Lemke's Method—a Recursive Approach*

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ABSTRACT

Let $\mathbf{M}$ be an nth-order matrix and $\mathbf{q}$ and $\mathbf{d}$ be vectors, both of order n, and let ϑ be a scalar variable. Then the linear complementarity problem LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$) is defined to be: Determine a vector $\mathbf{x}\geqslant\mathbf{0}$ such that $\mathbf{w}=\mathbf{M}\mathbf{x}+\mathbf{q}+\vartheta\mathbf{d}\geqslant\mathbf{0}$ and $\mathbf{x}^T\mathbf{w}=0$. It is shown that if $\mathbf{M}$ and $\mathbf{d}$ satisfy certain conditions, then every solution of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$), for all values of ϑ, lies on a graph where the solutions corresponding to nodes are degenerate and those corresponding to arcs are nondegenerate. Moreover, nodes are associated with particular values, while arcs are associated with open intervals of ϑ. Arcs emanating from a given node are shown to be determined by the solutions of two smaller LCPs involving Schur complements of $\mathbf{M}$, and this enables inductive proofs to be constructed for some of the standard results, establishing relationships between the number of solutions for two different values of ϑ and, in the case where $\mathbf{M}$ is a P-matrix, uniqueness of the solutions.

1. INTRODUCTION

Let $\mathbf{M}$ be a square matrix and $\mathbf{q}$ a vector, both of order n. Then the *linear complementarity problem* LCP($\mathbf{q},\mathbf{M}$) is defined as:

Determine a vector $\mathbf{x}\geqslant\mathbf{0}$ such that

$$\mathbf{M}\mathbf{x}+\mathbf{q}=\mathbf{w}, \tag{1}$$

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where $\mathbf{w} \geqslant \mathbf{0}$ and

$$\mathbf{x}^T\mathbf{w} = 0. \tag{2}$$

The requirements that $\mathbf{x} \geqslant \mathbf{0}$, $\mathbf{w} \geqslant \mathbf{0}$, and $\mathbf{x}^T\mathbf{w} = 0$ imply that if $\mathbf{x} = [x_i]$ and $\mathbf{w} = [w_i]$, then $w_i > 0$ only if $x_i = 0$, and $x_i > 0$ only if $w_i = 0$—the eponymous complementarity of the problem. Depending on the properties of $\mathbf{M}$, it can be shown that either no, one, or many solutions of LCP($\mathbf{q}, \mathbf{M}$) exist. One algorithm, due to Lemke [5, 6], has been the principal means of solving the problem in practice.

Lemke's algorithm is based on a homotopy and solves LCP($\mathbf{q} + \vartheta\mathbf{d}, \mathbf{M}$) where ϑ is a scalar parameter and $\mathbf{d}$ is a positive vector, normally the vector of ones. It works as follows. Let $\mathbf{x}$ be a solution of LCP($\mathbf{q} + \vartheta\mathbf{d}, \mathbf{M}$) for a particular value of ϑ, and assume that m elements of $\mathbf{x}$ and $n - m$ elements of $\mathbf{w}$ are positive. If, then, in order to preserve complementarity, the n zero elements of $\mathbf{x}$ and $\mathbf{w}$ are kept at that value, Equation (1) consists effectively of n equations in $n + 1$ unknowns (the n positive elements of $\mathbf{x}$ and $\mathbf{w}$ together with ϑ). Provided that the appropriate conditions of linear independence are satisfied, n of these $n + 1$ variables (the basic variables) may then be regarded as functions of the remaining one, which may then be varied at will while the n basic variables are changed in such a way that Equation (1) remains satisfied. Initially ϑ is selected to be the independent variable and is given a value large enough for $\mathbf{q} + \vartheta\mathbf{d} > \mathbf{0}$ so that $\mathbf{x} = \mathbf{0}$, $\mathbf{w} = \mathbf{q} + \vartheta\mathbf{d}$ is a solution of LCP($\mathbf{q} + \vartheta\mathbf{d}, \mathbf{M}$). The variable ϑ is then reduced until some positive element of $\mathbf{w}$, w_j say, becomes zero. w_j is then kept at this value while ϑ assumes basic variable status (which it retains thereafter), and the "entering variable" x_j, the variable complementary to w_j, is taken to be the new independent variable, thereby ensuring that Equation (2) remains satisfied when the new independent variable is changed. This variable is then increased until another positive element of $\mathbf{w}$, w_k say, becomes zero, when it is in turn held at that value while the previous entering variable x_j becomes a basic variable and x_k becomes the new entering (independent) variable. Should a positive element x_k of $\mathbf{x}$ become zero instead of w_k, then the roles of x_k and w_k are reversed. This process is repeated either until ϑ becomes zero, when the current solution of LCP($\mathbf{q} + \vartheta\mathbf{d}, \mathbf{M}$) will be the required solution, or until any increase of the entering variable causes no decrease in any of the basic variables. In this case, no new entering variable can be identified and the algorithm is aborted (ray termination).

Since the role of just one basic variable is changed at each stage, the mechanics can be readily handled by the exchange algorithm, and the computational details of the method closely resemble those of the simplex method. The algorithm is effective and, as described above, unambiguous.

Problems arise if at some stage of the process more than one basic variable become simultaneously zero (degeneracy), for then there is no clear indication as to which of the possible candidates should be taken to be the entering variable. Lemke, in his seminal paper [5], assumes that no degeneracy occurs, while Eaves [2] effectively removed it by using the lexical inequalities borrowed from linear programming [3]. We show that any ambiguities associated with degeneracy may be resolved by solving the two auxiliary problems LCP($\pm\mathbf{h},\mathbf{G}$), where $\mathbf{G}$ is a Schur complement derived from the original matrix $\mathbf{M}$. This lends itself to an inductive analysis which we suggest is not appreciably more difficult than those based on other devices. The nature of this analysis, however, does lead us to concentrate on the case where $\mathbf{M}$ is nondegenerate (a degenerate matrix is not to be confused with a degenerate solution of an LCP—see the definitions below) and so effectively excludes P_0-matrices [2] from consideration. It is possible that these will be dealt with in a subsequent paper.

NOTE. The nature of an LCP is not changed by identical row and column permutations of $\mathbf{M}$, with corresponding changes to $\mathbf{x}$ and $\mathbf{q}$, since these preserve complementarity. It is convenient to permute $\mathbf{M}$ so that $\mathbf{x}^T=[\mathbf{x}_1^T\ \ \mathbf{0}^T]$ and $\mathbf{w}^T=[\mathbf{0}^T\ \ \mathbf{w}_2^T]$ or, in the case of degeneracy, $\mathbf{x}^T=[\mathbf{x}_1^T\ \ \mathbf{0}^T\ \ \mathbf{0}^T]$ and $\mathbf{w}^T=[\mathbf{0}^T\ \ \mathbf{0}^T\ \ \mathbf{w}_3^T]$, where all nonnull partitions are strictly positive, and this is done in what follows.

2. DEGENERACY AND THE AUXILIARY PROBLEMS

We begin with some definitions. A *solution* of LCP($\mathbf{q},\mathbf{M}$) is a vector $\mathbf{x}\geqslant\mathbf{0}$ such that $\mathbf{Mx}+\mathbf{q}=\mathbf{w}\geqslant 0$ and $\mathbf{x}^T\mathbf{w}=0$. This solution is said to be *degenerate* if $\mathbf{x}=[x_i]$ and $\mathbf{w}=[w_i]$ and $x_i=w_i=0$ for at least one value of i. A solution is *simply degenerate* if $x_i=w_i=0$ for just one value of i, *multiply degenerate* if $x_i=w_i=0$ for more than one value of i, and *totally degenerate* if $\mathbf{x}=\mathbf{w}=\mathbf{0}$. The *support* $S(\mathbf{x})$ of $\mathbf{x}$ is defined by $S(\mathbf{x})=\{i\mid x_i>0\}$. A matrix $\mathbf{M}$ is said to be *degenerate* if at least one of its principal minors is zero, and if all are positive, then it is called a *P-matrix*. Clearly P-matrices are nondegenerate.

A linear complementarity problem may have more than one solution, but if $\mathbf{M}$ is nondegenerate, then the solutions of LCP($\mathbf{q},\mathbf{M}$) are distinct in the sense that no strictly convex combination of two or more solutions is itself a solution. This important result, due originally to Mangasarian [7, 8], is true even if one or more solutions of LCP($\mathbf{q},\mathbf{M}$) are degenerate and means that the problem must have a finite number of solutions. We shall see subsequently that the numbers of solutions of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$) for two different values of ϑ differ by some multiple of two (or zero) provided that every solution for both values of ϑ is nondegenerate, and we begin by establishing sufficient conditions for all the solutions of LCP($\mathbf{q}, \mathbf{M}$) to have this essential property.

LEMMA 1. *Let* $\mathbf{M}$ *be nondegenerate. Then the solutions of* LCP($\mathbf{q}, \mathbf{M}$) *are nondegenerate if no principal submatrix* $\mathbf{K}$ *of* $\mathbf{M}$ *and corresponding elements* $\mathbf{p}$ *of* $\mathbf{q}$ *exist such that* $\mathbf{K}^{-1}\mathbf{p}$ *has a zero element.*

Proof. By contradiction. Assume, without loss of generality (see note at end of Section 1 above), that $\mathbf{x}$ denotes a degenerate solution and that it satisfies

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_2 \end{bmatrix},$$

where $\mathbf{x}_1 \geqslant \mathbf{0}$ and has at least one element equal to zero, and $\mathbf{w}_2 > \mathbf{0}$. Thus

$$\mathbf{x}_1 = -\mathbf{M}_{11}^{-1}\mathbf{q}_1,$$

so that if $\mathbf{K} = \mathbf{M}_{11}$ and $\mathbf{p} = \mathbf{q}_1$, then $\mathbf{K}^{-1}\mathbf{p}$ has at least one zero element. ∎

That vectors $\mathbf{q}$ exist such that, for any nondegenerate matrix $\mathbf{M}$, the vector-matrix pair ($\mathbf{q}, \mathbf{M}$) satisfies the above conditions was essentially proved by Eaves [2], who showed that "almost any $\mathbf{q}$ will do." See also [1] for a more elementary discussion. Note that if ($\mathbf{q}, \mathbf{M}$) does not satisfy these conditions, then neither does ($-\mathbf{q}, \mathbf{M}$). However, they are not necessary for the solutions of LCP($\mathbf{q}, \mathbf{M}$) to be nondegenerate. If

$$\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

then LCP($\mathbf{q}, \mathbf{M}$) has a single nondegenerate solution $\mathbf{x} = \mathbf{0}$ but LCP($-\mathbf{q}, \mathbf{M}$) has two solutions, one nondegenerate ($x^T = [1, 0]$) and one degenerate ($x^T = [0, \frac{1}{2}]$). The existence of a degenerate solution of LCP($-\mathbf{q}, \mathbf{M}$) shows that the conditions of Lemma 1 are not satisfied (in fact, $\mathbf{M}^{-1}\mathbf{q} = [0, \frac{1}{2}]^T$), but this does not prevent the existence of a unique nondegenerate solution of LCP($\mathbf{q}, \mathbf{M}$).

The existence of conditions guaranteeing that all the solutions of LCP($\mathbf{q}, \mathbf{M}$) are nondegenerate suggests the following definition:

DEFINITION. Let $\mathbf{q}$ and $\mathbf{M}$ satisfy the conditions of Lemma 1. Then LCP($\mathbf{q}, \mathbf{M}$) will be called a *nondegenerate* LCP.

We now turn to the problem LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$), where $\mathbf{M}$ is nondegenerate and $\mathbf{d}$ is an arbitrary vector. We know from the behavior of Lemke's algorithm that for certain values of ϑ one or more solutions of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$) may become degenerate, and that if this degeneracy is multiple the basic form of the algorithm breaks down. In order to deal with this we introduce the idea of auxiliary problems. Let $\mathbf{p}=\mathbf{q}+\vartheta\mathbf{d}$, and let $\mathbf{x}$ be a degenerate solution of LCP($\mathbf{p},\mathbf{M}$), so that Equation (1) becomes (see note at end of Section 1 above)

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{w}_3 \end{bmatrix}. \tag{3}$$

Then if

$$\mathbf{G}=\mathbf{M}_{22}-\mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12} \tag{4}$$

and

$$\mathbf{h}=\mathbf{d}_2-\mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{d}_1, \tag{5}$$

the *auxiliary problems* of the solution $\mathbf{x}$ are defined to be LCP($\mathbf{h},\mathbf{G}$) and LCP($-\mathbf{h},\mathbf{G}$). Note that, since they involve $\mathbf{d}$, the auxiliary problems are specific to the particular Lemke homotopy, and since this is to some extent arbitrary, then so are the auxiliary problems. This implies that if $\mathbf{M}$ is nondegenerate, the conditions of Theorem 3 (below) can always be met. The utility of the auxiliary problems stems from the following two theorems.

THEOREM 1. *If* $\mathbf{M}$ *is nondegenerate (a P-matrix) and* $\mathbf{G}$ *is as defined above, then* $\mathbf{G}$ *is nondegenerate (a P-matrix).*

Proof. See e.g. Tucker [9], or [1]. ■

THEOREM 2. *If* LCP($\mathbf{d},\mathbf{M}$) *is nondegenerate and* $\mathbf{h}$ *and* $\mathbf{G}$ *are as defined above for some degenerate solution* $\mathbf{x}$ *of* LCP($\mathbf{p},\mathbf{M}$), *then* LCP($\mathbf{h},\mathbf{G}$) *is nondegenerate.*

Proof. By contradiction. Let $\mathbf{x}$ satisfy Equation (3), so that $\mathbf{G}$ and $\mathbf{h}$ are given by Equations (4) and (5), and let LCP($\mathbf{h},\mathbf{G}$) be degenerate. Then, without loss of generality and with obvious partitioning (see note at end of Section 1 above), we may assume that $\mathbf{G}_{11}^{-1}\mathbf{h}_1$ has at least one zero element (note that since $\mathbf{M}$ is nondegenerate, then from Theorem 1, $\mathbf{G}$, is also nondegenerate, so that $\mathbf{G}_{11}$ is nonsingular). Let $\mathbf{M}$ be partitioned as in Equation (3), and further partition the following submatrices thus:

$$\mathbf{M}_{12}=\begin{bmatrix}\mathbf{K}_{12} & \mathbf{K}_{13}\end{bmatrix},\qquad \mathbf{M}_{21}=\begin{bmatrix}\mathbf{K}_{21}\\ \mathbf{K}_{31}\end{bmatrix},$$

$$\mathbf{M}_{22}=\begin{bmatrix}\mathbf{K}_{22} & \mathbf{K}_{23}\\ \mathbf{K}_{32} & \mathbf{K}_{33}\end{bmatrix},\quad \text{and}\quad \mathbf{d}_2=\begin{bmatrix}\mathbf{g}_2\\ \mathbf{g}_3\end{bmatrix},$$

where $\mathbf{K}_{22}$ has the same dimensions as $\mathbf{G}_{11}$. Then, from (4) and (5),

$$\mathbf{G}_{11}=\mathbf{K}_{22}-\mathbf{K}_{21}\mathbf{M}_{11}^{-1}\mathbf{K}_{12},$$

$$\mathbf{h}_1=\mathbf{g}_2-\mathbf{K}_{21}\mathbf{M}_{11}^{-1}\mathbf{d}_1,$$

so that, if $\mathbf{x}_2$ is defined by

$$\mathbf{G}_{11}\mathbf{x}_2=\mathbf{h}_1, \tag{6}$$

then

$$\mathbf{K}_{21}\mathbf{M}_{11}^{-1}(\mathbf{d}_1-\mathbf{K}_{12}\mathbf{x}_2)+\mathbf{K}_{22}\mathbf{x}_2=\mathbf{g}_2. \tag{7}$$

If $\mathbf{x}_1$ is now defined by $\mathbf{x}_1=\mathbf{M}_{11}^{-1}(\mathbf{d}_1-\mathbf{K}_{12}\mathbf{x}_2)$, it follows immediately from (7) that

$$\begin{bmatrix}\mathbf{M}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=\begin{bmatrix}\mathbf{d}_1\\ \mathbf{g}_2\end{bmatrix}. \tag{8}$$

Now, by assumption and (6), $\mathbf{x}_2$ has at least one zero element. Since the matrix and the right-hand side of Equation (8) are submatrices of $\mathbf{M}$ and $\mathbf{d}$ respectively, it follows that LCP($\mathbf{d},\mathbf{M}$) is degenerate, contrary to hypothesis. The assumption that LCP($\mathbf{h},\mathbf{G}$) is degenerate is thus incorrect. ■

3. THE SOLUTION GRAPHS OF LEMKE'S ALGORITHM

We now turn to the problem LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$), where ϑ is arbitrary and LCP($\mathbf{d},\mathbf{M}$) is nondegenerate. Our concern will be the variation of the solution set of this problem with ϑ, and we continue with some further definitions.

Let $\mathbf{x}$ be a solution of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$). Suppose there is some neighborhood N of $(\mathbf{x},\vartheta)$ such that for all $(\mathbf{z},\sigma)\in N$ for which $\mathbf{z}$ is a solution of LCP($\mathbf{q}+\sigma\mathbf{d},\mathbf{M}$), at least one of the following three conditions holds:

(a) $S(\mathbf{z})\neq S(\mathbf{x})$,
(b) $S(\mathbf{Mz}+\mathbf{q}+\sigma\mathbf{d})\neq S(\mathbf{Mx}+\mathbf{q}+\vartheta\mathbf{d})$,
(c) $\mathbf{z}=\mathbf{x}$.

Then $\mathbf{x}$ will be called the *node* and ϑ a *nodal value*.

Let $\mathbf{x}$ be a solution of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$). Suppose that for all neighborhoods N of $(\mathbf{x},\vartheta)$ there is a $(\mathbf{z},\sigma)\in N$ such that $\mathbf{z}$ is a solution of LCP($\mathbf{q}+\sigma\mathbf{d},\mathbf{M}$) satisfying all of the following three conditions:

(a) $S(\mathbf{z})=S(\mathbf{x})$,
(b) $S(\mathbf{Mz}+\mathbf{q}+\sigma\mathbf{d})=S(\mathbf{Mx}+\mathbf{q}+\vartheta\mathbf{d})$,
(c) $\mathbf{z}\neq\mathbf{x}$.

Then $\mathbf{x}$ will be said to lie on an *arc* of the solution graph of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$). The totality of points $\mathbf{z}$ satisfying

(a) $\mathbf{z}$ is a solution of LCP($\mathbf{q}+\sigma\mathbf{d},\mathbf{M}$) for some σ,
(b) $S(\mathbf{z})=S(\mathbf{x})$ and $S(\mathbf{Mz}+\mathbf{q}+\sigma\mathbf{d})=S(\mathbf{Mx}+\mathbf{q}+\vartheta\mathbf{d})$

will be called an *arc* of the solution graph.

The totality of arcs and nodes as defined above will be called the *solution graph* of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$).

We shall see that the terms *node* and *arc* correspond to normal graph-theoretic usage with the exception that semiinfinite arcs, originating at a particular node and extending indefinitely, are possible. Nodes correspond to particular values of ϑ, while arcs correspond to open intervals. This graph-theoretic correspondence even extends to some nodes being the termination points of more than two arcs, and we shall see that every solution of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$) for any value of ϑ lies either on an arc or on a node of the solution graph. Note that if LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$) has more than one solution, some may be nodes while others may lie on arcs. The above definition implies that ϑ is a nodal value if at least one such solution is a node.

These definitions seek to make precise the nature of the changes undergone by the solutions of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$) as ϑ is varied. At a node, the supports of $\mathbf{x}$ and $\mathbf{w}$ change and the corresponding solution is degenerate, and we shall see that on an arc the supports of $\mathbf{x}$ and $\mathbf{w}$ remain constant and, in general, the corresponding solution is nondegenerate. All of these results stem from our principal theorem, which we now prove.

THEOREM 3. *Let* LCP($\mathbf{d}, \mathbf{M}$) *be nondegenerate, let* $\mathbf{p} = \mathbf{q} + \vartheta \mathbf{d}$, *and let* $\mathbf{x}$ *be a degenerate solution of* LCP($\mathbf{p}, \mathbf{M}$). *Then*

(a) *every solution of the auxiliary problems corresponds to an arc emanating from* $\mathbf{x}$, *and conversely, and*
(b) $\mathbf{x}$ *is a node.*

Moreover, all solutions on arcs are nondegenerate

Proof. Let $\mathbf{u}_2$ be a solution of LCP($\mathbf{h}, \mathbf{G}$), so that

$$\mathbf{G}\mathbf{u}_2 + \mathbf{h} = \mathbf{y}_2, \tag{9}$$

where

$$\mathbf{u}_2^T \mathbf{y}_2 = \mathbf{0} \tag{10}$$

and, from Theorem 2,

$$\mathbf{u}_2 + \mathbf{y}_2 > 0. \tag{11}$$

If we define $\mathbf{u}_1$ by

$$\mathbf{u}_1 = -\mathbf{M}_{11}^{-1}(\mathbf{M}_{12}\mathbf{u}_2 + \mathbf{d}_1) \tag{12}$$

(where, since $\mathbf{M}$ is nondegenerate, $\mathbf{M}_{11}$ is nonsingular), then Equations (4), (5), (9), and (12) yield

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y}_2 \end{bmatrix}. \tag{13}$$

Since $\mathbf{x}$ is a degenerate solution of LCP($\mathbf{p}, \mathbf{M}$), it satisfies Equation (3), so that, from Equation (13),

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 + \varepsilon \mathbf{u}_1 \\ \varepsilon \mathbf{u}_2 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_1 + \varepsilon \mathbf{d}_1 \\ \mathbf{p}_2 + \varepsilon \mathbf{d}_2 \\ \mathbf{p}_3 + \varepsilon \mathbf{d}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \varepsilon \mathbf{y}_2 \\ \mathbf{w}_3 + \epsilon \mathbf{y}_3 \end{bmatrix}, \tag{14}$$

where ε is a scalar and $\mathbf{y}_3 = \mathbf{M}_{31}\mathbf{u}_1 + \mathbf{M}_{32}\mathbf{u}_2 + \mathbf{d}_3$. Now since $\mathbf{x}_1 > \mathbf{0}$ and $\mathbf{w}_3 > \mathbf{0}$, for all ε satisfying $|\varepsilon| < \varepsilon_0$ and some positive ε_0 we have $\mathbf{x}_1 + \varepsilon \mathbf{u}_1 > \mathbf{0}$ and $\mathbf{w}_3 + \varepsilon \mathbf{y}_3 > \mathbf{0}$, so that since $\mathbf{u}_2$ satisfies Equation (10), it follows from (14) that $\mathbf{x}+\varepsilon\mathbf{u}$ is a solution of LCP($\mathbf{p}+\varepsilon\mathbf{d},\mathbf{M}$) for $0<\varepsilon<\varepsilon_0$. Thus each solution of the auxiliary problem LCP($\mathbf{h},\mathbf{G}$) corresponds to some arc leaving $\mathbf{x}$ with ϑ increasing. Moreover, from Equations (11) and (14), the solutions on these arcs are nondegenerate. Similarly, each solution of LCP($-\mathbf{h},\mathbf{G}$) corresponds to some arc leaving $\mathbf{x}$ with ϑ decreasing.

Conversely, let $\mathbf{x}+\mathbf{v}$ be the solution of LCP($\mathbf{p}+\varepsilon\mathbf{d},\mathbf{M}$), so that

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1+\mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{p}_1+\varepsilon\mathbf{d}_1 \\ \mathbf{p}_2+\varepsilon\mathbf{d}_2 \\ \mathbf{p}_3+\varepsilon\mathbf{d}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{w}_3+\mathbf{y}_3 \end{bmatrix} \tag{15}$$

for some vector $\mathbf{y}$. Now $\mathbf{w}_1>\mathbf{0}$ and $\mathbf{w}_3>\mathbf{0}$, so assume that ε and $\|\mathbf{v}\|$ are sufficiently small for $\mathbf{x}_1+\mathbf{v}_1>\mathbf{0}$ and $\mathbf{w}_3+\mathbf{y}_3>\mathbf{0}$. Since $\mathbf{x}+\mathbf{v}$ is the solution of an LCP, it must satisfy the complementarity condition, and this implies, from Equation (15), that $\mathbf{y}_1=\mathbf{0}$ and $\mathbf{v}_3=\mathbf{0}$. Substituting these values in Equation (15) and appealing once again to Equation (3) then yields

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} + \begin{bmatrix} \varepsilon\mathbf{d}_1 \\ \varepsilon\mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y}_2 \end{bmatrix}, \tag{16}$$

and eliminating $\mathbf{v}_1$ gives, from (4) and (5),

$$\mathbf{G}\mathbf{v}_2+\varepsilon\mathbf{h}=\mathbf{y}_2. \tag{17}$$

Since, from (15), $\mathbf{v}_2\geqslant\mathbf{0}$, $\mathbf{y}_2\geqslant\mathbf{0}$, and $\mathbf{v}_2^T\mathbf{y}_2=0$, and since $\varepsilon\neq0$, we have that $\mathbf{v}_2$ is a solution of one of the auxiliary problems of $\mathbf{x}$; and since, from Theorem 2, it is nondegenerate, the original solution of LCP($\mathbf{p}+\varepsilon\mathbf{d},\mathbf{M}$) is nondegenerate. Thus $\mathbf{x}$ is a node, because the solution of LCP($\mathbf{p},\mathbf{M}$) is degenerate by hypothesis, implying a change in the support of at least one of $\mathbf{x}$ and $\mathbf{Mx}+\mathbf{p}$ when $\mathbf{p}$ is replaced by $\mathbf{p}+\varepsilon\mathbf{d}$, $\varepsilon\neq0$. ∎

Note that, in the second part of the above proof, it is not sufficient to assume only that ε is small, since if LCP($\mathbf{p},\mathbf{M}$) has more than one solution, $\mathbf{x}+\mathbf{v}$ could lie in the neighborhood of a solution other than $\mathbf{x}$. In this case a small value of ε would not imply a small value of $\|\mathbf{v}\|$, so two independent assumptions about the magnitudes of ε and $\|\mathbf{v}\|$ are necessary.

The proof of Theorem 3 implicitly assumes that the orders of the principal submatrices $\mathbf{M}_{11}$, $\mathbf{M}_{22}$, and $\mathbf{M}_{33}$ are all at least unity, but this is not necessarily the case. The following three theorems give variations of Theorem 3 appropriate to different patterns of degeneracy. In all cases, $\mathbf{M}$ is assumed to be nondegenerate and $\mathbf{x}$ is assumed to be a degenerate solution of LCP($\mathbf{p}, \mathbf{M}$). The proofs are identical, *mutatis mutandis*, to that of Theorem 3 and are omitted.

THEOREM 4. *If Equation* (3) *may be written*

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

then Theorem 3 *holds with* $\mathbf{G} = \mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12}$ *and* $\mathbf{h} = \mathbf{d}_2 - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{d}_1$.

THEOREM 5. *If Equation* (3) *may be written*

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_2 \end{bmatrix}$$

then Theorem 3 *holds with* $\mathbf{G} = \mathbf{M}_{11}$ *and* $\mathbf{h} = \mathbf{d}_1$.

THEOREM 6. *If* $\mathbf{x}$ *is a totally degenerate solution of* LCP($\mathbf{p}, \mathbf{M}$), *then Theorem* 3 *holds with* $\mathbf{G} = \mathbf{M}$ *and* $\mathbf{h} = \mathbf{d}$.

Theorem 6 addresses the extreme case of degeneracy, i.e. total degeneracy. In this case the order of the matrix of the auxiliary problems is the same as that of the main problem. There is no reduction as there is for Theorems 3–5, and without a strict reduction in the order of the matrix involved the induction proofs of the next section are invalid. For this reason, the homotopies there chosen avoid totally degenerate solutions.

We are now in a position to make certain deductions about the solution graphs of Lemke's method. The first is that arcs must terminate at a node where the corresponding solution is degenerate. This follows from Equation (14), where ε may be increased until an element of either $\mathbf{x}_1 + \varepsilon \mathbf{u}_1$ or $\mathbf{w}_3 + \varepsilon \mathbf{y}_3$ becomes zero (if $\mathbf{u}_1 \geqslant \mathbf{0}$ and $\mathbf{y}_3 \geqslant \mathbf{0}$, then ε may be increased indefinitely, corresponding to ray termination). The second deduction that may be made is that it is not necessarily the case that arcs leave a node in the directions of both ϑ increasing and ϑ decreasing. If either of the auxiliary problems for the node has no solution, then no arcs leave the node in that direction. It is also possible that neither of the auxiliary problems has a solution, in which case the node is an isolated point of the solution graph (though it will not necessarily be so for other homotopies, i.e. for other choices of $\mathbf{d}$). This is the case if

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ 2 & 3 & 2 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}, \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For this problem, Lemke's algorithm gives the solution $[0\ \ 0\ \ 1]^T$, but $[1\ \ 0\ \ 0]^T$ is also a solution which is isolated for the given value of $\mathbf{d}$.

Thus, if LCP($\mathbf{d},\mathbf{M}$) is nondegenerate, the solution graph of LCP($\mathbf{q}+\vartheta\mathbf{d},\mathbf{M}$) consists of a collection of nodes connected by arcs, where the solutions at the nodes are always degenerate and those on arcs are nondegenerate. The possibility also exists of isolated nodes. The graph may consist of more than one component, i.e. subgraphs not connected to other subgraphs by any arc, and if this occurs, then only solutions corresponding to one such subgraph may be found by Lemke's algorithm, since it essentially traverses the graph along a sequence of connected arcs.

4. NONDEGENERATE HOMOTOPIES

We may refer to homotopies $\mathbf{q}+\vartheta\mathbf{d}$ for which LCP($\mathbf{d},\mathbf{M}$) is nondegenerate as *nondegenerate homotopies*. Since this designation is independent of $\mathbf{q}$, such a homotopy may be nondegenerate even if one or more of the solutions of LCP($\mathbf{q},\mathbf{M}$) are highly degenerate, and we show now that any two LCPs having the same $\mathbf{M}$ may be linked by a nondegenerate homotopy.

THEOREM 7. *Let* $\mathbf{M}$ *be nondegenerate, and let* $\mathbf{q}_1$ *and* $\mathbf{q}_2$ *be arbitrary vectors. Then there is a nondegenerate homotopy that takes* LCP($\mathbf{q}_1,\mathbf{M}$) *into* LCP($\mathbf{q}_2,\mathbf{M}$) *avoiding* ($\mathbf{0},\mathbf{M}$), *even if one or more solutions of the first two problems are degenerate.*

Proof. Let $\mathbf{d}=\mathbf{q}_2-\mathbf{q}_1$. If $\mathbf{q}_1$ and $\mathbf{q}_2$ are linearly independent and LCP($\mathbf{d},\mathbf{M}$) is nondegenerate, then the homotopy LCP($\mathbf{q}_1+\vartheta\mathbf{d},\mathbf{M}$) suffices. If one of the above conditions is not satisfied, define $\mathbf{d}_1$ and $\mathbf{d}_2$ by $\mathbf{d}_i=\mathbf{q}_i-\mathbf{d}$, $i=1,2$, where $\mathbf{d}$ is now chosen such that LCP($\mathbf{d},\mathbf{M}$), LCP($\mathbf{d}_1,\mathbf{M}$), and LCP($\mathbf{d}_2,\mathbf{M}$) are nondegenerate (that such a $\mathbf{d}$ exists follows essentially from a result of Eaves [2], or see [1]). Then the homotopies LCP($\mathbf{q}_1-\vartheta\mathbf{d}_1,\mathbf{M}$), $0\leqslant\vartheta\leqslant 1$, and LCP($\mathbf{d}+(\vartheta-1)\mathbf{d}_2,\mathbf{M}$), $1\leqslant\vartheta\leqslant 2$, take the solutions of LCP($\mathbf{q}_1,\mathbf{M}$) first to those of LCP($\mathbf{d},\mathbf{M}$) and then to those of LCP($\mathbf{q}_2,\mathbf{M}$).

Since, by construction, these homotopies are nondegenerate and avoid $(\mathbf{0}, \mathbf{M})$, the theorem follows. ■

Theorem 7 is similar to various theorems proved by Lemke and others [5, 6, 2, 8], with the exception that they did not use the notion of a solution graph. We now show that this idea, together with the auxiliary problems, can be used to derive certain semiquantitative results about the number of solutions of a linear complementarity problem. We start by proving some results first due to Murty [8], who obtained them through a study of complementary cones.

THEOREM 8. *Let* $\mathrm{LCP}(\mathbf{d}, \mathbf{M})$ *be nondegenerate. Then the difference between the numbers of solutions of* $\mathrm{LCP}(\mathbf{d}, \mathbf{M})$ *and* $\mathrm{LCP}(-\mathbf{d}, \mathbf{M})$ *is even.*

Proof. By induction. Let $\mathbf{M}$ be nth order, assume that the theorem holds for matrices of order r, $1 \leqslant r \leqslant n-1$, and set up a nondegenerate homotopy taking $\mathrm{LCP}(\mathbf{d}, \mathbf{M})$ into $\mathrm{LCP}(-\mathbf{d}, \mathbf{M})$ and avoiding $(\mathbf{0}, \mathbf{M})$. Since, by hypothesis, $\mathrm{LCP}(\mathbf{d}, \mathbf{M})$ is nondegenerate, by Theorem 3 its solutions will lie on arcs. Vary ϑ to take $\mathrm{LCP}(\mathbf{d}, \mathbf{M})$ to $\mathrm{LCP}(-\mathbf{d}, \mathbf{M})$ until either $\mathrm{LCP}(-\mathbf{d}, \mathbf{M})$ is attained or ϑ assumes a nodal value. If the former, the theorem is proved. If the latter, since the homotopy avoids $(\mathbf{0}, \mathbf{M})$, the order of the auxiliary problems of the nodal solution is less than n. Moreover, from Theorems 1 and 2, these auxiliary problems are nondegenerate, so from Theorem 3 and the induction hypothesis, the difference between the numbers of arcs entering and leaving the node is even. Repeating this argument for every nodal value of ϑ traversed shows that, since the solutions of $\mathrm{LCP}(-\mathbf{d}, \mathbf{M})$ are nondegenerate and thus lie on arcs, the difference between the number of solutions of this problem and $\mathrm{LCP}(\mathbf{d}, \mathbf{M})$ is even, establishing the induction.

To initiate the induction we consider the solutions of $\mathrm{LCP}(d, m)$ where d and m are of order 1, i.e. scalars. We have, since $m \neq 0$ and $d \neq 0$ from the nondegeneracy assumptions, four cases:

(1) $m > 0$ and $d > 0$. One solution, $x = 0$, $w = d$.
(2) $m > 0$ and $d < 0$. One solution, $x = |d|/m$, $w = 0$.
(3) $m < 0$ and $d > 0$. Two solutions, $x = 0$, $w = d$ and $x = d/|m|$, $w = 0$.
(4) $m < 0$ and $d < 0$. No solutions.

If $m > 0$, there is no difference between the numbers of solutions of $\mathrm{LCP}(d, m)$ and $\mathrm{LCP}(-d, m)$, but if $m < 0$ the difference is ± 2. Thus the theorem is true for LCPs of order 1 and hence, by induction, for LCPs of all orders. ■

We are now in a position to derive several results concerning the number of solutions of LCPs where the matrix is nondegenerate.

THEOREM 9. *Let* $\mathbf{M}$ *be nondegenerate, and let all the solutions of* LCP($\mathbf{q}_1, \mathbf{M}$) *and* LCP($\mathbf{q}_2, \mathbf{M}$), *where* $\mathbf{q}_1$ *and* $\mathbf{q}_2$ *are arbitrary, be nondegenerate. Then the difference between the numbers of solutions of these two problems is even.*

Proof. Set up a nondegenerate homotopy taking LCP($\mathbf{q}_1, \mathbf{M}$) into LCP($\mathbf{q}_2, \mathbf{M}$). Since all the solutions of each problem are nondegenerate, they lie on arcs. Let ϑ vary to take LCP($\mathbf{q}_1, \mathbf{M}$) into LCP($\mathbf{q}_2, \mathbf{M}$). As it passes a nodal value, the number of arcs, by Theorems 3 and 8, changes by an even number, and repetition of this argument for each nodal value of ϑ traversed proves the theorem. ∎

COROLLARY. *If LCP*($\mathbf{q}_1, \mathbf{M}$) *has an odd number of nondegenerate solutions and no degenerate ones for some* $\mathbf{q}_1$, *then LCP*($\mathbf{q}, \mathbf{M}$) *has at least one (possibly degenerate) solution for any* $\mathbf{q}$.

Proof. Trivial. ∎

Murty [8] proved Theorem 9 using analysis based on complementary cones. We now show how the above techniques can be employed to prove certain results for P-matrices which were first obtained by Lemke, Eaves, and others [5, 6, 2] using analysis based on convex sets.

THEOREM 10. *If* $\mathbf{M}$ *is a P-matrix, then* LCP($\mathbf{q}, \mathbf{M}$) *has a unique solution for any* $\mathbf{q}$.

Proof. By induction. Let $\mathbf{M}$ be nth order, and assume that the theorem holds for matrices of order r, $1 \leqslant r \leqslant n-1$. Consider any nondegenerate homotopy avoiding $(\mathbf{0}, \mathbf{M})$ and taking LCP($\mathbf{q}, \mathbf{M}$) into LCP($\mathbf{d}, \mathbf{M}$), where $\mathbf{d} > \mathbf{0}$ but is otherwise arbitrary. As ϑ traverses a nodal value, it follows from the induction hypothesis, Theorem 1, and Theorem 3 that only one arc enters and leaves the node, so that the solution graph consists at most of a set of disconnected "chains," where each chain consists of alternate arcs and nodes. Now $\mathbf{x} = \mathbf{0}$ is clearly, since $\mathbf{d} > \mathbf{0}$, a solution of LCP($\mathbf{d}, \mathbf{M}$), and any other solution must satisfy

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_2 \end{bmatrix},$$

implying the existence of a P-matrix $\mathbf{M}_{11}$ and a vector $\mathbf{x}_1 > \mathbf{0}$ such that $\mathbf{M}_{11}\mathbf{x}_1 < 0$, contradicting a result of Fiedler and Pták [4]. Hence $\mathbf{x} = \mathbf{0}$ is the only solution of LCP($\mathbf{d}, \mathbf{M}$), and only one chain of solutions is possible.

To initiate the induction we need to consider only the first two cases of the corresponding section of the proof of Theorem 8. From these we deduce that LCP(d, m), where d is nonzero and m is positive, has a unique solution. ∎

5. CONCLUSIONS AND FURTHER OUTLOOK

We have shown in this paper that the solutions of LCP($\mathbf{q} + \vartheta\mathbf{d}, \mathbf{M}$), where $\mathbf{M}$ is nondegenerate, lie on a graph where, if $\mathbf{d}$ satisfies certain not particularly stringent conditions, all solutions on the arcs of the graph are nondegenerate and all solutions at the nodes are degenerate. If $\mathbf{M}$ is a P-matrix, this graph consists of a single chain (even if some of the solutions of the LCP are multiply degenerate), but if $\mathbf{M}$ is not a P-matrix, then if a solution is multiply degenerate at a node, it is possible for that node to be the termination point of any even number of arcs.

If, because of the nature of the problem or because of the use of lexical inequalities, only simple degeneracies occur, the solution graph consists of a set of disconnected chains with each chain consisting of alternate arcs and nodes. In general, these chains terminate with infinite arcs (ray termination), but the existence of closed loops cannot be ruled out. When Lemke's algorithm is used to solve such a problem, no ambiguities occur, as the entering variable is always uniquely determined. At such simple degeneracies, the auxiliary problems LCP(h, g) are one-dimensional with g and h scalars. If $g > 0$, as is the case when $\mathbf{M}$ is a P-matrix, then as the node is traversed, ϑ will continue decreasing (or increasing). If, on the other hand, $g < 0$, traversal of a node causes ϑ to increase if it was previously decreasing, and vice versa. It is this behavior that leads to the possibility of multiple solutions for a given ϑ when $\mathbf{M}$ is not a P-matrix and that gives rise to the possibility of more than two arcs emanating from the same node.

We are now in a position to examine the effect of using lexical inequalities to resolve cases of multiple degeneracy. If, in the unperturbed system, more than two arcs leave a given node, the effect of lexical (or any other) perturbation is to convert the *compound* node (more than two arcs) into a number of *simple* nodes (just two arcs), which can then be traversed using Lemke's algorithm. However, this process can easily lead to solutions not being found. Consider the solution graph of Figure 1. Here there is one compound node and two solutions of the problem LCP($\mathbf{q}, \mathbf{M}$) corresponding to $\vartheta = 0$. Perturbation will convert the compound node into two simple ones, and if the solver is fortunate, these will resemble those of Figure 2 when one

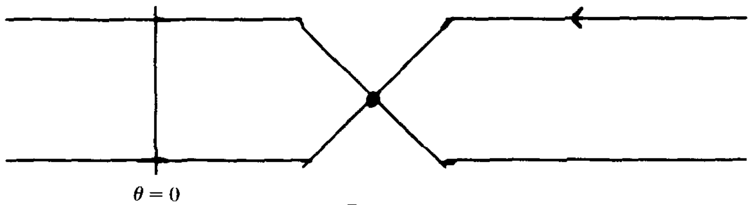

FIG. 1.

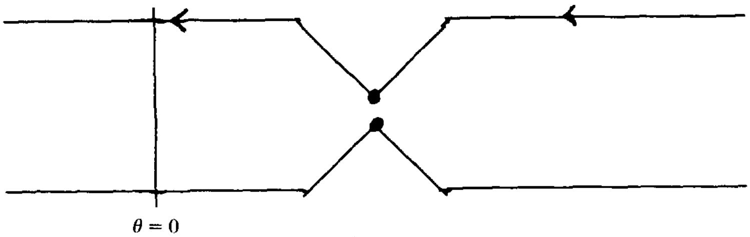

FIG. 2.

solution will, and one will not, be found. If, on the other hand, the perturbed solution graph resembles that of Figure 3, no solution will be detected and immediate ray termination will occur.

Clearly one area for future exploration lies in alternative methods of resolving multiple degeneracies, possibly by solving the auxiliary problems. This would appear to be a somewhat daunting task and would not, in any case, enable Lemke's method to find solutions that lie on disconnected chains or isolated loops of the solution graph.

Another area for further research would be the extension of the above analysis to P_0-matrices and to matrices derived from bimatrix games. P_0-matrices have all their principal minors nonnegative, and under certain circumstances can have a continuum of solutions. It is to be expected that this will give rise to a "sheet" of solutions joining two arcs of the solution graph. Finally, it should be possible to determine, for the matrices derived from bimatrix games and which have the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix}$$

where **A** and **B** are rectangular matrices having the same dimensions, whether or not Lemke's algorithm generates a solution graph similar to those described above or, because of the multiple degeneracies of **M**, a totally different type of solution exists.

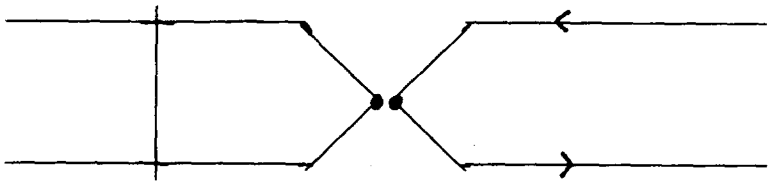

FIG. 3.

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