| = \tag{71}$$

$$= \left| \sum_{k=M+1}^{\infty} \alpha_k \zeta_k(\bar{z}_1, \ldots, \bar{z}_n) \right| \leqslant \tag{72}$$

$$\leqslant \sum_{k=M+1}^{\infty} |\alpha_k| |\zeta_k(\bar{z}_1, \ldots, \bar{z}_n)| \leqslant \tag{73}$$

$$\leqslant C \sum_{k=M+1}^{\infty} r^k \zeta_k(|\bar{z}_1|, \ldots, |\bar{z}_n|) \leqslant \tag{74}$$

$$\leqslant C \sum_{k=M+1}^{\infty} r^k \zeta_k(q, \ldots, q) \leqslant \tag{75}$$

$$\leqslant C \sum_{k=M+1}^{\infty} \binom{k+n-1}{k} (qr)^k \overset{!}{=} \tag{76}$$

$$\overset{!}{=} C \binom{M+n}{M} \frac{\sum_{l=1}^{n} (-1)^{l+1} \binom{n}{l} \frac{l}{M+l} (qr)^{M+l}}{(1-qr)^n} = \tag{77}$$

$$= C \binom{M+n}{M} \frac{n \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \frac{1}{M+l+1} (qr)^{M+l+1}}{(1-qr)^n} = \tag{78}$$

$$= C \binom{M+n}{n} \frac{n \int_0^{qr} x^M (1-x)^{n-1} dx}{(1-qr)^n} \leqslant \tag{79}$$

$$\leqslant C \binom{M+n}{n} \frac{n \int_0^{qr} x^M dx}{(1-qr)^n} = \tag{80}$$

$$= C \binom{M+n}{n} \frac{n (qr)^{M+1}}{(M+1)(1-qr)^n} = \tag{81}$$

$$= C \binom{M+n}{n-1} \frac{(qr)^{M+1}}{(1-qr)^n}. \tag{82}$$

Equality $\overset{!}{=}$ can be proven using induction by $M$ as follows. Consider $S_{n,M} = S_{n,M}(x) = \sum_{k=M+1}^{\infty} \binom{k+n-1}{k} x^k$. For $M = 0$, we get

$$S_{n,0} = \left( \sum_{j=0}^{\infty} x^j \right)^n - 1 = \frac{1}{(1-x)^n} - 1 = \frac{\sum_{l=1}^{n} (-1)^{l+1} \binom{n}{l} x^l}{(1-x)^n}. \tag{83}$$

Now suppose we have proved the statement for $S_{n,M-1}$. Consider $S_{n,M}$:

$$S_{n,M}(1-x)^n = \left( S_{n,M-1} - \binom{M+n-1}{M} x^M \right) (1-x)^n = \tag{84}$$

$$= \binom{M+n-1}{M-1} \sum_{l=1}^{n} (-1)^{l+1} \binom{n}{l} \frac{l}{M+l-1} x^{M+l-1} - \tag{85}$$

$$- \binom{M+n-1}{M} x^M (1-x)^n = \tag{86}$$

$$= \binom{M+n-1}{M-1} \sum_{l=0}^{n-1} (-1)^l \binom{n}{l+1} \frac{l+1}{M+l} x^{M+l} - \tag{87}$$

$$- \binom{M+n-1}{M} \sum_{l=0}^{n} (-1)^l \binom{n}{l} x^{M+l} = \tag{88}$$

$$= \sum_{l=0}^{n} (-1)^l \frac{(M+n-1)!(n-l)}{(M-1)!l!(n-l)!(M+l)} x^{M+l} - \tag{89}$$

$$- \sum_{l=0}^{n} (-1)^l \frac{(M+n-1)!n}{M!l!(n-l)!} x^{M+l} = \tag{90}$$

$$= \sum_{l=0}^{n} (-1)^l \frac{(M+n-1)!}{(M-1)!l!(n-l)!} \left( \frac{n-l}{M+l} - \frac{n}{M} \right) x^{M+l} = \tag{91}$$

$$= \sum_{l=0}^{n} (-1)^{l+1} \frac{(M+n)!}{M!l!(n-l)!} \frac{l}{M+l} x^{M+l} = \tag{92}$$

$$= \binom{M+n}{M} \sum_{l=1}^{n} (-1)^{l+1} \binom{n}{l} \frac{l}{M+l} x^{M+l}. \tag{93}$$

# Appendix E Proof of Lemma 3

In this section, we prove Lemma 3. Recall that we need to prove that there exists a point $z^* \in D_1$ such that functions $f_j(z)$ after a linear transformation give functions $\tilde{f}_i(z)$ that are locally equal to $(z - z^*)^i + o((z - z^*)^i)$.

Functions $f_j(z)$, $j = 0, \dots, n$, are analytic and linearly independent. Therefore, the Wronskian

$$\det W(z) = \det \begin{pmatrix} f_0(z) & f_1(z) & \dots & f_n(z) \\ f_0'(z) & f_1'(z) & \dots & f_n'(z) \\ \dots & \dots & \dots & \dots \\ f_0^{(n)}(z) & f_1^{(n)}(z) & \dots & f_n^{(n)}(z) \end{pmatrix} \tag{94}$$

is not identically 0, see [3]. Note that $\det W(z)$ is an analytic function in $D_1$ and, therefore, it has not more than a finite number of zeros in $D_1$. Let $z^* \in D_1$ be such that $\det W(z^*) \neq 0$. Define functions $\tilde{f}_0(z), \dots, \tilde{f}_n(z)$ by the following linear combination:

$$\begin{pmatrix} 0! \cdot \tilde{f}_0(z) & 1! \cdot \tilde{f}_1(z) & \dots & n! \cdot \tilde{f}_n(z) \end{pmatrix} = \begin{pmatrix} f_0(z) & f_1(z) & \dots & f_n(z) \end{pmatrix} (W(z^*))^{-1}. \tag{95}$$

Consider the matrix $\tilde{W}(z)$ for functions $0!\tilde{f}_0(z), \dots, n!\tilde{f}_n(z)$ defined as

$$\tilde{W}(z) = \begin{pmatrix} \tilde{f}_0(z) & \tilde{f}_1(z) & \dots & n! \cdot \tilde{f}_n(z) \\ \tilde{f}_0'(z) & \tilde{f}_1'(z) & \dots & n! \cdot \tilde{f}_n'(z) \\ \dots & \dots & \dots & \dots \\ \tilde{f}_0^{(n)}(z) & \tilde{f}_1^{(n)}(z) & \dots & n! \cdot \tilde{f}_n^{(n)}(z) \end{pmatrix}. \tag{96}$$

At point $z^*$, we get that $\tilde{W}(z^*) = W(z^*)(W(z^*))^{-1}$ becomes an identity matrix. Thus, $\tilde{f}_k^{(l)}(z^*) = 0$ for $l < k$ and $k!\tilde{f}_k^{(k)}(z^*) = 1$, which means that functions $\tilde{f}_0(z), \dots, \tilde{f}_n(z)$ satisfy (51).

**Remark 9.** Given condition $(*)$, one can choose $z^*$ and the functions so that for a particular index $k$, $\tilde{f}_k(1) \neq 0$. To see this, let us first consider the case $k = n$. Note that $f_j(1) \neq 0$ for at least one $j$; otherwise the matrix $M(1, \hat{z}_1, \ldots, \hat{z}_n)$ would be singular, which contradicts $(*)$. We can consider a linear transformation of functions $f_k(z)$ such that $f_0(1) \neq 0$, and $f_k(1) = 0$ for all $k \neq 0$. Then $\tilde{f}_n(1) \neq 0$ if and only if the coefficient of $f_0(z)$ in the definition of $\tilde{f}_n(z)$, see (95), is non-zero. This coefficient, up to multiplication by $\det W(z^*)$ and a sign, is equal to the Wronskian for functions $f_1(z), \ldots, f_n(z)$ at point $z^*$, which is non-zero for all possible $z^*$ except a finite set. Note that if $\tilde{f}_n(1) \neq 0$, then either $\tilde{f}_k(1) \neq 0$ or $\tilde{f}_k(1) + \tilde{f}_n(1) \neq 0$. Therefore, the result for all $k$ follows from the fact that $\tilde{f}_k(z) + \tilde{f}_n(z)$ satisfies (51) for $i = k$.

# Appendix F  Proof of Lemma 4

In this section, we prove Lemma 4. We need to prove (52), i.e.,

$$\frac{h(z, z_1, \ldots, z_k, 0, \ldots, 0)}{h(1, z_1, \ldots, z_k, 0, \ldots, 0)} = \frac{\det \Lambda_k(z, z_1, \ldots, z_k)}{\det \Lambda_k(1, z_1, \ldots, z_k)}, \tag{97}$$

where $\Lambda_k(z, z_1, \ldots, z_k)$ is an alternant matrix constructed using functions $f_{n-k}(z)$, $\ldots, f_n(z)$ and points $z, z_1, \ldots, z_k$.

Recall that function $h(z, z_1, \ldots, z_k, 0, \ldots, 0)$ is the determinant of matrix $\bar{M}(z, z_1, \ldots, z_k, 0, \ldots, 0)$, which entries are $F^n_{j,m}(z_1, \ldots, z_k, 0, \ldots, 0)$. From Property 2, we get $F^n_{j,m}(z_1, \ldots, z_k, 0, \ldots, 0) = F^k_{j,m}(z_1, \ldots, z_k)$. Now, according to Property 3,

$$F^k_{j,m} + \sum_{l=1}^{k} (-1)^l \sigma_l(z_1, \ldots, z_k) F^k_{j,m+l} = \alpha_{j,m}, \tag{98}$$

where $f_j(z) = \sum_{l=0}^{\infty} \alpha_{j,l} z^l$. Thus, after a linear transformation, we get that matrix $\bar{M}(z, z_1, \ldots, z_k, 0, \ldots, 0)$ changes to

$$\begin{pmatrix} f_0(z) & \ldots & f_n(z) \\ \alpha_{0,0} & \ldots & \alpha_{n,0} \\ \ldots & \ldots & \ldots \\ \alpha_{0,n-k-1} & \ldots & \alpha_{n,n-k-1} \\ F^k_{0,n-k} & \ldots & F^k_{n,n-k} \\ \ldots & \ldots & \ldots \\ F^k_{0,n-1} & \ldots & F^k_{n,n-1} \end{pmatrix}. \tag{99}$$

Here, we added the $(m+l+2)^{\text{nd}}$ row multiplies by $(-1)^l \sigma_l(z_1, \ldots, z_k)$ to the $(m+2)^{\text{nd}}$ row for $l = 1, \ldots, k$ and $m = 0, \ldots, n-k-1$. Now, note that $\alpha_{j,k} = 0$ for $j > k$ and $\alpha_{j,j} = 1$, which means that the matrix in (99) is equal to

$$\begin{pmatrix} f_0(z) & \ldots & f_{n-k-1}(z) & f_{n-k}(z) & \ldots & f_n(z) \\ 1 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{0,n-k-1} & \ldots & 1 & 0 & \ldots & 0 \\ F^k_{0,n-k} & \ldots & F^k_{n-k-1,n-k} & F^k_{n-k,n-k} & \ldots & F^k_{n,n-k} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ F^k_{0,n-1} & \ldots & F^k_{n-k-1,n-1} & F^k_{n-k,n-1} & \ldots & F^k_{n,n-1} \end{pmatrix}. \tag{100}$$

Hence, the determinant is equal to

$$h(z, z_1, \dots, z_k, 0, \dots, 0) = (-1)^{n-k} \det \begin{pmatrix} f_{n-k}(z) & \dots & f_n(z) \\ F^k_{n-k,n-k} & \dots & F^k_{n,n-k} \\ \dots & \dots & \dots \\ F^k_{n-k,n-1} & \dots & F^k_{n,n-1} \end{pmatrix}. \tag{101}$$

At this moment, we can use Lemma 1, to find that up to multiplication by the Vandermonde determinant $V(z_1, \dots, z_k)$, the determinant of the matrix on the right-hand side of (101) is equal to the determinant of an almost alternant matrix

$$V(z_1, \dots, z_k) \det \begin{pmatrix} f_{n-k}(z) & \dots & f_n(z) \\ F^k_{n-k,n-k} & \dots & F^k_{n,n-k} \\ \dots & \dots & \dots \\ F^k_{n-k,n-1} & \dots & F^k_{n,n-1} \end{pmatrix} = \det \begin{pmatrix} f_{n-k}(z) & \dots & f_n(z) \\ \frac{f_{n-k}(z_1)}{z_1^{n-k}} & \dots & \frac{f_n(z_1)}{z_1^{n-k}} \\ \dots & \dots & \dots \\ \frac{f_{n-k}(z_k)}{z_k^{n-k}} & \dots & \frac{f_n(z_k)}{z_k^{n-k}} \end{pmatrix}, \tag{102}$$

which is not identically equal to 0 due to the linear independence of the functions $f_{n-k}(z), \dots, f_n(z)$. Here, we used the fact that $f_l(z)$, $l = n-k, \dots, n$, satisfies (51), and, therefore, has first $n-k-1$ coefficients in the Taylor expansion equal to 0. Hence, the $(m,k)$-transformation of function $f_l(z)/z^{n-k} = \sum_{j=n-k}^{\infty} \alpha_j z^{j-n+k}$ is equal to $F^k_{l,n-k+m} = \sum_{j=n-k+m}^{\infty} \alpha_j \zeta_{j-n+k-m}$. Combining (101) and (102), gives (52) and, therefore, concludes the proof.

# Appendix G  Proof of Lemma 5

In this section, we prove Lemma 5. In this lemma, we assume that equation (57) holds for $z_{k+1} = \dots = z_n = 0$, i.e.,

$$\frac{\partial}{\partial z} \frac{h(z, z_1, \dots, z_k, 0, \dots, 0)}{h(1, z_1, \dots, z_k, 0, \dots, 0)}\Big|_{z=1} = \frac{f'_{n-1}(1)}{f_{n-1}(1)} - (k-1)\frac{C'(1)}{C(1)} + \sum_{j=1}^{k} \frac{C'(1)}{C(1) - C(z_k)}, \tag{103}$$

and that $f_{j+1}(z) = C(z) f_j(z)$ for $j = n-k+1, \dots, n-1$. We need to prove that function $f_{n-k}(z)$ is equal (up to a linear combination of functions $f_{n-k+1}(z), \dots, f_n(z)$) to $\beta_0 f_{n-k+1}/C(z)$.

First, we apply Lemma 4 and get

$$\frac{h(z, z_1, \dots, z_k, 0, \dots, 0)}{h(1, z_1, \dots, z_k, 0, \dots, 0)} = \frac{f_{n-k+1}(z) \det M_G(z, z_1, \dots, z_k)}{f_{n-k+1}(1) \det M_G(1, z_1, \dots, z_k)}, \tag{104}$$

where

$$M_G(z, z_1, \dots, z_k) = \begin{pmatrix} \frac{1}{G(z)} & 1 & \dots & C(z)^{k-1} \\ \frac{1}{G(z_1)} & 1 & \dots & C(z_1)^{k-1} \\ \dots & \dots & \dots & \dots \\ \frac{1}{G(z_k)} & 1 & \dots & C(z_k)^{k-1} \end{pmatrix}, \tag{105}$$

and $G(z) = f_{n-k+1}(z)/f_{n-k}(z)$.

Second, we find the derivative of (104) at 1:

$$\frac{\partial}{\partial z} \frac{h(z, z_1, \dots, z_k, 0, \dots, 0)}{h(1, z_1, \dots, z_k, 0, \dots, 0)}\Big|_{z=1} = \frac{f'_{n-k+1}(1)}{f_{n-k+1}(1)} + \frac{\partial}{\partial z} \frac{\det M_G(z, z_1, \dots, z_k)}{\det M_G(1, z_1, \dots, z_k)}\Big|_{z=1}. \tag{106}$$

Note that $f_{n-1}(z) = f_{n-k+1}(z)C(z)^{k-2}$. Thus,

$$\frac{f'_{n-1}(1)}{f_{n-1}(1)} = \frac{f'_{n-k+1}(1)}{f_{n-k+1}(1)} + (k-2)\frac{C'(1)}{C(1)}. \tag{107}$$

Combining equations (57), (106) and (107), we get that

$$\frac{\partial}{\partial z}\frac{\det M_G(z, z_1, \ldots, z_k)}{\det M_G(1, z_1, \ldots, z_k)}\bigg|_{z=1} = -\frac{C'(1)}{C(1)} + \sum_{l=1}^{k}\frac{C'(1)}{C(1)-C(z_l)}. \tag{108}$$

Third, we prove that functions $C(z_1)/G(z_1)$, $1, \ldots, C(z_1)^k$ are linearly dependent if $C'(1) \neq 0$, which we show later. Let $\mu_j(z)$ be the determinant of matrix $M_G(z, z_1, \ldots, z_k)$ without the second row and $(j+1)^{\text{st}}$ column. Note that $\mu_j(z)$ does not depend on $z_1$. Fix any $z_2, \ldots, z_k$ such that $\mu_0(1) \neq 0$, which is possible since function $C(z)$ is not constant. By multiplying both sides of (108) by $\det M_G(1, z_1, \ldots, z_k)$, we get

$$\begin{aligned}\frac{1}{G(z_1)}\mu'_0(1) + \sum_{l=0}^{k-1}\mu'_{l+1}(1)C(z_1)^l &= -\frac{1}{G(z_1)}\mu_0(1)\frac{C'(1)}{C(1)} + \\ &+ \frac{1}{G(z_1)}\mu_0(1)\sum_{l=1}^{k}\frac{C'(1)}{C(1)-C(z_l)} + \\ &+ \sum_{l=0}^{k-1}\mu_{l+1}(1)C(z_1)^l\left(-\frac{C'(1)}{C(1)} + \sum_{l=1}^{k}\frac{C'(1)}{C(1)-C(z_l)}\right).\end{aligned} \tag{109}$$

Note that $\mu_0(z) = V(C(z), C(z_2), \ldots, C(z_k))$. Hence,

$$\mu'_0(1) = \mu_0(1)\sum_{l=2}^{k}\frac{C'(1)}{C(1)-C(z_l)}. \tag{110}$$

Therefore, we can rewrite equation (109) as

$$\begin{aligned}&\frac{C(z_1)}{G(z_1)}\mu_0(1)\frac{C'(1)}{C(1)(C(1)-C(z_1))} = \\ &= \sum_{l=0}^{k-1}C(z_1)^l\left(\mu'_{l+1}(1) + \mu_{l+1}(1)\frac{C'(1)}{C(1)} - \mu_{l+1}(1)\sum_{l=1}^{k}\frac{C'(1)}{C(1)-C(z_l)}\right).\end{aligned} \tag{111}$$

Note that multiplying both sides by $C(1) - C(z_1)$ will give a linear dependency between functions $C(z_1)/G(z_1)$, $1, \ldots, C(z_1)^k$ with a non-zero coefficient for $C(z_1)/G(z_1)$ (if $C'(1) \neq 0$). To get (58), one needs to multiply both sides of equation (111) by $f_{n-k+1}(z_1)\, C(1)\, (C(1) - C(z_1))/(C(z_1)\, \mu_0(1)\, C'(1))$.

Finally, we prove that $C'(1) \neq 0$. Suppose $C'(1) = 0$. We will prove that this contradicts $(**)$. Note that $c = f'_{n-1}(1)/f_{n-1}(1)$ and, due to (57),

$$h'(1, z_1, \ldots, z_n) = c\, h(1, z_1, \ldots, z_n). \tag{112}$$

This means that matrix

$$\begin{pmatrix} f'_0(1) - c\, f_0(1) & \ldots & f'_n(1) - c\, f_n(1) \\ f_0(z_1) & \ldots & f_n(z_1) \\ \cdots & \cdots & \cdots \\ f_0(z_1) & \ldots & f_n(z_1) \end{pmatrix} \tag{113}$$

is singular for all $z_1, \ldots, z_n$. Since functions $f_0(z), \ldots, f_n(z)$ are linearly independent, we get that $f_i'(1) = cf_i(1)$ for $i = 0, \ldots, n$, which contradicts with $(**)$.

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