Well-posedness of infinite-dimensional linear systems with nonlinear feedback

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A B S T R A C T

We study existence of solutions, and in particular well-posedness, for a class of inhomogeneous, nonlinear partial differential equations (PDE's). The main idea is to use system theory to write the nonlinear PDE as a well-posed infinite-dimensional linear system interconnected with a static nonlinearity. By a simple example, it is shown that in general well-posedness of the closed-loop system is not guaranteed. We show that well-posedness of the closed-loop system is guaranteed for linear systems whose input to output map is coercive for small times interconnected to monotone nonlinearities. This work generalizes the results presented in [1], where only globally Lipschitz continuous nonlinearities were considered. Furthermore, it is shown that a general class of linear port-Hamiltonian systems satisfies the conditions asked on the open-loop system. The result is applied to show well-posedness of a system consisting of a vibrating string with nonlinear damping at the boundary.

1. Introduction

The notion of well-posedness for infinite-dimensional linear systems has been much studied in the last years, see e.g. [2,3]. More recently, existence of solution and in particular well-posedness of nonlinear partial differential equations (PDE's), has been addressed using system theory, see [4]. In the survey [1], conditions for the well-posedness of infinite-dimensional linear systems are provided in detail. In that work, also the case with static nonlinear feedback has been addressed for globally Lipschitz continuous nonlinearities.

The problem of well-posedness for only locally Lipschitz continuous nonlinearities has been considered in the discussion paper [4], where some issues related to this open problem were addressed.

The paper [5] provides conditions on a nonlinear boundary feedback interconnected with a linear port-Hamiltonian system to determine a nonlinear contraction semigroup. Even if those nonlinearities comprise some classes of locally Lipschitz continuous functions, well-posedness in the sense of Tucsnak and Weiss [1] is not addressed for the closed-loop system.

In this work, we introduce a more general class of closed-loop well-posed systems composed of a well-posed linear infinite-dimensional system whose input to output map is coercive for small times interconnected with static and monotone nonlinear feedback, which includes the class of locally Lipschitz continuous functions considered in [5].

This paper is organized as follows. In Section 2, the necessary background is presented and a motivating example which introduces the problem is provided. In particular, we recall the notion of well-posedness, both for linear and nonlinear systems. Section 3 is dedicated to the statement and the proof of the main result. In Section 4, it is shown that the assumptions required on the linear open-loop system are satisfied for an important class of port-Hamiltonian systems. The result is applied to show the well-posedness of a vibrating string with a nonlinear damper at the boundary. Section 6 contains conclusions and future work.

2. Background and problem statement

As said in the introduction we follow the idea of Tucsnak and Weiss [1]. That is, we consider an inhomogeneous, non-linear system as the interconnection of an inhomogeneous linear system with a static nonlinearity as depicted in Fig. 1. Furthermore, it is assumed that the linear part, denoted by $\Sigma^0$ is well-posed, of which we recall the definition first.
Consider the linear system $\Sigma^P$, with input space $U$, state space $X$, and output space $Y$ (all real Hilbert spaces), described by the equations

$$\Sigma^P: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ x(0) = x_0 \end{cases} \quad (1)$$

where $A, B, C$ and $D$ are in general unbounded operators.

**Definition 2.1.** The system $\Sigma^P$ is said to be well-posed if for every $u \in L^2_{\text{loc}}([0, \infty); U)$ (input) and for every $x_0 \in X$ (initial state), the abstract differential equation (1) possesses a unique (mild) solution $x \in C([0, \infty); X)$ (state trajectory) and $y \in L^1_{\text{loc}}([0, \infty); Y)$ (output function). Hence, if $\Sigma^P$ is well-posed, then the solution of (1) can be written using four families of bounded linear operators as follows:\footnote{For a positive $t$, $P_t$ denotes the operator of truncation to the interval $[0, t]$ of a function defined on a larger set than $[0, t]$, see [1].}

$$x(t) = \mathcal{T}_t x_0 + \Phi_t u, \quad P_t y = \Psi_t x_0 + \Upsilon_t u, \quad \Phi_t, \Psi_t \in B(X; Y), \quad \Upsilon_t \subseteq B(U; Y)$$

for all $t \in [0, \infty)$. Moreover, on any bounded time interval $[0, \tau]$, $0 < \tau < \infty$, $(x(t)$) and $P_t y$ depend continuously on $x_0$ and on $P_t$. By definition, the system $\Sigma^P$ is said to be well-posed if for any input $v \in L^2_{\text{loc}}([0, \infty); U)$ and any $x_0 \in X$ (initial state) there exist $t_f \in (0, \infty]$ and unique functions $x \in C^1([0, t_f); X)$ (state trajectory) and $y \in L^2_{\text{loc}}([0, t_f); Y)$ (output function) such that

$$x(t) = \mathcal{T}_t x_0 + \Phi_t v - \Phi_t f(y), \quad P_t y = \Psi_t x_0 + \Upsilon_t v - \Upsilon_t f(y), \quad \text{for all } t < t_f, \quad \text{and moreover, on any bounded time interval } [0, \tau], \quad \text{for all } t < t_f, \quad x(\tau) \text{ and } P_t y \text{ depend continuously on } x_0 \text{ and on } P_t.$$
Fig. 2. Nonlinear feedback interconnection of a pure shift and a static nonlinearity.

For $0 < t < 1$ we have from (11)

$$
\int_0^1 x(\zeta, t)^2 \, d\zeta \geq \int_{1-t}^1 x(\zeta, t)^2 \, d\zeta

= \int_{1-t}^1 \frac{1}{\sqrt{(\zeta + t - 1)^2}} \, d\zeta = \infty.
$$

Hence there does not exist any $t > 0$ such that state lies in the state space, and so the system merged by this simple interconnection is not well-posed.

The above example implies that if we want/have to consider connection as in Fig. 1 with $f$ (only) locally Lipschitz, then we have to impose extra condition on $\Sigma^T$ and $f$. In the following we assume $U$ and $Y$ to be the same real Hilbert space, i.e., $U = Y$. On the system we impose the following, where $F_t$ was introduced in Definition 2.1.

**Assumption 2.1.** There exists $t^* > 0$ such that for all $t < t^*$, the operator $F_t$ is coercive, i.e., there exists $\tilde{c} > 0$ such that for all $u \in L^2([0, t^*]; U)$, it holds

$$
\langle F_t u, u \rangle \geq \tilde{c} \|u\|^2, \text{ for all } t < t^*.
$$

This condition can be interpreted as being strict input passive on small time intervals and for finite-dimensional systems it is satisfied if and only if $D + D^T > 0$.

For the nonlinear function $f(\cdot)$ we assume the following.

**Assumption 2.2.** The nonlinearity satisfies the following properties:

- $f$ is continuous
- $\forall y_1, y_2, (f(y_1) - f(y_2), y_1 - y_2)_U \geq 0$,
- $f(0) = 0$.

**Remark 2.1.** The class of considered nonlinear functions $f(\cdot)$ comprises strictly increasing, positive and unbounded locally Lipschitz continuous (scalar) functions like odd polynomials (e.g. $f(y) = y^3$).

We end this section with a result on $m$-dissipativity.

**Definition 2.3.** The (nonlinear) operator $J$ on domain $D(J) \subset X$ is called $m$-dissipative if

- $J$ is dissipative, i.e., $(kx - Jx, x - \bar{x})_X \leq 0$ for $x, \bar{x} \in D(J)$,
- For all $\lambda > 0$, the operator $J$ satisfies the range condition
  $$
  X = \{y \in X | \exists x \in D(J), y = (\lambda I - J)(x)\}
  = \text{Ran}(\lambda I - J).
  $$

Notice that since the operator $J$ is dissipative, the solution $x$ of the equation $(\lambda I - J)(x) = y$ for a given $y \in X$ and a given $\lambda > 0$ is unique. In fact, suppose there are two solutions, $x_1$ and $x_2$, respectively. We have $y = \lambda x_1 - J(x_1), y = \lambda x_2 - J(x_2)$ so that

$$
\lambda \|x_1 - x_2\|^2 = \lambda \langle x_1 - x_2, x_1 - x_2 \rangle

= \langle (x_1) - J(x_1), x_1 - x_2 \rangle \leq 0,
$$

which is possible if and only if $x_1 = x_2$.

**Lemma 2.1.** Let $f : Y \mapsto Y$ be a function satisfying the conditions in Assumption 2.2, then for every $\lambda > 0$ the range of $\lambda I + f$ equals $Y$, and thus $-f$ is $m$-dissipative. Furthermore,

$$
\|\langle \lambda I + f(y) \rangle\| \geq \lambda \|y\|.
$$

**Proof.** Since the domain of $-f$ equals the whole space $Y$, it is maximally dissipative, i.e., it does not have a proper (dissipative) extension. Since $Y$ is a Hilbert space this gives that $-f$ is $m$-dissipative see [7, Section 2.3]. For the norm inequality (12) we use the inequality in Assumption 2.2 with $y_1 = y$ and $y_2 = 0$,

$$
\|\langle \lambda I + f(y) \rangle\|^2 = (\langle \lambda I + f(y) \rangle, \langle \lambda I + f(y) \rangle)

= \lambda^2 \|y\|^2 + \lambda \langle f(y), y \rangle + \|f(y)\|^2 \geq \lambda^2 \|y\|^2.
$$

Taking the square root on both sides ends the proof.

3. Main result

First we state and prove some lemmas. For any continuous $f : U \mapsto Y$ and any $t^* > 0$, we define the operator $A_t$ by

$$(A_t)(y)(\cdot) = f(y(\cdot)) \text{ for } y \text{ in } D(A_t) = \{y \in L^2([0, t^*]; Y) | f(y(\cdot)) \in L^2([0, t^*]; U)\}.
$$

Since $D(A_t) = D(-A_t)$, the domain $D(A_t)$ will be used in the following.

**Lemma 3.1.** Under Assumption 2.2 the operator $-A_t$ on the domain $D(A_t)$ is $m$-dissipative.

**Proof.** Let first prove that $-A_t$ is dissipative. Taking $x, \bar{x} \in D(A_t)$, we have

$$
(-A_t(x) - (-A_t(\bar{x})), x - \bar{x})_X = -(A_t(x) - A_t(\bar{x}), x - \bar{x})_X \leq 0
$$

since by assumption the last inequality holds pointwise.

It remains to prove that $\text{Ran}(\lambda I - (-A_t)) = L^2([0, t^*]; U)$ for all $\lambda > 0$. So given $\lambda > 0$, we have to show that for all $u \in L^2([0, t^*]; U)$, there exists $y \in D(A_t)$ such that $u = (\lambda I - (A_t))y$.

For $u \in L^2([0, t^*]; U)$, we define

$$
y(t) = (\lambda I + f)^{-1}(u(t)), \quad t \in [0, t^*]
$$

By Lemma 2.1 this inverse exists. Furthermore, using (12) we obtain that $y \in L^2([0, t^*]; U)$. Now since

$$
f(y(t)) = (\lambda I + f)(y(t)) - \lambda y(t) = u(t) - \lambda y(t)
$$

we find that $f(y(\cdot)) \in L^2([0, t^*]; U)$. Concluding, $-A_t$ is $m$-dissipative.

**Lemma 3.2.** Under Assumptions 2.1 and 2.2 the operator $\epsilon I - F_t^{-1} - A_t$ on the domain $D(\epsilon I - F_t^{-1} - A_t) = D(A_t)$ is dissipative for sufficiently small $\epsilon > 0$. 

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Note that since $F_t$ is coercive, it is boundedly invertible, see e.g. [6, Example A.4.2].
Proof. By Assumption 2.1 it follows that $\Sigma_i^{−1}$ exists and since $\Sigma_i$ is coercive, $\Sigma_i^{−1}$ is also coercive, i.e., there exists $c > 0$ such that for all $y \in L^2([0, t^*])$, it holds (33) below.

Let us now consider $y, \tilde{y} \in \mathcal{D}(\epsilon_1 - \Sigma_i^{−1} - A_i$). It yields
\[
\begin{align*}
\langle (\epsilon_1 - \Sigma_i^{−1} - A_i) y \rangle - \langle (\epsilon_1 - \Sigma_i^{−1} - A_i)\tilde{y} \rangle, y - \tilde{y} \rangle

\end{align*}
\]
\[
= - \langle A_i y - A_i\tilde{y}, y - \tilde{y} \rangle - \langle \Sigma_i^{−1}, \epsilon \cdot y \rangle, y - \tilde{y} \rangle

\end{align*}
\]
\[
\leq (c + \epsilon) \|y - \tilde{y}\|^2.
\]

With the help of above lemmas we show that the nonlinear system $\Sigma'$ is well-posed on $[0, \infty)$. We begin by showing that this holds in $[0, t^*]$. Here $t^*$ is the constant introduced in Assumption 2.1.

Lemma 3.3. Under Assumptions 2.1 and 2.2, the state trajectory of $\Sigma'$, $x(t)$, and the output of $\Sigma'$, $y$, exist in $X$ and in $L^2([0, t^*]; Y)$ respectively for $t < t^*$. For every $x_0 \in X$ and every $v \in L^2([0, t^*]; U)$ there exists a unique solution of (5) and (6). Moreover, $f(y) \in L^2([0, t^*]; U)$.

Proof. For the $t^*$ of Assumption 2.1, we start by proving the existence of $y \in L^2([0, t^*]; Y)$, the output of the closed-loop system. Consider the operator $-A_i$ on the domain $\mathcal{D}(A_i)$ defined by (13). It is easy to see that $\mathcal{D}(A_i)$ is dense in $L^2([0, t^*]; Y)$, i.e., $\overline{\mathcal{D}(A_i)} = L^2([0, t^*]; Y)$. Let us also define operator $\epsilon_1 - \Sigma_i^{−1}$ on the domain $\mathcal{D}(\epsilon_1 - \Sigma_i^{−1}) = L^2([0, t^*]; Y)$. Notice that $\mathcal{D}(\epsilon_1 - \Sigma_i^{−1}) = \mathcal{D}(\epsilon_1 - \Sigma_i^{−1})$ and that $\epsilon_1 - \Sigma_i^{−1}$ is a continuous operator on $\mathcal{D}(A_i)$.

By Lemma 3.1, operator $-A_i$ is $m$-disjoint. Moreover, $\epsilon_1 - \Sigma_i^{−1} - A_i$ is disjoint by Lemma 3.2. Hence, $\epsilon_1 - \Sigma_i^{−1} - A_i$ is $m$-disjoint by Miyadera [7, Corollary 6.19]. It means that for all $\lambda > 0$,

$\text{Ran} (\lambda I - \epsilon_1 + \Sigma_i^{−1}) = L^2([0, t^*]; Y)$.

Taking $\lambda = \epsilon$, it shows that the equation
\[
(\lambda I + \Sigma_i^{−1}) y = \omega
\]
has a unique solution $y \in \mathcal{D}(\lambda I)$ for all $\omega \in L^2([0, t^*]; U)$. Choosing $\omega = \Sigma_i^{−1} \Psi_s x_0 + v \in L^2([0, t^*]; U)$, we find
\[
\phi_T y = \Psi_s x_0 + \nabla_T v - \nabla_T A_i(y),
\]
which is equivalent to
\[
y = \Psi_s x_0 + \nabla_T v - \nabla_T (\epsilon_1 - \Sigma_i^{−1} - A_i) y.
\]

Hence, the output equation (16) has a unique solution $y \in L^2([0, t^*]; Y)$ for which $A_i y \in L^2([0, t^*]; U)$, such that $y$, denoted by $x$, is obtained by injecting (16) in (5). Using (3), we have from (16) that
\[
P_i y = \Psi_i x_0 + P_i \nabla_T v - P_i \nabla_T A_i(y).
\]

For ease of reading, we will now (often) replace $A_i$ by $f(y)$. Moreover, there exist positive constants $\gamma_1, \ldots, \gamma_4$ such that for all $t \leq t^*$ the following inequalities hold
\[
\|f(x) - \tilde{f}(t)\| \leq \gamma_1 \|x_0 - \tilde{x}_0\| + \gamma_2 \|P_i v - P_i \tilde{v}\|.
\]
\[
\|P_i y - P_i \tilde{y}\| \leq \gamma_3 \|x_0 - \tilde{x}_0\| + \gamma_4 \|P_i v - P_i \tilde{v}\|.
\]

Proof. Consider two initial conditions $x_0$ and $\tilde{x}_0$ in $X$, two external inputs $v$ and $\tilde{v}$ in $L^2([0, t^*]; U)$ and $t < t^*$. The two corresponding state trajectories are given by
\[
x(t) = x_0 + f_x v - \Phi f(y)
\]
and the corresponding outputs are given by
\[
P_i y = \Psi_i x_0 + P_i \nabla v - P_i \nabla f(y).
\]

We start by proving the continuous dependence for the output. From (19), it holds
\[
\|P_i y - P_i \tilde{y}\| \leq \epsilon_1 \|y_0 - \tilde{y}_0\| + \|P_i v - P_i \tilde{v}\|
\]
where the causality of $\Sigma_i$ has been used, i.e., $f(x) = F_i f(y)$. Using the coercivity of $\epsilon_1$ and the inequality of $f$ (or $A_i$), we find
\[
\|f(x) - f(\tilde{x})\| \leq \gamma_1 \|x_0 - \tilde{x}_0\| + \gamma_2 \|P_i v - P_i \tilde{v}\|
\]
for some $\epsilon > 0$. Moreover, by the Cauchy–Schwarz inequality, we find
\[
\|f(x) - f(\tilde{x})\| \leq \gamma_1 \|x_0 - \tilde{x}_0\| + \gamma_2 \|P_i v - P_i \tilde{v}\|
\]
Combining (20), (21), and (22) yields
\[
\|f(x) - f(\tilde{x})\| \leq \gamma_1 \|x_0 - \tilde{x}_0\| + \gamma_2 \|P_i v - P_i \tilde{v}\|
\]
which is the second inequality of (17). Moreover, from (19)
\[
\|f(x) - f(\tilde{x})\| \leq \gamma_1 \|x_0 - \tilde{x}_0\| + \gamma_2 \|P_i v - P_i \tilde{v}\|
\]
Using (23), it holds
\[
\|f(x) - f(\tilde{x})\| \leq \gamma_1 \|x_0 - \tilde{x}_0\| + \gamma_2 \|P_i v - P_i \tilde{v}\|
\]

Putting (18) and (24) together yields

\[
\begin{align*}
\|x(t) - \tilde{x}(t)\| & \leq \|x_0 - \tilde{x}_0\| + \|\Phi_t \cdot (P_t v - P_t \tilde{v})\| + \\
&\leq \|x_0 - \tilde{x}_0\| + \|\Phi_t \cdot (P_t v - P_t \tilde{v})\| + \\
&\leq \|x_0 - \tilde{x}_0\| + \|\Phi_t \cdot (P_t v - P_t \tilde{v})\| + \\
&\leq \|x_0 - \tilde{x}_0\| + \|\Phi_t \cdot (P_t v - P_t \tilde{v})\| + \\
&\leq \|x_0 - \tilde{x}_0\| + \|\Phi_t \cdot (P_t v - P_t \tilde{v})\|
\end{align*}
\]

which is the first inequality of (17). ■

We are ready now to prove the well-posedness of \(\Sigma^t\).

**Theorem 3.1.** Under Assumptions 2.1 and 2.2, the system \(\Sigma^t\) is well-posed in the sense of Definition 2.2 with \(t_f = \infty\). Furthermore, inequalities, like (17) with \(\gamma^t\) depending on \(t\), hold for all \(t > 0\).

**Proof.** We prove this by induction. That is, we show that the system is well-posed on the interval \([0, k t^*]\), with \(k \in \mathbb{N}\) and that inequalities, like (17) with \(\gamma^t\) depending on \(k\), hold in Lemmas 3.3 and 3.4 we showed that this holds for \(k = 1\). Assuming now that \(k = K + 1\), we show the correctness for \(k = K + 1\). Let \(x_0 \in X\) and \(v \in L^2([0, (K + 1)t^*]; U)\) be given. For \(t \in (0, t^*)^t\) the assertion holds by the induction hypothesis, so we assume that \(t \in (K^*, (K + 1)t^*)\). We show first that we have a solution, and next we show the continuous dependence on the initial condition and external input.

By the induction hypothesis, the state and the output exist until \(K^*\), i.e.,

\[
x(t) = \begin{array}{c}
\left(T_t x_0 + \Phi_t v - \Phi f(y) + \\
P_t y \Phi_t x_0 + P_t f(y)
\end{array}
\]

for \(t \in (0, K^*)\). For \(v_{K^*} \in \mathbb{R}^{n(K + 1)t^*} \in L^2([0, (K + 1)t^*]; U)\) and \(t \in (0, t^*)^t\), we define

\[
x_{K^*}(t) = \begin{array}{c}
\left(T_t x_0 + \Phi_t v - \Phi f(y) + \\
P_t y \Phi_t x_0 + P_t f(y)
\end{array}
\]

Thus \(x_{K^*}\) and \(y_{K^*}\) are the state trajectory and the output generated by the initial condition \(x(K^*)\) and the external input \(v_{K^*}\) in \(\Sigma^t\). Again by the induction hypothesis this exists.

We extend the solutions \(x, y\) to the time interval \([K^*, (K + 1)t^]*) by defining

\[
x(t + K^*) = x_{K^*}(t)
\]

\[
P(t) = P_{K^*} y, y_{K^*}
\]

for \(t \in [0, t^*]\). Developing (26) for \(t \in [0, t^*]\), we find

\[
x(t + K^*) = \begin{array}{c}
\left(T_t x_0 + \Phi_t v - \Phi f(y) + \\
P_t y \Phi_t x_0 + P_t f(y)
\end{array}
\]

which describes the homogeneous port-Hamiltonian system, i.e., (28). Further, we assume that \(y_{K^*}(t)\) is a contraction semigroup on the state space \(X\). Here \(X\) is \(L^2([a, b]; \mathbb{R}^n)\) equipped with the inner product

\[
(f, g)_X = \int_a^b f(\xi)g(\xi)d\xi.
\]
Furthermore, it follows by [8, Theorem 11.3.2] that (28)–(29) is a boundary control system in the sense of [8, Definition 11.1.1].

The energy associated to (28) is given by $E(t) = \frac{1}{2} [x(t)]^2$. Along classical solutions of (28), an expression of the time derivative of the energy is provided in [8, Theorem 7.1.5] and is given by

$$\frac{dE}{dt}(t) = \frac{1}{2} \left( [H(\xi) x(\xi, t)]^T P_1 H(\xi) x(\xi, t) \right)^b.$$  (31)

We suppose that (28)–(29) is impedance passive, i.e., that $\frac{dE}{dt}(t) \leq u^T(t) y(t)$ holds along classical solutions.

**Lemma 4.1.** Let us consider the impedance passive boundary control system (28)–(29). Assume that $P_1 H(\xi)$ is diagonalizable, i.e., there exist $\Delta(\xi)$, a diagonal matrix-valued function and $S(\xi)$, a matrix-valued function, both continuously differentiable on $[a, b]$ such that

$$P_1 H(\xi) = S^{-1}(\xi) \Delta(\xi) S(\xi), \quad \xi \in [a, b].$$  (32)

Furthermore, assume that

$$\text{rank} \begin{bmatrix} W_{y_1} & W_{y_2} \end{bmatrix} = n = \text{rank}(W_{\xi}).$$  (33)

Then, the system (28)–(29) is regular, well-posed and satisfies $\lim_{s \to \infty} G(s) = \lim_{s \to \infty} \|G(s)\| =: D$, where $G(s)$ is the transfer function of (28)–(29). Moreover, the feed-through term $D$ is coercive.

**Proof.** The regularity and the well-posedness are provided by [8, Theorem 13.2.2].

By [8, Lemma 13.2.5], the diagonal matrix $\Delta(\xi)$ has the form

$$\Delta(\xi) = \begin{bmatrix} A(\xi) & 0 \\ 0 & \Theta(\xi) \end{bmatrix},$$  (34)

where $A(\xi)$ is a diagonal real matrix-valued function with strictly positive functions on the diagonal and $\Theta(\xi)$ is a diagonal real matrix-valued function with strictly negative functions on the diagonal.

We consider the state transformation

$$z(\xi, t) = \begin{bmatrix} z_1(\xi, t) \\ \tilde{z}_2(\xi, t) \end{bmatrix} := S(\xi) x(\xi, t).$$  (35)

In this way, the PDE (28) becomes

$$\frac{dz}{dt}(\xi, t) = \frac{\partial}{\partial \xi} (\Delta z(\xi, t)) + S(\xi) \frac{dS^{-1}(\xi)}{dt} \Delta z(\xi, t) + \tilde{S}(\xi) \tilde{P}_0 S^{-1}(\xi) \tilde{z}(\xi, t),$$  (36)

and (29) becomes

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = K u(t) + Q y(t), \quad y(t) = O_1 u_1(t) + O_2 y_2(t),$$  (37)

where

$$u(t) = \begin{bmatrix} \frac{d}{dt} z_1(\xi, t) \\ \Theta(\xi) \frac{d}{dt} \tilde{z}_2(\xi, t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} \Theta(\xi) \frac{d}{dt} z_1(\xi, t) \\ \Theta(\xi) \frac{d}{dt} \tilde{z}_2(\xi, t) \end{bmatrix}.$$  (38)

$K$ and $Q$ are two square $n \times n$ matrices with $\Theta(\xi)$ of rank $n$ and $O_1$ and $O_2$ are $k \times n$ matrices, see [8, Section 13.4]. By [8, Lemma 13.1.14], the limit of the transfer function of (36)–(37) for $\Re(s) \to \infty$ is equal to the same limit of the transfer function of

$$\frac{dz}{dt}(\xi, t) = \frac{\partial}{\partial \xi} (\Delta z(\xi, t)),$$  (39)

with the boundary input and output (37). If we write $O_1 K^{-1}$ as $\begin{bmatrix} * & D \end{bmatrix}$, with $D k \times k$, then by [8, Theorem 13.3.1], this $D$ is the feedthrough operator of our system, i.e., $\lim_{s \to \infty} G(s) = \lim_{s \to \infty} S(\xi) \tilde{P}_0 S^{-1}(\xi) \tilde{z}(\xi, t) = D$. Since the system is impedance passive, the transfer function is positive real, see e.g., [8, Example 12.2.3]. Hence $D$ satisfies $D + D^T \geq 0$. To prove that this inequality is strict, we begin by showing that $D$ is invertible. Suppose by contradiction that $D$ is not invertible. By the relation with $O_1 K^{-1}$ this implies that there exists a non-zero $u \in \mathbb{R}^k$ such that $O_1 K^{-1} u = 0$. Let us define

$$\begin{bmatrix} A(\xi) z_1(b) \\ \Theta(\xi) z_2(b) \end{bmatrix} := K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad \begin{bmatrix} z_1(b) \\ z_2(b) \end{bmatrix} := \begin{bmatrix} 0 \\ u \end{bmatrix}.$$  (40)

In this way,

$$y = O_1 u_1(t) + O_2 u_2(t) = O_1 K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix} + 0 = 0.$$  (41)

It can be shown that the energy balance combined with the impedance passivity gives

$$\frac{1}{2} [z^T(b) H_5(b) A(b) z(b) - z^T(a) H_5(a) A(a) z(a)] \leq u^T y,$$  (42)

where $H_5(\xi) = S^{-T}(\xi) H(\xi) S^{-1}(\xi)$. Using (40) and (41), (42) gives

$$z_1^T(b) H_5(b) A(b) z_1(b) - z_1^T(a) H_5(a) A(a) z_1(a) \leq 0,$$  (43)

where the decomposition $H_5(\xi) = \begin{bmatrix} H_{11}(\xi) & H_{12}(\xi) \\ H_{21}(\xi) & H_{22}(\xi) \end{bmatrix}$ has been used.

Since $H_{11}(\xi)$ and $H_{22}(\xi)$ are two principal matrices of $H_5(\xi)$, since $H_{11}(\xi) A(\xi) = A(\xi) H_{11}(\xi)$, $H_{22}(\xi) \Theta(\xi) = \Theta(\xi) H_{22}(\xi)$, and since $H_{5}(\xi)$ is positive definite, the relations

$$H_{11}(\xi) A(\xi) > 0, \quad H_{22}(\xi) \Theta(\xi) < 0$$  (44)

hold. Combining this with (43) implies that $z_1(b) = 0$ and $z_1(a) = 0$. Using (40), this yields $K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix} = 0$. This means that $u$ is identically zero, which is a contradiction. Hence, $D$ is invertible. So we have shown the invertibility of the feedthrough matrix for any impedance passive port-Hamiltonian system for which the Hamiltonian can be written as (32). If we put all inputs except the first one equal to zero and we only consider the first output, then this scalar input–output system is impedance passive. The above result implies that its feedthrough is invertible. It is easy to see that this (new) feedthrough equals $D_{11}$. We can repeat this argument for all components of the input vector, and so we find that $D_{1i} \neq 0$ for $i = 1, \ldots, k$. Since the system is impedance passive, we even know that $D_{1i} > 0$ for $i = 1, \ldots, k$.

Let $u$ be a unitary $k \times k$ matrix, and define $\tilde{u} = u t u$ and $\tilde{y} = i t y$. Then $\tilde{u}^T \tilde{y} = u^T y$, and so the port-Hamiltonian system with the new input $\tilde{u}$ and new output $\tilde{y}$ is still impedance passive. The feedthrough matrix $\tilde{D}$ of this system is related to $D$ via $\tilde{D} = u D t u^T$.

Since the port-Hamiltonian system with input $u$ and output $y$ still satisfies all the assumptions, we have that the diagonal elements of $\tilde{D}$ are strictly positive.

Assume now that $D$ is not coercive. Thus there exists a non-zero $w_0 \in \mathbb{R}^k$ such that $u_0^T D w_0 = 0$. Without loss of generality, we may assume that $w_0$ has norm one. Let $u$ be a unitary matrix which maps this vector onto $[1, 0, \ldots, 0]^T$. Then

$$0 = u_0^T D w_0 = [1 \ 0 \ldots \ 0] D t u^T \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} = \tilde{D}_{11}.$$  (45)

This is a contradiction, and so $D$ is coercive. \[\square\]

The following theorem characterizes closed-loop systems resulting from the interconnection of a linear system comprises in the class of linear port-Hamiltonian systems introduced in this section with a static nonlinearity that satisfies Assumption 2.2.
Theorem 4.1. Consider the first order port-Hamiltonian system described by (28)–(29) that satisfies the assumptions of Lemma 4.1. Furthermore, consider the interconnection \( u(t) = -f(y(t)) \) where \( f(\cdot) \) is a nonlinear function that satisfies Assumption 2.2. Then, the resulting nonlinear system is well-posed.

Proof. By Lemma 4.1, the linear system is well-posed, regular and the corresponding feedthrough operator, \( D \), is coercive. Moreover, since
\[
\lim_{\Re(s) \to -\infty} G(s) = D, \tag{45}
\]
there exists a sufficiently large \( \alpha \in \mathbb{R} \) such that \( G(s) \) is boundedly invertible on \( C_{\alpha} := \{ s \in \mathbb{C} | \Re(s) > \alpha \} \). By the Paley–Wiener Theorem, see e.g. [8, Theorem A.2.9], there exists a sufficiently small \( t^* > 0 \) such that the operator \( F_\gamma \) is boundedly invertible for all \( t < t^* \). Moreover, since the port-Hamiltonian system (28)–(29) is impedance passive, the operator \( F_\gamma \) is positive, i.e., along any solution on \([0,t^*)\) it holds \( \langle F_\gamma u,u \rangle \geq 0 \). This fact together with the invertibility, implies coercivity of the operator \( F_\gamma \) for \( t < t^* \), i.e. Assumption 2.1 is satisfied.

Since the considered nonlinearity satisfies Assumption 2.2, Theorem 3.1 provides the well-posedness of the closed-loop system. ■

5. Example: The vibrating string with a nonlinear damper at the boundary

In this section, Theorem 4.1 is illustrated with a vibrating string with a nonlinear damper attached to it. This system can be described by means of the following PDE
\[
\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left( T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right), \quad w(\zeta, 0) = w_0(\zeta),
\]
\[
- f \left( \frac{\partial w}{\partial t}(1, t) \right) = T(1) \frac{\partial w}{\partial \zeta}(1, t), \quad \frac{\partial w}{\partial \zeta}(0, t) = 0, \tag{46}
\]
where \( \zeta \in [0,1] \) is the spatial variable, \( w(\zeta, t) \) is the vertical position of the string at position \( \zeta \) and at time \( t \), \( T(\zeta) \) and \( \rho(\zeta) \) represent the Young's modulus and the mass density respectively and are supposed to be positive, continuously differentiable functions. Eq. (46) can be seen as the linear PDE
\[
\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left( T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right), \tag{47}
\]
with boundary input and output
\[
u(t) = T(1) \frac{\partial w}{\partial \zeta}(1, t), \quad y(t) = \frac{\partial w}{\partial \zeta}(1, t) \tag{48}
\]
connected by the nonlinear feedback \( u(t) = -f(y(t)) \). Defining the state variables \( x_1(\zeta, t) = \rho \frac{\partial w}{\partial \zeta} \) (the momentum) and \( x_2(\zeta, t) = \frac{\partial w}{\partial \zeta} \) (the strain), the linear PDE (47) admits a port-Hamiltonian representation in the form
\[
\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} \left( H(\zeta) x(\zeta, t) \right), \tag{49}
\]
where \( x(\zeta, t) = \begin{bmatrix} x_1(\zeta, t) & x_2(\zeta, t) \end{bmatrix}^T \), \( P_1 = \begin{bmatrix} 0 & 1 \\ \frac{T(\zeta)}{\rho(\zeta)} & 0 \end{bmatrix} \) and \( H(\zeta) = \begin{bmatrix} 0 & -1 \\ 1 & \frac{T(\zeta)}{\rho(\zeta)} \end{bmatrix} \). This PDE falls in the well-established class of linear port-Hamiltonian systems on 1-D spatial domain, whose properties are considered in the previous section. For this system, \( P_1 H(\zeta) \) can be expressed as \( P_1 H(\zeta) = S^{-1}(\zeta) \Delta(\zeta) S(\zeta) \) where
\[
S(\zeta) = \begin{bmatrix} \frac{\partial w}{\partial \zeta} \end{bmatrix}, \quad \Delta(\zeta) = \begin{bmatrix} A(\zeta) & 0 \\ \rho(\zeta) & 0 \end{bmatrix}, \tag{50}
\]
with \( A(\zeta) = \Theta(\zeta) = \gamma(\zeta) = \sqrt{\frac{T(\zeta)}{\rho(\zeta)}} \). Then Theorem 4.1 establishes well-posedness of (46) for any function \( f(\cdot) \) that satisfies Assumption 2.2, e.g. \( f(y) = y^2 \) or any odd polynomial representing nonlinear damping at the end of the string.

6. Conclusion and future work

In this paper, well-posedness of a class of infinite-dimensional linear systems interconnected with a static nonlinearity has been proven. The problem has been introduced with a simple (counter) example. As main result, sufficient conditions on the linear system to end up with a well-posed closed-loop system are provided, extending the class of admissible nonlinearities presented in [1]. Moreover, it is shown that impedance passive port-Hamiltonian systems satisfy the necessary conditions of the well-posed linear system. Finally, the result has been applied on a vibrating string with a nonlinear damper at the boundary.

Future work aims at extending the class of nonlinearities for which a closed-loop system is well-posed to dynamical systems.

Conflict of interest statement

None.

Declaration of conflicting interests

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