Degree, Toughness and Subgraph Conditions for Hamiltonian Properties of Graphs

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DEGREE, TOUGHNESS AND SUBGRAPH CONDITIONS FOR HAMILTONIAN PROPERTIES OF GRAPHS

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Preface

As the title of this thesis suggests, it contains research results in the area of hamiltonian graph theory, in particular on sufficient conditions for hamiltonian properties. Such conditions guarantee that a graph has a specific hamiltonian property if the condition is imposed on the graph.

After an introductory chapter (Chapter 1), the reader will find seven chapters (Chapters 2-8), each containing results involving different types of sufficient conditions for a variety of hamiltonian properties. Apart from the introductory chapter, all chapters have the structure of a journal paper. Chapter 2 is mainly based on research that was done while the author was working as a PhD student at Northwestern Polytechnical University in Xi'an, China. The other chapters are mainly based on research that was done when the author was a visiting joint PhD student at the University of Twente.

Apart from a general background and an overview of our contributions to the field, in the introductory chapter we also present most of the necessary terminology and notation that will be used in the subsequent chapters. There are several more specific terms and notations that are not defined in the introductory chapter, but they can be found in the chapters where they are used.

Chapters 2-4 investigate hamiltonian properties implied by conditions involving implicit degrees, in two distinct ways. One approach is to put implicit degree conditions on specific induced subgraphs, and falls into the category that is commonly known under the name of heavy subgraph conditions. The other approach is to consider minimum implicit degree conditions restricted to some specific classes of graphs.

Chapters 5-7 deal with three different hamiltonian properties implied by conditions involving the toughness and forbidden induced subgraphs.

In Chapter 8, we present partial solutions to an interesting, and still wide open conjecture due to Thomassen [74] from 1996.

Papers underlying this thesis

[1] On implicit heavy subgraphs and hamiltonicity of 2-connected graphs, *Discussiones Mathematicae Graph Theory*, DOI: 10.7151/dmgt.2170 (with W. Wideł and L. Wang). (Chapter 2)

[2] Implicit heavy subgraph conditions for hamiltonicity of almost distance-hereditary graphs (with H.J. Broersma and L. Wang). (Chapter 3)

[3] Implicit minimum degree conditions for hamiltonicity of claw-free graphs (with H.J. Broersma, E. Flandrin, H. Li and Y. Zhu). (Chapter 4)

[4] Toughness, forbidden subgraphs, and hamiltonian-connected graphs (with H.J. Broersma and L. Wang). (Chapter 5)

[5] Toughness, forbidden subgraphs and pancyclicity (with H.J. Broersma and L. Wang). (Chapter 6)

[6] On hamiltonicity of 1-tough triangle-free graphs (with H.J. Broersma and L. Wang). (Chapter 7)

[7] On Thomassen's Conjecture for some classes of graphs (with H.J. Broersma and L. Wang). (Chapter 8)

Contents

Chapter 1

Introduction

In order to explain in layman's terms what this thesis is about, and provide some intuition for the concepts, we start this introduction with an illustrative small example. The next section contains formal definitions of the frequently used graph-theoretical concepts and properties.

Example 1.1. Suppose we have a round table, and we want to arrange the seats of a number of participants of an international conference around this table in such a way that all the participants can talk to their two neighbors. For the sake of simplicity say we have seven participants who we name A, B, C, D, E, F, and G, and suppose they master the following languages:

A speaks English;

B speaks English and Chinese;

C speaks English, Italian and Russian;

D speaks Japanese and Chinese;

E speaks German and Italian;

F speaks French, Japanese and Russian;

G speaks French and German.

Is it possible to arrange the seats of $A - G$ around the table in the required way? And how to check this and find a suitable solution if there is one?

In order to check whether this is possible, the only relevant information is which of the pairs of participants master a common language. We can visualize this information graphically by drawing seven points on a piece of paper, putting the labels A to G next to the corresponding points, and drawing lines between two points whenever the corresponding pairs of participants share a common language.

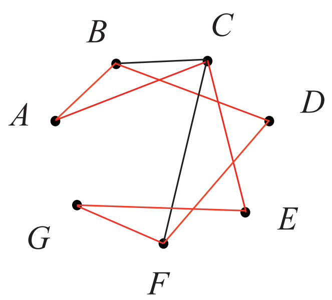

FIGURE 1.1

Once this has been completed (see Figure 1.1), we can easily check that there is a suitable arrangement, and in fact find such an arrangement, for example the arrangement $A - B - D - F - G - E - C - A$. For larger numbers of participants one can imagine that this becomes much harder.

The drawing in the above example is a visualization of what we call a graph in mathematics. Such a graph consists of a set of vertices (representing the points or the participants in the above example) and a set of edges (representing the lines or the pairs of participants that share a common language in the above example). For the formal definitions we refer to the next section and the references we give there. We continue here with some intuitive descriptions of the graph concepts that play a key role in this thesis.

It is clear that a graph can represent any set of objects together with the existing relationships between pairs of these objects. This makes graphs very widely applicable as mathematical abstractions in a diversity of practical settings and a huge range of scientific areas. Like many other disciplines, graph theory originates from practical needs and also serves as a tool in solving practical problems. However, far beyond a tool in life, graph theory has developed into a mature independent branch within mathematics and plays an important role in scientific research work.

After modeling general problems using graphs, exploring the properties of graphs becomes a natural demand. For instance, in the above example, we need to determine whether the corresponding graph has the property that it admits a cyclic way of passing through each vertex exactly once. In fact, this kind of cyclic arrangement is known within graph theory as a Hamilton cycle. The decision problem to determine whether a given graph contains a Hamilton cycle (or not) is one of the most well-known decision problems within graph theory and computational complexity, and is one of the notorious NP-complete problems. People have approached this problem from many different angles, but no one has come up with an easy criterium in terms of a necessary and sufficient condition yet. Part of this thesis deals with sufficient conditions that guarantee that a graph admits a Hamilton cycle. In the other parts, we focus on related sufficient conditions for graph properties that are stronger than the property of having a Hamilton cycle, and are commonly known as hamiltonian properties.

One of the stronger hamiltonian properties we consider in this thesis is called hamiltonian-connectedness, and requires that every pair of distinct vertices of the graph is connected by a Hamilton path, i.e., a path passing through each vertex of the graph exactly once. Another stronger hamiltonian property called pancyclicity requires that the graph contains cycles of any length from 3 up to the number of vertices.

The graph theoretical concepts of degree, subgraph and toughness are three commonly used concepts in studying conditions for hamiltonian properties. Sufficiency results in terms of these concepts impose certain restrictions on the degrees, subgraphs or toughness in order to guarantee the existence of a Hamilton cycle, but in many papers we see that these concepts appear in conjunction.

The degree of a vertex is the number of other vertices to which the vertex is related, in the above sense. Intuitively, increasing the degree of the vertices of a graph makes it more likely that the graph has a Hamilton cycle. This idea was the basis for many results that have appeared since the 1950s, and degree conditions are still among the most studied conditions for hamiltonian properties. In Chapters 2 to 4 of this thesis we focus on conditions involving the more recently introduced concept of the implicit degree of a vertex.

Subgraph conditions impose restrictions on the graph structure, e.g., that some fixed smaller graphs are not allowed to appear as a (subgraph) copy in the larger graph. Next to degree conditions, these forbidden subgraph conditions have attracted a lot of attention in the context of hamiltonian properties. Forbidden subgraph conditions are involved in almost all the chapters of this thesis.

Toughness conditions were introduced in the 1970s as another means to analyse hamiltonian properties. These conditions take into account how well the graph fits together. A larger toughness intuitively reflects that one has to remove more vertices from the graph to let it fall apart into many unrelated parts. In Chapters 5 to 7 of this thesis, we involve toughness in our study of hamiltonian properties of graphs.

Chapter 8 deals with an open conjecture that we will describe in more detail after we have introduced the necessary terminology. We show that this conjecture is valid for several classes of graphs that are defined by forbidden subgraph conditions.

The next section contains formal definitions of some of the most frequently used concepts throughout this thesis. We assume that the reader is familiar with basic mathematical concepts and elementary graph theory.

1.1 Terminology and notation

All graphs we consider in this thesis are finite, simple and undirected graphs. For terminology, notation and concepts not defined here, we refer the reader to [10].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If there is an edge $e \in E(G)$ that joins two distinct vertices $u, v \in V(G)$, then we usually write uv (or vu) instead of e, and we say that u and v are *adjacent* vertices or that they are *neighbors*. We also say that e is *incident* with u (and v) and, vice versa that u (and v) are incident with e.

A *path* of G is a sequence of distinct adjacent vertices of G, i.e., more precisely a sequence $v_1v_2 \dots v_k$ such that $\{v_1, v_2, \dots, v_k\} \subseteq V(G)$, $v_i \neq v_j$ for all $i \neq j$, and $v_iv_{i+1} \in E(G)$ for all $i \in \{1, 2, \dots, k-1\}$. The *length* of such a path is its number of edges, i.e., $k-1$ in the definition. The graph that consists of a path on n vertices (and no additional edges) is denoted by P_n. The graph G is called *connected* if there exist paths in G between any two distinct vertices of G. A *cycle* of G is a sequence $v_1v_2 \dots v_kv_1$ such that $k \geq 3$, $\{v_1, v_2, \dots, v_k\} \subseteq V(G)$, $v_i \neq v_j$ for all $i \neq j$, $v_iv_{i+1} \in E(G)$ for all $i \in \{1, 2, \dots, k-1\}$, and additionally $v_kv_1 \in E(G)$. As with a path, the length of a cycle is also its number of edges, i.e., k in the definition. The graph that consists of a cycle on n vertices (and no additional edges) is denoted by C_n.

For the hamiltonian properties that we intuitively described in the beginning of this introductory chapter, we now give the formal definitions. A cycle in a graph G is called a *Hamilton cycle* (or *hamiltonian cycle*), if it contains all the vertices of G, and G is called *hamiltonian* if it contains a Hamilton cycle. Similarly, a *Hamilton path* in a graph G is a path that contains all the vertices of G. We say a graph is *traceable* if it contains a Hamilton path, and is *hamiltonian-connected* if every pair of distinct vertices of the graph is connected by a Hamilton path. A graph G of *order* $|V(G)| = n$ is called *pancyclic* if G contains cycles of any length from 3 up to n. Obviously, a pancyclic graph or a hamiltonian-connected graph is also a hamiltonian graph, and a hamiltonian graph is also a traceable graph, but the reverse statements do not hold in general.

We next turn to several definitions related to the number of vertices that are adjacent to a given vertex. We first define what we mean with a (proper) subgraph and an induced subgraph of a given graph G. A graph H with vertex set $V(H)$ and edge set $E(H)$ is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. This subgraph H is said to be a *proper* subgraph of G if $H \neq G$, i.e., if at least one of the inclusions is a proper inclusion. For a nonempty set $S \subseteq V(G)$, the subgraph of G *induced by* S, denoted by $G[S]$ or $\langle S \rangle$, is the graph with vertex set S and all edges of $E(G)$ that join pairs of vertices of S. If $S = \{x_1, x_2, ..., x_{|S|}\}$, then $G[S] = G[\{x_1, x_2, ..., x_{|S|}\}]$ is also written as $G[x_1, x_2, ..., x_{|S|}]$. A graph H is called an *induced subgraph* of G if H is isomorphic to $G[S]$ for some $S \subseteq V(G)$.

For a vertex $u \in V(G)$ and a subgraph H of G, the *neighborhood* of u in H is denoted by $N_H(u) = \{v \in V(H) \mid uv \in E(G)\}$, and the *degree* of u in H, denoted as $d_H(u)$, is defined by $d_H(u) = |N_H(u)|$. For two vertices $u, v \in V(H)$ in a connected graph H, the *distance* between u and v in H, denoted by $d_H(u, v)$, is the length of a shortest (u, v)-path in H, which is a path connecting u and v. When there is no danger of ambiguity, we will use $N(u)$, $d(u)$ and $d(u, v)$ instead of $N_G(u)$, $d_G(u)$ and $d_G(u, v)$, respectively. We use $\delta(G)$ to denote the minimum degree of (the vertices of) G, and $\sigma_k(G) = \min\{\Sigma_{i=1}^{k} d(v_i) \mid \{v_1, \ldots, v_k\}$ is a set of mutually nonadjacent vertices of $G\}$. We use $N_2(u)$ to denote the set of vertices which are at distance two from u, i.e., $N_2(u) = \{v \in V(G) \mid d(u, v) = 2\}$.

Apart from the above more or less standard and traditional definitions based on the neighbors and the degrees of the vertices of a graph, we will focus in particular on the more recent concept of implicit degree that was introduced by Zhu, Li and Deng at the end of the 1980s [79]. Since the definition of this concept is rather technical and nonstandard but essential in some of our chapters, we give it explicitly for later reference.

Definition 1.1 (Zhu et al. [79]). Let $u \in V(G)$ and suppose $d(u) = \ell + 1$ for some integer ℓ. Set $M_2(u) = \max\{d(v) \mid v \in N_2(u)\}$. If $N_2(u) \neq \emptyset$ and $d(u) \geq 2$, then let $d_1^u \leq d_2^u \leq d_3^u \leq \ldots \leq d_\ell^u \leq d_{\ell+1}^u \leq \ldots$ be the degree sequence of the vertices of $N(u) \cup N_2(u)$. Define

$$d^*(u) = \begin{cases} d_{\ell+1}^u, & \text{if } d_{\ell+1}^u > M_2(u); \\ d_\ell^u, & \text{otherwise.} \end{cases}$$

Then the implicit degree of u is defined as $id(u) = \max\{d(u), d^*(u)\}$. If $N_2(u) = \emptyset$ or $d(u) \leq 1$, then we define $id(u) = d(u)$.

Clearly, the above definition implies that $id(u) \geq d(u)$ for every vertex u of G. This is important for the purpose of improving existing results based on conditions that impose a lower bound on the degrees of the vertices of a graph: if one can prove that the same condition imposed on the implicit degrees instead of the degrees guarantees that the same concluding statement holds, it is very likely that the result is more generally applicable. We will encounter several examples of this phenomenon in the sequel of this thesis.

We continue with some additional definitions that we need in order to describe the results in the sections that follow. The *complete graph* on n vertices, i.e., the graph in which all $\binom{n}{2}$ pairs are adjacent, is denoted by K_n. The *complete bipartite graph* on $m+n$ vertices, denoted by $K_{m,n}$, consists of a vertex set $A \cup B$ with $|A| = m > 0$, $|B| = n > 0$, and $A \cap B = \emptyset$, and edge set $\{uv \mid u \in A, v \in B\}$. The special case $K_{1,3}$ with vertex set $\{u, v, w, x\}$ and edge set $\{uv, uw, ux\}$ is commonly known as the *claw*, and we call u the *center*, and v, w, x the *end vertices* of this claw. For a given graph H, we say that G is *H-free* if G does not contain an induced copy of H. In the special case that H is the claw, we use *claw-free* instead of H-free. More generally, for a set of graphs $\mathscr{H}$, G is said to be *$\mathscr{H}$-free* if G is H-free for every $H \in \mathscr{H}$.

For a proper subset $S \subsetneq V(G)$, we use $G - S$ to denote the subgraph of G induced by the vertices of $V(G) \setminus S$. In case $S = \{v\}$, we write $G - v$ instead of $G - \{v\}$. A *cut* or *vertex cut* of a graph G is a proper subset $S \subsetneq V(G)$ such that $G - S$ is disconnected. Note that S may be the empty set, in which case G is a *disconnected* graph. In that case, the (inclusion) maximal connected subgraphs of G are called the *components* of G. The number of components of a graph G is denoted by $\omega(G)$. The *connectivity* or *vertex connectivity* $\kappa(G)$ of G is the smallest size of a vertex cut of G in case G is not a complete graph, and $\kappa(K_n) = n - 1$ by definition. A graph G is called *k-connected* or *k-vertex-connected* if $\kappa(G) \geq k$.

The following variant on connectivity introduced by Chvátal [32] in the 1970s is more relevant than connectivity in the context of hamiltonian properties. He defined a noncomplete graph G to be *t-tough* if $t \cdot \omega(G - S) \leq |S|$ for all vertex cuts S of G. The *toughness* of G, denoted by $\tau(G)$, is the maximum value of t such that G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). It is an easy exercise to show that every hamiltonian graph is 1-tough, but that the reverse statement does not hold. It is still an open problem whether there exists a constant t_0 such that every t_0-tough graph on $n \geq 3$ vertices is hamiltonian. This was already conjectured by Chvátal in [32], and is currently usually referred to as Chvátal's Conjecture. In this thesis we will derive several hamiltonicity results that involve conditions on the toughness.

In the next three sections, we will briefly present the main contributions of this thesis, together with some background, motivation and related results. More details and the proofs of our results can be found in the subsequent chapters. The next three sections deal with subgraph conditions, implicit degree conditions, and toughness conditions for hamiltonian properties, respectively.

1.2 Subgraph conditions for hamiltonian properties

Next to degree conditions, subgraph conditions and mixes of the two types of conditions are the most commonly found conditions within the graph theory research that deals with path and cycle properties. There have been many results established on hamiltonian properties related to subgraph conditions. The overwhelming majority of these results deal with the class of claw-free graphs, its subclass of line graphs, and superclasses of claw-free graphs in which claws are allowed as induced subgraphs, but additional conditions are imposed on these claws. These results are motivated by early conjectures on claw-free graphs and line graphs, and many more recent conjectures, most of which turned out to be equivalent with the earlier conjectures. We refer the interested reader to [14] and [18] for more background and details.

We apply forbidden subgraph conditions in each of the chapters of this thesis. In Chapter 2, we impose implicit degree conditions on different pairs of induced subgraphs to study the hamiltonicity of graphs. In Chapter 3, we consider the hamiltonicity of superclasses of claw-free graphs by imposing implicit degree conditions on induced claws. In Chapter 4, we establish minimum implicit degree conditions that guarantee the hamiltonicity of claw-free graphs. We will give more details of the results we obtain in Chapters 2 to 4 in the next section.

In Chapters 5 and 6, we consider two hamiltonian properties of graphs that do not contain an induced copy of a subgraph of $K_1 \cup P_4$. We obtain that for any proper subgraph H of $K_1 \cup P_4$, every H-free graph with toughness larger than one is hamiltonian-connected and every H-free 1-tough graph is pancyclic except for a few specific classes of graphs. In Chapter 7, we consider the hamiltonicity of $\{\triangle, K_1 \cup K_{1,3}\}$-free graphs and $\{\triangle, K_1 \cup P_4\}$-free graphs (where $\triangle$ is shorthand for a K_3), and the two results we obtain there give partial answers to two conjectures in [66]. More details of the results we obtain in Chapters 5 to 7 can be found in the last section of this chapter.

In Chapter 8, we present and prove two results that deal with affirmative answers to the conjecture due to Thomassen [74] that every hamiltonian graph G of minimum degree at least 3 contains an edge e such that $G-e$ and G/e are both hamiltonian. Here $G-e$ denotes the graph obtained from G by deleting the edge e, while G/e denotes the graph obtained from G by contracting the edge $e = uv \in E(G)$. Hence, G/e is obtained from G by replacing u and v by a new vertex x and making x adjacent to the vertices of $N_G(u) \cup N_G(v)$ (so avoiding multiple edges). We show that the conjecture is valid for $K_1 \cup P_4$-free graphs and for $K_1 \cup K_{1,3}$-free graphs, where the latter result extends a result due to Bielak [8].

1.3 Implicit degree and hamiltonian properties

The oldest existing sufficiency conditions that guarantee hamiltonicity are based on the traditional degrees of the vertices, and date back to the work of Dirac [33] from the 1950s and extensions due to Ore [69] from the 1960s. Since then, many extensions and generalizations of their results have appeared, and degree conditions are still the most popular conditions in hamiltonian graph theory to date. All these results are based on the intuitive idea that a denser graph (in terms of the edge density) is more likely to be hamiltonian than a less dense (sparser) graph. Lower bounds on the minimum degree, degree sums, cardinalities of neighborhood unions, and other variants of degree-like parameters are a natural and effective way of imposing a certain density (and reasonable distribution) of the edges of a graph. The concept of implicit degree due to Zhu et al. [79], that we introduced in Definition 1.1, is based on the same intuition. As we noted before, it is obvious from the definition that for every vertex of a graph, the implicit degree is greater than or equal to its degree. Relaxing a degree condition to an implicit degree condition is one way to try in order to improve existing results. Results such as Ore's Theorem [69], Fan's Theorem [36] and Bondy's Theorem [11], have all been extended in this sense. For details we refer the reader to [79], [26] and [61], respectively.

Conditions in terms of degree-like parameters have also often been combined with other conditions, e.g., structural conditions like forbidden subgraph conditions. As an example, it is known that lower bounds on degree-like parameters that guarantee hamiltonicity of general graphs can be relaxed considerably if one restricts oneself to claw-free graphs. But a degree condition can also help to relax the forbidden subgraph condition itself, by allowing the subgraphs in the graph, but with some requirements regarding neighborhoods or degrees of their vertices imposed on them. The earliest ideas in this direction date back to the 1990s by accounts in [19] and [17]. These ideas gave rise to the notion of a *heavy subgraph*, and according to the different requirements of the degree on subgraphs people studied different notions, such as the notions of an *f–heavy* [68], *o–heavy* [21, 68] and *c–heavy* subgraph [55]. Just to give the flavor of these results, without going into detail and without defining the special graphs that appear in the below statements, we mention a few examples of results in which different groups of authors have fully characterized the pairs that imply hamiltonicity. For the definitions of the notions and special graphs mentioned here we refer to Chapter 2.

Theorem 1.1 (Li, Ryjáček, Wang and Zhang [53])**.** *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$–o–heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, Z_1, Z_2, B, N$ or W.*

Theorem 1.2. *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$–f–heavy implies that G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*

- P_4, P_5, P_6 (Chen, Wei and Zhang [29]),
- Z_1 (Bedrossian, Chen and Schelp [7]),
- B (Li, Wei and Gao [58]),

- N (Chen, Wei and Zhang [27]),
- Z_2, W (Ning and Zhang [68]).

Theorem 1.3 (Li and Ning [55]). *Let S be a connected graph of order at least three and let G be a 2-connected claw–o–heavy graph. Then G being S–c–heavy implies that G is hamiltonian if and only if $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

As analogous counterparts of o–heavy, f–heavy and c–heavy graphs, one can define *implicit o–heavy*, *implicit f–heavy* and *implicit c–heavy* graphs by replacing the degree condition in the corresponding definition by an implicit degree condition. In Chapter 2 we pose the following problems and give partial answers.

Problem 1.1. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–o–heavy graph is hamiltonian.

Problem 1.2. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–f–heavy graph is hamiltonian.

Problem 1.3. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–c–heavy graph is hamiltonian.

A graph G is *almost distance-hereditary* if each connected induced subgraph H of G has the property $d_H(x,y) \leq d_G(x,y)+1$ for any pair of vertices $x, y \in V(H)$. Combining the condition of almost distance-hereditary and heavy subgraph conditions, Chen and Ning [25] gave the following results.

Theorem 1.4 (Chen and Ning [25]). *Let G be a 2-connected claw-heavy graph. If G is almost distance-hereditary, then G is hamiltonian.*

Theorem 1.5 (Chen and Ning [25]). *Let G be a 3-connected 1-heavy graph. If G is almost distance-hereditary, then G is hamiltonian.*

In Chapter 3, we extend the above two results by replacing the heavy subgraph conditions by implicit heavy subgraph conditions, respectively.

As we noted before, Dirac's result of 1952 has inspired multiple lines of research aimed at finding milder sufficient conditions for hamiltonicity of graphs that can either be applied to a larger class of general graphs, or to a specific class of graphs. We mentioned two of these lines of research here, namely the restriction to claw-free graphs and the introduction of the notion of the implicit degree. Our aim in Chapter 4 was to combine the two approaches, motivated by the following result.

Theorem 1.6 (Matthews and Sumner [64]). *Let G be a 2-connected claw-free graph of order n such that $\delta(G) \geq \frac{n-2}{3}$. Then G is hamiltonian.*

Note that the lower bound on the minimum degree in Dirac's result can be improved considerably from $\frac{n}{2}$ to $\frac{n-2}{3}$, by restricting the statement to claw-free graphs. Like Dirac's result, the above theorem is best-possible, in the sense that there are infinitely many nonhamiltonian 2-connected claw-free graphs in which all vertices have degree at least $\frac{n-1}{3}$. Unlike Dirac's result however, it is known that the lower bound in the above result can be relaxed considerably further by imposing a larger connectivity, or by excluding specific infinite families of claw-free graphs. The degree condition can even be omitted completely if the imposed connectivity is sufficiently large. For a discussion on such results and several open problems on claw-free graphs we refer the interested reader to Section 4 in [14] and to Section 7 in [18].

We use $\delta_1(G)$ to denote the minimum implicit degree of a graph G. In [79], Zhu, Li and Deng presented generalizations of many degree condition results, implying the following analogue of Dirac's result.

Theorem 1.7 (Zhu, Li and Deng [79]). *Let G be a graph on n vertices with $\delta_1(G) \geq \frac{n}{2}$. Then G is hamiltonian.*

Motivated by the above results, it is natural to expect that the following counterpart of Theorem 1.6 holds: any 2-connected claw-free graph G of order n with $\delta_1(G) \geq \frac{n-2}{3}$ is hamiltonian. However, unfortunately this is not the case. The following infinite class of nonhamiltonian claw-free graphs shows that this statement is false. Suppose n is a sufficiently large integer that is divisible by 3. Let G be the graph of order n consisting of three vertex-disjoint complete graphs on $\frac{n}{3}$ vertices each, with the additional edges of two vertex-disjoint triangles, each containing one vertex from each of the three complete graphs. Then G is claw-free, nonhamiltonian and $\delta_1(G) \geq \frac{n+3}{3}$. So, there does not exist a straightforward counterpart of Theorem 1.6 in which

$\delta(G)$ is simply replaced by $\delta_1(G)$. Exploring the structure of the above examples in more detail, note that for every 2-cut $\{u,v\}$ of G, $G-\{u,v\}$ has a complete component of order $\frac{n-6}{3}$. Excluding this by imposing an additional condition, we prove the following result in Chapter 4.

Theorem 1.8. *Let G be a 2-connected claw-free graph on n vertices with $\delta_1(G) \geq \frac{n+3}{3}$. If G contains no 2-cut $\{u,v\}$ such that $G-\{u,v\}$ has a complete component of order less than $\frac{n-5}{3}$, then G is hamiltonian.*

We give examples to show that the condition on the 2-cuts in the above result cannot be omitted and is sharp. We clearly get rid of this condition if we restrict ourselves to 3-connected claw-free graphs. However, in this case we can lower the degree bound by a factor if we add a similar condition on the 3-cuts, and obtain the following result.

Theorem 1.9. *Let G be a 3-connected claw-free graph on n vertices with $\delta_1(G) \geq \frac{n+6}{4}$. If G contains no 3-cut $\{u,v,w\}$ such that $G-\{u,v,w\}$ has a complete component of order less than $\frac{n-7}{4}$, then G is hamiltonian.*

1.4 Toughness and hamiltonian properties

Since its introduction by Chvátal [32] in the 1970s, the toughness notion has received a lot of attention, mainly inspired by what we referred to earlier as Chvátal's Conjecture. The survey paper [5] deals with a large number of results that have been established until more than ten years ago. A more recent survey of results and open problems appeared a few years ago [13].

The notions of toughness and (vertex) connectivity have a lot of commonality, but are essentially different in the context of hamiltonicity as well as computational complexity. Both toughness and connectivity are measures that capture how well the parts of the graph are tight together, but the clear difference is that the connectivity only deals with the minimum number of vertices that need to be removed to separate the remaining vertices into more than one component, while the toughness also takes into account how many components there are in the resulting graph.

From the definitions we have given earlier, it is obvious that every hamiltonian graph is 1-tough and 2-connected, so the latter are clearly necessary conditions for hamiltonicity. It is natural to wonder whether there are sufficient conditions for hamiltonicity in terms of the connectivity or the toughness. However, it is easy to come up with examples of nonhamiltonian graphs with an arbitrarily high connectivity, while it is still open whether there exist nonhamiltonian graphs with an arbitrarily high toughness. In fact, Chvátal's Conjecture states that this is not the case.

In the context of computational complexity, there is a striking difference between connectivity and toughness. It is well-known that there exists a polynomial algorithm to determine the connectivity of a graph, whereas it is NP-hard to determine the toughness of a graph. We omit the details. Nevertheless, from a theoretical viewpoint it is interesting to study whether imposing that the graphs are 1-tough instead of 2-connected (which is implied by 1-toughness) enables relaxations of other conditions that imply that a 2-connected graph is hamiltonian. We now turn to examples in the light of forbidden subgraph conditions.

Over the years, researchers have established full characterizations of all possible single forbidden graphs and pairs of forbidden subgraphs ensuring that every 2-connected graph is hamiltonian. We refer the reader to [6], [38] and [57] for details. It can be observed that many of the nonhamiltonian graph families that show the necessity of forbidding certain subgraphs are not 1-tough. This fact caused researchers to think about using the necessary condition of being 1-tough instead of 2-connected. In a recent study, Li et al. [54] considered single forbidden subgraphs under the condition of 1-toughness, and came up with the following partial solution.

Theorem 1.10 (Li, Broersma and Zhang [54]). *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

Here $K_1 \cup P_3$ denotes the disjoint union of a complete graph on one vertex and a path on three vertices, and $2K_1$ denotes two disjoint copies of a complete graph on one vertex. The other graphs are similarly defined. Moreover, in [54] the authors gave an almost complete characterization of single forbidden subgraphs ensuring hamiltonicity of 1-tough graphs by proving the following complementary result.

Theorem 1.11 (Li, Broersma and Zhang [54]). *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

The two results together leave $K_1 \cup P_4$ as the only open case. As far as we are aware it is still open whether every 1-tough $K_1 \cup P_4$-free graph is hamiltonian. In fact, this is a conjecture due to Nikoghosyan [66].

Various sufficient conditions for a graph to be hamiltonian are so strong that they imply considerably more about the cycle structure of the graph. Based on this observation, Bondy [9] presented a metaconjecture in 1971 in which he stated that almost any nontrivial condition on a graph which implies that the graph is hamiltonian also implies that it is pancyclic (except for maybe a simple family of exceptional graphs). Inspired by Bondy's metaconjecture, we examined whether the condition in Theorem 1.10 in fact implies pancyclicity. The results of our findings are presented in Chapter 6.

Turning to hamiltonian-connectedness, analogous to hamiltonicity one easily checks that every hamiltonian-connected graph is 3-connected and has toughness strictly larger than 1. Again, it is easy to show that no level of connectivity is large enough to ensure a graph is hamiltonian-connected. In 1978, Jung [51] presented the following result, in which he shows that for P_4-free graphs, the toughness condition $\tau(G) > 1$ is a necessary and sufficient condition for hamiltonian-connectivity.

Theorem 1.12 (Jung [51]). *Let G be a P_4-free graph. Then G is hamiltonian-connected if and only if $\tau(G) > 1$.*

Chen and Gould [28] concluded in 2000 that if $\{S, T\}$ is a pair of graphs such that every 2-connected $\{S, T\}$-free graph is hamiltonian, then every 3-connected $\{S, T\}$-free graph is hamiltonian-connected. Following up on this idea, we considered the following question. Suppose R is a graph such that every 1-tough R-free graph is hamiltonian. Is then every R-free graph G with $\tau(G) > 1$ hamiltonian-connected? For the purpose of answering this question, we have checked every case of Theorem 1.10, and report on our positive findings in Chapter 5. Note that the case with $R = P_4$ had already been solved by Jung in Theorem 1.12.

In 2013, Nikoghosyan [66] posed several related conjectures that we listed at the end of this introductory chapter, including the open case that we mentioned in the context of Theorem 1.10. In Chapter 7, we will reflect on the following conjectures and present partial solutions.

Conjecture 1.1. Every 1-tough $K_1 \cup P_4$-free graph is hamiltonian.

Conjecture 1.2. Every $K_1 \cup K_{1,3}$-free graph with $\tau > 4/3$ is hamiltonian.

Conjecture 1.3. Every $K_2 \cup K_2$-free graph with $\tau > 1$ is hamiltonian.

Chapter 2

Implicit heavy subgraphs and hamiltonicity

In this chapter, we study various implicit degree conditions, including but not limited to Ore–type and Fan–type conditions. We will prove that imposing these conditions on specific induced subgraphs of a 2–connected implicit claw–heavy graph ensures its hamiltonicity. In particular, we improve a recent result of Huang [49], and complete the characterization of pairs of o–heavy and f–heavy subgraphs for hamiltonicity of 2–connected graphs.

2.1 Introduction

As we mentioned before, forbidden subgraph conditions and degree conditions are two important and well-studied types of sufficient conditions for the existence of Hamilton cycles in graphs. It is known that P_3 is the only connected graph of order at least three forbidding of which in a 2–connected graph G implies hamiltonicity of G (recall that we use P_n to denote a path on n vertices). When disconnected subgraphs are also considered, forbidding an induced $3K_1$ also ensures hamiltonicity. The former fact can be deduced from results in [38] and the latter follows directly from a classical theorem due to Chvátal and Erdős [31]. In fact, the graphs P_3 and $3K_1$ are the only graphs of order at least three having this property. In [57], Li and Vrána proved the necessity part of the following theorem.

Theorem 2.1 (Li and Vrána [57]). *Let G be a 2–connected graph and S be a graph of order at least three. Then G being S–free implies that G is hamiltonian if and only if S is P_3 or $3K_1$.*

Before we continue, let us first present the symbols and illustrations for some frequently used forbidden induced subgraphs. We refer to Figure 2.1 for these graphs and refrain from giving formal definitions, as the structure of these graphs is clear from the illustrations.

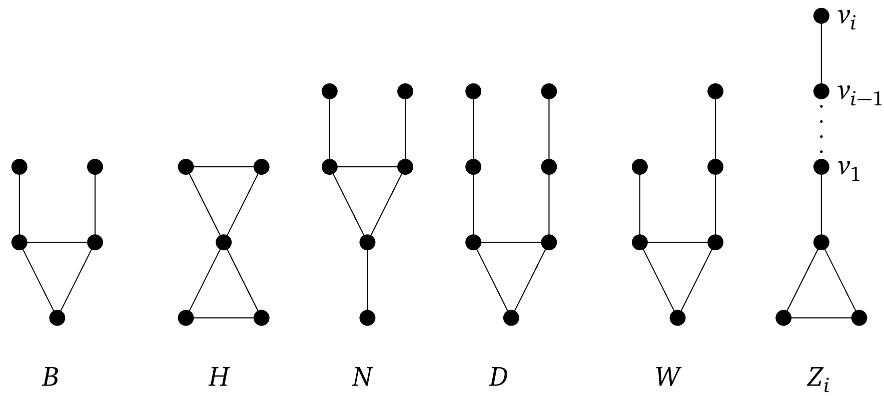

FIGURE 2.1: Some frequently used forbidden induced subgraphs: B (the bull), H (the hourglass), N (the net), D (the deer), W (the wounded) and Z_i.

The case with pairs of forbidden subgraphs other than P_3 and $3K_1$ is much more interesting. The complete characterization of forbidden pairs of connected subgraphs for hamiltonicity, based partially on results from [19], [34], [44] and [46], was obtained by Bedrossian in [6]. The 'only if' part of the following theorem is due to Faudree and Gould [38].

Theorem 2.2 (Bedrossian [6]; Faudree and Gould [38]). *Let R and S be connected graphs with $R, S \neq P_3$, and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

In [57], Li and Vrána considered pairs of forbidden subgraphs that are not necessarily connected.

Theorem 2.3 (Li and Vrána [57]). *Let R and S be graphs of order at least three other than P_3 and $3K_1$, and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of P_6, W, N or $K_2 \cup P_4$.*

A widely studied way of relaxing the forbidden induced subgraph conditions for hamiltonicity is allowing the subgraphs in the graph, but with some requirements regarding degrees of their vertices imposed on them. Some of these extensions exploit the concept of implicit degree, as it was introduced in Definition 1.1 in Chapter 1.

Some of the (implicit) degree conditions suitable for relaxing the forbidden subgraph conditions originate from the following classical results.

Theorem 2.4 (Fan [36]). *Let G be a 2-connected graph of order $n \geq 3$. If*

$$d(u, v) = 2 \Rightarrow \max\{d(u), d(v)\} \geq n/2$$

for every pair of vertices u and v in G, then G is hamiltonian.

Theorem 2.5 (Ore [69]). *Let G be a graph of order $n \geq 3$. If every pair of nonadjacent vertices of G has degree sum at least n, then G is hamiltonian.*

The authors of [79] prove a counterpart of Ore's Theorem 2.5, where the degree sum condition is replaced by an implicit degree sum condition. A similar extension of Theorem 2.4 can be found in [26]. Theorems 2.4 and 2.5, and their extensions, gave rise to the notions of f–heavy graphs [68], o–heavy graphs [21], [68], implicit f–heavy graphs [24], and implicit o–heavy graphs. Here, we adopt the definitions of o–heavy graphs and f–heavy graphs from [68].

Let G be a graph of order n. A vertex v of G is called *heavy* (or *implicit heavy*) if $d(v) \geq n/2$ (or $id(v) \geq n/2$). If v is not heavy (or not implicit heavy), we call it *light* (or *implicit light*, respectively). For a given graph H we say that G is *H–o–heavy* (or *implicit H–o–heavy*) if in every induced subgraph of G isomorphic to H there are two nonadjacent vertices with degree sum (implicit degree sum, respectively) at least n. And G is said to be *H–f–heavy* (or *implicit H–f–heavy*), if for every subgraph S of G isomorphic to H, and every two vertices $u, v \in V(S)$, we have

$$d_S(u, v) = 2 \Rightarrow \max\{d(u), d(v)\} \geq n/2$$

($\max\{id(u), id(v)\} \geq n/2$, respectively).

For a family of graphs $\mathscr{H}$, G is said to be *(implicit) $\mathscr{H}$–o–heavy*, if G is (implicit) H–o–heavy for every $H \in \mathscr{H}$. The classes of *$\mathscr{H}$–f–heavy* and *implicit $\mathscr{H}$–f–heavy* graphs are defined similarly. We note that the above definitions of H–f–heavy, $\mathscr{H}$–o–heavy, and $\mathscr{H}$–f–heavy are also all taken from [68]. When a graph is implicit $K_{1,3}$–o–heavy we will call it *implicit claw–heavy*.

Observe that every H–free graph is trivially H–o–heavy and H–f–heavy. Furthermore, every H–o–heavy (or H–f–heavy) graph is implicit H–o–heavy (implicit H–f–heavy, respectively). Replacing forbidden subgraph conditions by conditions expressed in terms of heavy subgraphs yielded the following extensions of Theorem 2.2.

Theorem 2.6 (B. Li et al. [53]). *Let R and S be connected graphs with $R \neq P_3$ and $S \neq P_3$, and let G be a 2-connected graph. Then G being $\{R, S\}$–o–heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, Z_1, Z_2, B, N$ or W.*

Theorem 2.7. *Let R and S be connected graphs with $R \neq P_3$ and $S \neq P_3$, and let G be a 2-connected graph. Then G being $\{R, S\}$–f–heavy implies that G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*
- P_4, P_5, P_6 (Chen et al. [29]),
- Z_1 (Bedrossian et al. [7]),
- B (G. Li et al. [58]),
- N (Chen et al. [27]),
- Z_2, W (Ning and S. Zhang [68]).

Recently, motivated by the main result of [48], Li and Ning [55] introduced another type of heavy subgraphs. We say that an induced subgraph H of G is *c–heavy* in G, if for every maximal clique C of H every non–trivial component of $H - C$ contains a vertex that is heavy in G. The graph G is said to be *H–c–heavy* if every induced subgraph of G isomorphic to H is c–heavy. For a family $\mathscr{H}$ of graphs, G is called *$\mathscr{H}$–c–heavy* if G is H–c–heavy for every $H \in \mathscr{H}$.

Observe that every graph is trivially $\{K_{1,3}, C_3, P_3\}$–c–heavy, since removal of a maximal clique from any of the three subgraphs results in a graph consisting of trivial components (or an empty graph). With that remark in mind, the authors of [55] extended Theorem 2.2 in the following way.

Theorem 2.8 (B. Li and Ning [55]). *Let S be a connected graph of order at least three, and let G be a 2-connected claw–o–heavy graph. Then G being S–c–heavy implies that G is hamiltonian if and only if $S = P_4$, P_5, P_6, Z_1, Z_2, B, N or W.*

Similarly to implicit o–heavy and implicit f–heavy graphs, we can define *implicit H–c–heavy* and *implicit $\mathscr{H}$–c–heavy* graphs by replacing the degree condition in the definition of c–heavy graphs by an implicit degree condition. In the light of the results presented so far, and noting that every implicit claw–f–heavy graph is implicit claw–heavy, it seems worthwhile to try to tackle the following problems.

Problem 2.1. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–o–heavy graph is hamiltonian.

Problem 2.2. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–f–heavy graph is hamiltonian.

Problem 2.3. Characterize all graphs S such that every 2–connected implicit claw–heavy and implicit S–c–heavy graph is hamiltonian.

As byproducts of the proof of our main result, we obtained the following partial answers to Problems 2.1– 2.3.

Theorem 2.9. *Let G be a 2–connected implicit claw-heavy graph. If G is implicit S–o–heavy for a subgraph S of $K_2 \cup P_4$, then G is hamiltonian.*

Theorem 2.10. *Let G be a 2–connected implicit claw-heavy graph. If G is implicit S–f–heavy, where S is one of the graphs $K_1 \cup P_3$, $K_2 \cup P_3$, $K_1 \cup P_4$, $K_2 \cup P_4, P_4, Z_1$ and Z_2, then G is hamiltonian.*

Theorem 2.11. *Let G be a 2–connected implicit claw-heavy graph. If G is implicit S–c–heavy, where S is one of the graphs $K_1 \cup K_2$, $2K_1 \cup K_2$, $K_1 \cup 2K_2$, $K_2 \cup K_2$, $K_1 \cup P_3$, $K_2 \cup P_3$, $K_1 \cup P_4$, $K_2 \cup P_4, P_4, P_5$ and P_6, then G is hamiltonian.*

Clearly, if S is any of the graphs $K_1 \cup K_2$, $2K_1 \cup K_2$, $K_2 \cup K_2$ and $K_1 \cup 2K_2$, then every graph is S–f–heavy. Observe also that each of the remaining subgraphs of $K_2 \cup P_4$ appears in each of Theorems 2.9–2.11. Hence, as corollaries from these theorems and Theorems 2.6–2.8, we get the following complete characterizations of heavy pairs of (not necessarily connected) subgraphs for hamiltonicity.

Corollary 2.12. *Let R and S be graphs other than P_3 and $3K_1$, and let G be a 2-connected graph. Then G being $\{R, S\}$–o–heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of P_5, W, N or $K_2 \cup P_4$.*

Corollary 2.13. *Let R and S be graphs other than P_3 and $3K_1$, and let G be a 2-connected graph. Then G being $\{R, S\}$–f–heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is one of $P_4, P_5, P_6, Z_1, Z_2, B, N, W$, $K_1 \cup P_3$, $K_2 \cup P_3$, $K_1 \cup P_4$ and $K_2 \cup P_4$.*

Corollary 2.14. *Let S be a graph of order at least three other than P_3 and $3K_1$, and let G be a 2-connected claw–o–heavy graph. Then G being S-c-heavy implies G is hamiltonian if and only if S is one of P_4, P_5, P_6, Z_1, Z_2, B, N, W, $K_1 \cup K_2$, $2K_1 \cup K_2$, $K_1 \cup 2K_2$, $K_2 \cup K_2$, $K_1 \cup P_3$, $K_2 \cup P_3$, $K_1 \cup P_4$ and $K_2 \cup P_4$.*

We note that the assumption of the graph S being of order at least three in Corollary 2.14 is necessary, since every graph is trivially $\{K_1, 2K_1, K_2\}$–c–heavy.

For triples of forbidden subgraphs there are also many results. The following are two well–known results of this type.

Theorem 2.15 (Broersma and Veldman [19]; Brousek [20])**.** *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, D\}$–free, then G is hamiltonian.*

Theorem 2.16 (Faudree et al. [37]; Brousek [20])**.** *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, H\}$–free, then G is hamiltonian.*

Note that the pair $\{K_{1,3}, P_6\}$ that is present in Theorem 2.2 is missing in Theorem 2.6. A construction of a 2–connected, claw–free and P_6–o–heavy graph that is not hamiltonian can be found in [53] [1]. Since every P_6–o–heavy graph is also implicit $\{P_7, D\}$–o–heavy, it is clear that Theorems 2.15 and 2.16 can not be improved by imposing the condition of implicit o–heaviness on all of their forbidden subgraphs. However, a slightly stronger implicit degree sum conditions is sufficient to ensure hamiltonicity. Our main result is the following.

Theorem 2.17. *Let G be a 2-connected, implicit claw–heavy graph of order n such that in every path $v_1v_2v_3v_4v_5v_6v_7$ induced in G at least one of the following conditions is satisfied:*

(a) $id(v_4) \geq n/2$, *or*

(b) $id(v_i) + id(v_j) \geq n$ *for some* $i \in \{1, 2\}$, $j \in \{6, 7\}$.

If

(1) in every induced D of G with vertex set $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and edge set $\{u_1u_2, u_2u_3, u_3u_4, u_3u_5, u_4u_5, u_5u_6, u_6u_7\}$ at least one of the following conditions is satisfied:

- *(a)* $id(u_4) \geq n/2$, *or*
- *(b)* $id(u_i) + id(u_j) \geq n$ *for some* $i \in \{1, 2, 4\}$, $j \in \{6, 7\}$, *or*

(2) in every induced H of G with vertex set $\{u_1, u_2, u_3, u_4, u_5\}$ and edge set $\{u_1u_2, u_2u_3, u_1u_3, u_3u_4, u_3u_5, u_4u_5\}$ at least one of the following conditions is satisfied:

- *(a) both u_1 and u_2 are implicit heavy, or*
- *(b)* $id(u_i) + id(u_j) \geq n$ *for some* $i \in \{1, 2\}$, $j \in \{4, 5\}$,

then G is hamiltonian.

[1] Nevertheless, the condition of P_6–o–heaviness can be replaced with other degree conditions on paths P_6 to ensure hamiltonicity of 2-connected claw–o–heavy graphs. We refer the interested reader to [56] for details.

Note that the conditions imposed on paths of order seven in Theorem 2.17 are satisfied in particular by implicit P_7–f–heavy and implicit P_7–c–heavy graphs. Similarly, the conditions imposed on induced deers are satisfied by implicit D–f–heavy graphs and implicit D–c–heavy graphs, and the conditions imposed on hourglasses are satisfied by implicit H–c–heavy graphs, implicit H–f–heavy graphs and implicit H–o–heavy graphs. Hence, Theorem 2.17 implies the following new results.

Corollary 2.18. *Let G be a 2-connected, implicit claw–heavy graph. If G is*

- *implicit* $\{P_7, D\}$*–c–heavy or implicit* $\{P_7, H\}$*–c–heavy, or*
- *implicit* P_7*–f–heavy and implicit D–c–heavy, or*
- *implicit* P_7*–f–heavy and implicit H–c–heavy, or*
- *implicit* P_7*–f–heavy and implicit H–o–heavy, or*
- *implicit* P_7*–c–heavy and implicit H–o–heavy, or*
- *implicit* P_7*–c–heavy and implicit H–f–heavy,*

then G is hamiltonian.

Some previously known results, including recent extensions of Theorem 2.15 and Theorem 2.16, can also be deduced from Theorem 2.17.

Corollary 2.19 (Huang [49])**.** *Let G be a 2–connected, implicit claw–heavy graph. If G is* P_6*–free, then G is hamiltonian.*

Corollary 2.20 (Broersma et al. [17])**.** *Let G be a 2–connected, claw–f–heavy graph. If G is* $\{P_7, D\}$*–free or* $\{P_7, H\}$*–free, then G is hamiltonian.*

Corollary 2.21 (Cai and H. Li [23])**.** *Let G be a 2–connected, implicit claw–f–heavy graph. If G is* $\{P_7, D\}$*–free or* $\{P_7, H\}$*–free, then G is hamiltonian.*

Corollary 2.22 (Ning [67])**.** *Let G be a 2–connected, claw–f–heavy graph. If G is* $\{P_7, D\}$*–f–heavy or* $\{P_7, H\}$*–f–heavy, then G is hamiltonian.*

Corollary 2.23 (Huang [50])**.** *Let G be a 2-connected, claw–f–heavy graph. If G is implicit* $\{P_7, D\}$*–f–heavy or implicit* $\{P_7, H\}$*–f–heavy, then G is hamiltonian.*

Corollary 2.24 (Cai and Zhang [24]). *Let G be a 2-connected, implicit claw–heavy graph. If G is implicit $\{P_7, D\}$–f–heavy or implicit $\{P_7, H\}$–f–heavy, then G is hamiltonian.*

The rest of this chapter is organized as follows. In Section 2.2 we define some auxiliary notions, and present lemmas that are used throughout the later proofs. The proofs of Theorems 2.9, 2.10, 2.11 and 2.17 are presented in Section 2.3.

2.2 Preliminaries

In this section, we present two lemmas that will be used throughout the proofs of our main results. They make use of the notion of an *implicit heavy cycle*, which is a cycle that contains all implicit heavy vertices of a graph. For a vertex $v \in V(G)$ lying on a cycle C with a given orientation, we denote by v^+ its immediate successor on C and by v^- its immediate predecessor. For a set $A \subset V(C)$, the sets A^+ and A^- are defined analogously, i.e., $A^+ = \{v^+ \mid v \in A\}$ and $A^- = \{v^- \mid v \in A\}$. We write xCy for the path from $x \in V(C)$ to $y \in V(C)$ following the orientation of C, whereas $x\overline{C}y$ denotes the path from x to y in the opposite direction. Similar notation is used for paths.

The next lemma is implicit in [60].

Lemma 2.25 (Li et al. [60]). *Every 2-connected graph contains an implicit heavy cycle.*

A cycle C is called *nonextendable* if there is no cycle longer than C in G containing all vertices of C. We use $E^*(G)$ to denote the set $\{xy \mid xy \in E(G)$ or $id(x) + id(y) \geq n\}$. We sometimes call the nonadjacent pairs that satisfy the second condition *pseudo-edges*.

Another useful lemma appeared in [49].

Lemma 2.26 (Huang [49]). *Let G be a 2-connected graph on $n \geq 3$ vertices, and let C be a nonextendable cycle of G of length at most $n-1$. If P is an xy–path in G such that $V(C) \subset V(P)$, then $xy \notin E^*(G)$.*

2.3 Proofs of Theorems 2.9–2.11 and 2.17

For a proof by contradiction suppose that a graph G satisfying the assumptions of any of the Theorems 2.9, 2.10, 2.11 or 2.17 is not hamiltonian. Then G is a 2-connected implicit claw-heavy graph. By Lemma 2.25, there is an implicit heavy cycle in G. Let C be a longest (nonextendable) implicit heavy cycle in G and give C an orientation. From the assumption of 2-connectivity of G it follows that there is a path P connecting two vertices $x_1, x_2 \in V(C)$ internally-disjoint with C such that $|V(P)| \geq 3$. Let $P = x_1u_1u_2\ldots u_rx_2$ be such a path of minimum length. Note that this implies that P is induced unless $x_1x_2 \in E(G)$. The following four claims also appeared in [24, 49, 50], since they are basic properties of a longest (nonextendable) implicit heavy cycle. We also use them to start our proof.

Claim 1. $u_kx_i^+ \notin E^*(G)$ and $u_kx_i^- \notin E^*(G)$ for every $k \in \{1,2,\ldots,r\}$ and $i \in \{1,2\}$.

Proof. Since $P' = x_1^+Cx_1Pu_k$ and $P'' = x_1^-\overline{C}x_1Pu_k$ are paths such that $V(C) \subset V(P')$ and $V(C) \subset V(P'')$, $u_kx_1^+ \notin E^*(G)$ and $u_kx_1^- \notin E^*(G)$ by Lemma 2.26. Similarly, $u_kx_2^+ \notin E^*(G)$ and $u_kx_2^- \notin E^*(G)$. □

Claim 2. $x_1^-x_1^+ \in E^*(G)$ and $x_2^-x_2^+ \in E^*(G)$.

Proof. If $x_1^-x_1^+ \in E(G)$, the first statement is obvious. Suppose $x_1^-x_1^+ \notin E(G)$. Then the set $\{x_1, x_1^-, x_1^+, u_1\}$ induces a claw. By Claim 1, we have $id(u_1) + id(x_1^-) < n$ and $id(u_1) + id(x_1^+) < n$. Since G is implicit claw–heavy, this implies that $id(x_1^-) + id(x_1^+) \geq n$. Thus, $x_1^-x_1^+ \in E^*(G)$. Similarly, $x_2^-x_2^+ \in E^*(G)$. □

Claim 3. $x_1^-x_2^- \notin E^*(G)$ and $x_1^+x_2^+ \notin E^*(G)$.

Proof. Observe that the paths $P' = x_1^-\overline{C}x_2\overline{P}x_1Cx_2^-$ and $P'' = x_1^+Cx_2\overline{P}x_1\overline{C}x_2^+$ are paths such that $V(C) \subset V(P')$ and $V(C) \subset V(P'')$. Thus, the claim follows from Lemma 2.26. □

Claim 4. $x_1^-x_1^+ \in E(G)$ or $x_2^-x_2^+ \in E(G)$.

Proof. Suppose to the contrary that $x_1^- x_1^+ \notin E(G)$ and $x_2^- x_2^+ \notin E(G)$. Then $id(x_1^-) + id(x_1^+) \geq n$ and $id(x_2^-) + id(x_2^+) \geq n$ by Claim 2. Thus, $id(x_1^-) + id(x_2^-) \geq n$ or $id(x_1^+) + id(x_2^+) \geq n$, contradicting Claim 3. □

By Claim 4, without loss of generality, we assume that $x_1^- x_1^+ \in E(G)$. The following two claims were proved in [24], hence we omit their proofs.

Claim 5 (Cai and Zhang [24]). For $i \in \{1,2\}$, $x_i x_{3-i}^- \notin E^*(G)$ and $x_i x_{3-i}^+ \notin E^*(G)$.

By Claim 5, there is a vertex in $x_i^+ C x_{3-i}^-$ not adjacent to x_i in G for $i = 1,2$. Let y_i be the first vertex in $x_i^+ C x_{3-i}^-$ not adjacent to x_i in G for $i = 1,2$. Let u be any vertex of P other than x_1 and x_2, and let z_i be an arbitrary vertex in $x_i^+ C y_i$ for $i = 1,2$.

Claim 6 (Cai and Zhang [24]). The (pseudo-)edges uz_1, uz_2, z_1x_2, z_2x_1, and z_1z_2 do not exist in G.

The proof of the different results now splits into cases, depending on the conditions satisfied by G in each of the results.

Case 1. G is implicit $K_2 \cup P_4$–o–heavy or implicit $K_2 \cup P_4$–f–heavy.
By Claim 6, we have that both of the vertex sets $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$ and $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$ induce a graph isomorphic to $K_2 \cup P_4$ in G.

First assume that G is implicit $K_2 \cup P_4$–f–heavy. Since none of the vertices u_1 and u_r belongs to C, both these vertices are implicit light. This implies that both y_2^- and y_1^- are implicit heavy, contradicting Claim 6. This contradiction proves the part of Theorem 2.10 regarding implicit $K_2 \cup P_4$–f–heavy graphs. By taking induced subgraphs from $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$ and $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$ corresponding to $K_1 \cup P_4$, P_4, $K_1 \cup P_3$ and $K_2 \cup P_3$, we get the same contradiction which can also prove the part of Theorem 2.10 regarding implicit $K_1 \cup P_4$–f–heavy graphs, implicit P_4–f–heavy graphs, implicit $K_1 \cup P_3$–f–heavy graphs and implicit $K_2 \cup P_3$–f–heavy graphs, respectively.

Next consider the case that G is implicit $K_2 \cup P_4$–o–heavy. Then there is a pair of nonadjacent vertices with implicit degree sum at least n in each of the vertex sets $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$ and $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$. Let us focus on the set $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$. Since $uz_1, z_1x_2, z_1z_2 \notin E^*(G)$ by Claim 6, it follows that the pair of nonadjacent vertices with implicit degree sum at least n belongs to the set $\{u_r, x_2, y_2^-, y_2\}$. Since $uz_2 \notin E^*(G)$ by Claim 6, we have $id(x_2) + id(y_2) \geq n$. Now, since $id(x_1) + id(y_1) + id(x_2) + id(y_2) \geq 2n$, we have $id(x_1) + id(y_2) \geq n$ or $id(x_2) + id(y_1) \geq n$, which contradicts Claim 6. This contradiction proves the part of Theorem 2.9 regarding implicit $K_2 \cup P_4$–o–heavy graphs. The remaining part regarding implicit S–o–heavy graphs for any proper subgraph S of $K_2 \cup P_4$ is implied by the validity of the theorem for $K_2 \cup P_4$. Thus, the proof of Theorem 2.9 is complete.

Case 2. G is implicit S–f–heavy for S being one of Z_1 and Z_2.
Suppose first that G is implicit Z_1–f–heavy. Then, since the vertex u_1 is implicit light by the choice of C and the set $\{x_1^-, x_1^+, x_1, u_1\}$ induces Z_1, both vertices x_1^- and x_1^+ are implicit heavy. Now it follows from Claim 3 that both x_2^- and x_2^+ are implicit light. Then $x_2^-x_2^+ \in E(G)$, by Claim 2. But now the set $\{x_2^-, x_2^+, x_2, u_r\}$ induces Z_1, a contradiction. Thus, G is Z_2–f–heavy.

Suppose that either $r \geq 2$ or $r = 1$ and $x_1x_2 \notin E(G)$. Then one of the sets $\{x_1^-, x_1^+, x_1, u_1, u_2\}$ or $\{x_1^-, x_1^+, x_1, u_1, x_2\}$ induces Z_2. Similarly to the previous paragraph, this implies that both x_1^- and x_1^+ are implicit heavy, and in consequence x_2^- and x_2^+ are implicit light vertices forming an edge in G. But then either $\{x_2^-, x_2^+, x_2, u_r, u_{r-1}\}$ or $\{x_2^-, x_2^+, x_2, u_r, x_1\}$ also induces a Z_2, a contradiction.

Thus, $r = 1$ and $x_1x_2 \in E(G)$. But now both sets $\{u_1, x_2, x_1, y_1^-, y_1\}$ and $\{u_1, x_1, x_2, y_2^-, y_2\}$ induce Z_2, implying that both y_1^- and y_2^- are implicit heavy. This contradicts Claim 6. Together with Case 1, this contradiction completes the proof of Theorem 2.10.

Case 3. G is implicit $K_1 \cup P_3$–c–heavy.
We continue the proof with a number of claims.

Claim 7. x_1 and x_2 are implicit heavy.

Proof. By Claim 6, we have that both sets $\{x_1^+, x_2, y_2^-, y_2\}$ and $\{x_2^+, x_1, y_1^-, y_1\}$ induce a graph isomorphic to $K_1 \cup P_3$ in G. Since G is implicit $K_1 \cup P_3$–c–heavy and the independent vertex of $K_1 \cup P_3$ is a maximal clique, there is an implicit heavy vertex in both sets $\{x_2, y_2^-, y_2\}$ and $\{x_1, y_1^-, y_1\}$. If y_1 or y_1^- is implicit heavy, then none of the vertices of $\{x_2, y_2^-, y_2\}$ can be implicit heavy by Claim 6, a contradiction. Hence, x_1 is implicit heavy. Similarly, x_2 is also implicit heavy. □

Claim 8. $x_2^- x_2^+ \in E(G)$.

Proof. By Claim 5 and Claim 7, we have that x_2^- and x_2^+ are implicit light. Since G is implicit claw-heavy, $x_2^- x_2^+ \in E(G)$. □

By Claim 3, there is a vertex in $x_i^+ C x_{3-i}^-$ not adjacent to x_i^- in G for $i = 1, 2$. Let w_i be the first vertex in $x_i^+ C x_{3-i}^-$ not adjacent to x_i^- in G for $i = 1, 2$. Note that $w_i \neq x_i^+$.

Claim 9. $uw_i^- \notin E(G)$ and $uw_i \notin E(G)$.

Proof. Suppose that $uw_1^- \in E(G)$. By Claim 1, we have that $w_1^- \neq x_1^+$. Then $C' = x_1 P u w_1^- C x_1^- w_1^{--} \bar{C} x_1$ is a cycle such that $V(C) \subset V(C')$, a contradiction. Hence, $uw_1^- \notin E(G)$. We also have that $uw_1 \notin E(G)$; otherwise, $C'' = x_1 P u w_1 C x_1^- w_1^- \bar{C} x_1$ is a cycle such that $V(C) \subset V(C'')$, a contradiction. By symmetry, we have that $uw_2^- \notin E(G)$ and $uw_2 \notin E(G)$. □

From Claim 1 and Claim 9 we have that $\{u, x_1^-, w_1^-, w_1\}$ induces a graph isomorphic to $K_1 \cup P_3$ in G. Since G is implicit $K_1 \cup P_3$–c–heavy, there is an implicit heavy vertex in the set $\{x_1^-, w_1^-, w_1\}$. By Claim 5 and Claim 7 we have that x_1^- is implicit light. If w_1^- is implicit heavy, then $w_1^- \neq x_1^+$ by Claim 5 and Claim 7. Thus $P' = w_1^- C x_2^- x_2^+ C x_1^- w_1^{--} \bar{C} x_1 P x_2$ is a path such that $V(C) \subset V(P')$ and $w_1^- x_2 \in E^*(G)$, contradicting Lemma 2.26. If w_1 is implicit heavy, then $P'' = w_1 C x_2^- x_2^+ C x_1^- w_1^- \bar{C} x_1 P x_2$ is a path such that $V(C) \subset V(P'')$ and $w_1 x_2 \in E^*(G)$, contradicting Lemma 2.26. Thus, the part of Theorem 2.11 regarding implicit $K_1 \cup P_3$–c–heavy graphs is finished by these contradictions. The proof of the validity of the remaining part of Theorem 2.11 will be completed after the next case.

Case 4. G satisfies the assumptions of Theorem 2.17.

Claim 10. $x_1 x_2 \in E(G)$.

Proof. Suppose that $x_1x_2 \notin E(G)$. By the choice of P, Claim 1 and Claim 6 we have that $P' = y_1y_1^-x_1u_1u_2\dots u_rx_2y_2^-y_2$ is an induced P_{r+6}, where $r \geq 1$. Let $y_1y_1^-x_1u_1v_5v_6v_7$ be the path induced by the first seven vertices of P'. Since u_1 is implicit light, it follows from the assumptions of Theorem 2.17 that for some $a \in \{y_1, y_1^-\}$ and $b \in \{v_6, v_7\}$ the inequality $id(a) + id(b) \geq n$ holds. Since $b \in V(P) \cup \{x_2, y_2^-, y_2\}$, this contradicts Claim 6. □

We complete the proof by considering two cases, depending on the value of r. When $r \geq 2$, we can use the method of the proof of Case 2 in [24] completely, because the proof does not involve any heavy subgraphs other than the claw. Here we omit this part of the proof and consider the case when $r = 1$.

Suppose that $r = 1$. Then the set $\{y_1, y_1^-, x_1, u_1, x_2, y_2^-, y_2\}$ induces a D. Since the vertex u_1 is implicit light, Claim 6 implies that G does not satisfy the conditions imposed on induced deers in Theorem 2.17. Hence, it satisfies the conditions imposed on H.

Observe that $\{x_1^-, x_1^+, x_1, u_1, x_2\}$ induces an H. Now, using Claim 5 and Claim 6, we get that both vertices x_1^- and x_1^+ are implicit heavy. Similarly as in Case 2, this implies that both x_2^- and x_2^+ are implicit light and $x_2^-x_2^+ \in E(G)$. But now the set $\{u_1, x_1, x_2, x_2^-, x_2^+\}$ induces an H. Using Claim 5 and Claim 6, we conclude that this contradicts the assumptions of Theorem 2.17. This final contradiction completes the proof of Theorem 2.17.

Observe that every 2–connected implicit–claw–heavy graph that is implicit S–c–heavy for S being one of $K_1 \cup K_2$, $2K_1 \cup K_2$, $K_1 \cup 2K_2$, $K_2 \cup K_2$, $K_2 \cup P_3$, $K_1 \cup P_4$, $K_2 \cup P_4, P_4, P_5$ and P_6 satisfies the assumptions of Theorem 2.17. Hence, together with Case 3, Case 4 also completes the proof of Theorem 2.11. ■

Chapter 3

Hamiltonicity of almost distance-hereditary graphs

Recall that a graph G is said to be almost distance-hereditary if each connected induced subgraph H of G has the property $d_H(x,y) \leq d_G(x,y)+1$ for any pair of vertices $x,y \in V(H)$. In this chapter, we give two sufficient conditions for hamiltonicity of almost distance-hereditary graphs. One states that every 2-connected implicit claw-heavy almost distance-hereditary graph is hamiltonian, and the other says that every 3-connected implicit 1-heavy almost distance-hereditary graph is hamiltonian. These results improve two recent results due to Chen and Ning [25].

3.1 Introduction

Considering the intersection of the classes of almost distance-hereditary graphs and claw-free graphs, about ten years ago Feng and Guo [40] proved the following result.

Theorem 3.1 (Feng and Guo [40]). *Let G be a 2-connected claw-free graph. If G is almost distance-hereditary, then G is hamiltonian.*

The same authors later improved this result by relaxing the condition of being claw-free by putting a Dirac–type degree condition on the end vertices of every induced claw, adopting ideas that originate from [17] and [21]. Let G be a graph on n vertices. A vertex v of G is called *heavy* if $d(v) \geq n/2$. The graph G is called *1-heavy* (respectively *2-heavy*) if at least one (respectively two) of the end vertices of each induced claw of G are heavy. By using this concept of a 2-heavy graph, Feng and Guo [41] extended Theorem 3.1 in the following way.

Theorem 3.2 (Feng and Guo [41]). *Let G be a 2-connected 2-heavy graph. If G is almost distance-hereditary, then G is hamiltonian.*

Replacing the Dirac–type degree condition by an Ore–type degree condition, Fujisawa and Yamashita [43] introduced the notion of a claw-heavy graph. A graph G on n vertices is called *claw-heavy* if each claw of G has a pair of end vertices with degree sum at least n. Chen and Ning [25] recently used the above notions to obtain the following two results related to Theorems 3.1 and 3.2.

Theorem 3.3 (Chen and Ning [25]). *Let G be a 2-connected claw-heavy graph. If G is almost distance-hereditary, then G is hamiltonian.*

Theorem 3.4 (Chen and Ning [25]). *Let G be a 3-connected 1-heavy graph. If G is almost distance-hereditary, then G is hamiltonian.*

Replacing the degree conditions in the above definitions by implicit degree conditions, we say that a graph G on n vertices is *implicit 1-heavy* if at least one end vertex of each claw of G is *implicit heavy*, i.e., has implicit degree at least $n/2$. We call G *implicit claw-heavy* if each claw of G has a pair of end vertices with implicit degree sum at least n. In this chapter we prove the following two improvements of Theorems 3.3 and 3.4.

Theorem 3.5. *Let G be a 2-connected implicit claw-heavy graph. If G is almost distance-hereditary, then G is hamiltonian.*

Theorem 3.6. *Let G be a 3-connected implicit 1-heavy graph. If G is almost distance-hereditary, then G is hamiltonian.*

The implicit degree conditions in Theorems 3.5 and 3.6 can not be omitted. From the complete bipartite graphs $K_{m,m+1}$ it is obvious that a large connectivity together with the condition of almost distance-hereditary cannot guarantee a graph to be hamiltonian.

We postpone the proofs of Theorems 3.5 and 3.6 to Sections 3.3 and 3.4, respectively, and present some useful auxiliary lemmas in Section 3.2. To complete this introductory section, we continue with presenting an infinite class of examples of graphs that do not satisfy the conditions of Theorems 3.3 and 3.4, but that can be easily verified to be hamiltonian using Theorem 3.5 or 3.6.

Let k be a nonnegative integer. For any $m \geq k+1$, let G_m denote the join of a complete graph K_{2m} and a graph H, where H is the disjoint union of a K_4 and m copies of a K_2. Then G_m is a $2m$-connected graph of order $n = 4m+4$, and it is easy to check that G_m is almost distance-hereditary and hamiltonian. The degrees and implicit degrees of the vertices of G_m can also be determined in a straightforward way. For any vertex u belonging to the m copies of a K_2 of H, we obtain that $d(u) = 2m+1 < \frac{n}{2}$ and $id(u) = 2m+3 > \frac{n}{2}$; for any vertex v belonging to the K_{2m}, we obtain that $id(v) \geq d(v) = 4m+3 > \frac{n}{2}$; for any vertex w belonging to the K_4 of H, we obtain that $id(w) \geq d(w) = 2m+3 > \frac{n}{2}$. Using this, it is easy to check that G_m is implicit claw-heavy (and so implicit 1-heavy) but not 1-heavy (and so not claw-heavy) for all $m \geq 3$.

3.2 Some auxiliary lemmas

We start this section with some additional notation and terminology. For a cycle C with a given orientation and a vertex $x \in V(C)$, with x^+ and x^- we denote the immediate successor and predecessor of x on C, respectively. Similarly, for any $I \subseteq V(C)$, we let $I^- = \{x^- \mid x \in I\}$ and $I^+ = \{x^+ \mid x \in I\}$. For two vertices $x, y \in V(C)$, xCy denotes the subpath of C from x to y in the direction specified by the orientation of C. The vertex set of xCy will be denoted as $C[x, y]$ if both the end vertices x and y are included, and as $C(x, y]$ ($C[x, y)$, respectively $C(x, y)$) if x (y, respectively x and y) are not included. We use $y\overline{C}x$ to denote the path from y to x in the reverse direction.

A similar notation is used for paths. We use (x, y)-path as shorthand for a path between a vertex x and a vertex y.

Let G be a graph on n vertices. A cycle C of G is called *implicit heavy* if it contains all implicit heavy vertices of G; it is said to be *extendable* if there exists a longer cycle in G containing all the vertices of C. We use $E^*(G)$ to denote the set $\{xy \mid xy \in E(G) \text{ or } id(x)+id(y) \geq n\}$. Our proofs of Theorems 3.5 and 3.6 are based on the following three lemmas, the first two of which are given without proofs.

Lemma 3.7 (H. Li et al. [60]). *Every 2-connected graph contains an implicit heavy cycle.*

Lemma 3.8 (Huang [49]). *Let G be a 2-connected graph on $n \geq 3$ vertices and C be a nonextendable cycle of G with length at most $n-1$. If P is an xy-path in G such that $V(C) \subset V(P)$, then $xy \notin E^*(G)$.*

The above results are used to state and prove the following useful statements. Let G be a 2-connected nonhamiltonian graph of order n. Then, by Lemma 3.7, there exists an implicit heavy cycle in G. Since G is nonhamiltonian, there exists a nonextendable implicit heavy cycle C of G with length at most $n-1$. We consider an arbitrary component R of $G - V(C)$, and let $A = \{v_1, v_2, \ldots, v_k\}$ denote the set of neighbors of R on C. Using Lemma 3.8, it is now easy to deduce the following result.

Lemma 3.9. *For any $u \in V(R)$ and $v_i, v_j \in A$, the following statements hold.*

(a) $uv_i^- \notin E^*(G)$ *and* $uv_i^+ \notin E^*(G)$*;*

(b) $v_i^-v_j^- \notin E^*(G)$ *and* $v_i^+v_j^+ \notin E^*(G)$*;*

(c) If $v_i^-v_i^+ \in E(G)$*, then* $v_iv_j^- \notin E^*(G)$ *and* $v_iv_j^+ \notin E^*(G)$*;*

(d) Let $p \in C[v_i, v_j^-] \cap N(v_i) \cap N(v_j^-)$*. If* $v_i^-v_i^+ \in E(G)$*, then* $p^-p^+ \notin E^*(G)$ *and* $v_ip^+ \notin E^*(G)$*;*

(e) Let $p \in C[v_i, v_j^-] \cap N(v_i)$*. If* $v_i^-v_i^+ \in E(G)$*, then* $p^-v_j^- \notin E^*(G)$ *and* $v_j^+p^+ \notin E^*(G)$*.*

If, moreover G is implicit claw-heavy, then the following statements also hold.

(f) $v_i^- v_i^+ \in E^*(G)$;

(g) $v_i^- v_i^+ \in E(G)$ *or* $v_j^- v_j^+ \in E(G)$;

(h) $v_i v_j^- \notin E^*(G)$ *and* $v_i v_j^+ \notin E^*(G)$.

Proof. If we write 'By Lemma 3.8 and the path P' (or in the opposite order) in the proofs that follow, we mean that P is an (x, y)-path in G such that $V(C)$ is a proper subset of $V(P)$, and we conclude from Lemma 3.8 that $xy \notin E^*(G)$.

(a) Since $v_i, v_j \in A$ and $u \in R$, there exists a (u, v_i)-path Q_1 and a (u, v_j)-path Q_2 such that all the internal vertices of Q_1 and Q_2 (if any) are in R. By Lemma 3.8 and the path $P = uQ_1v_iCv_i^-$, we obtain that $uv_i^- \notin E^*(G)$. Similarly, $uv_i^+ \notin E^*(G)$.

(b) By Lemma 3.8 and the path $P = v_i^-\overline{C}v_j\overline{Q_2}uQ_1v_iCv_j^-$, we obtain that $v_i^-v_j^- \notin E^*(G)$. Similarly, $v_i^+v_j^+ \notin E^*(G)$.

(c) If $v_i^-v_i^+ \in E(G)$, then by the path $P = v_i\overline{Q_1}uQ_2v_jCv_i^-v_i^+Cv_j^-$ and using Lemma 3.8, we obtain that $v_iv_j^- \notin E^*(G)$. Similarly, $v_iv_j^+ \notin E^*(G)$.

(d) If $v_i^-v_i^+ \in E(G)$ and $p^- \neq v_i$, then by Lemma 3.8 and the path $P = p^-\overline{C}v_i^+v_i^-\overline{C}v_j\overline{Q_2}uQ_1v_ipv_j^-\overline{C}p^+$, we obtain that $p^-p^+ \notin E^*(G)$. If $v_i^-v_i^+ \in E(G)$ and $p^- = v_i$, then by the path $P = v_i\overline{Q_1}uQ_2v_jCv_i^-v_i^+v_j^-\overline{C}p^+$ and Lemma 3.8, we obtain that $v_ip^+ \notin E^*(G)$, i.e., $p^-p^+ \notin E^*(G)$. If $v_i^-v_i^+ \in E(G)$, then by the path $P = v_i\overline{Q_1}uQ_2v_jCv_i^-v_i^+Cpv_j^-\overline{C}p^+$ and Lemma 3.8, we obtain that $v_ip^+ \notin E^*(G)$.

(e) If $v_i^-v_i^+ \in E(G)$, then $p \neq v_j^-$ by Lemma 3.9(c). If $p = v_i^+$, then $p^-v_j^- \notin E^*(G)$ by Lemma 3.9(c). If $p \in C(v_i^+, v_j^-)$, by Lemma 3.8 and the path $P = p^-\overline{C}v_i^+v_i^-\overline{C}v_j\overline{Q_2}uQ_1v_ipCv_j^-$, we obtain that $p^-v_j^- \notin E^*(G)$. By Lemma 3.8 and the path $P = v_j^+Cv_i^-v_i^+Cpv_i\overline{Q_1}uQ_2v_j\overline{C}p^+$, we obtain that $v_j^+p^+ \notin E^*(G)$.

If, moreover G is implicit claw-heavy, then the statements in (f), (g), and (h) correspond to Claims 2, 1, and 4 in [24], respectively. Therefore, we omit the proofs of these statements. This completes the proof of Lemma 3.9. □

3.3 Proof of Theorem 3.5

Suppose, to the contrary, that G is a graph satisfying the conditions of Theorem 3.5, but that G is nonhamiltonian. Since G is a 2-connected implicit claw-heavy graph, by Lemma 3.7, G contains an implicit heavy cycle. Let C be a longest implicit heavy cycle and give C a cyclic orientation. Since G is nonhamiltonian, $V(G)\setminus V(C)\neq\emptyset$. Let R be a component of $G-V(C)$, and let $A=\{v_1,v_2,\ldots,v_k\}$ be the set of neighbors of R on C. Since G is 2-connected, there exists a path P connecting two vertices $v_i,v_j\in V(C)$ such that P is internally-disjoint with C, and with $|V(P)|\geq 3$. Let $P=v_iu_1u_2\ldots u_rv_j$ be such a path, chosen subject to the following rules:

(1) $|C(v_i,v_j)|$ is as small as possible;

(2) $|V(P)|$ is as small as possible subject to (1).

By Lemma 3.9(g), without loss of generality, we may assume that $v_i^-v_i^+\in E(G)$. We first prove the following simple claim.

Claim 1. $r=1$, i.e., $V(P)=\{v_i,u_1,v_j\}$.

Proof. Suppose that $r\geq 2$. Consider the subgraph $H=\langle V(P)\cup C[v_i,v_j)\rangle$. By Lemma 3.9(h), $v_iv_j^-\notin E(G)$. Thus $d_H(v_j^-,v_i)\geq 2$. By the choice of P, we have $d_H(v_j^-,u_r)=d_H(v_j^-,v_i)+d_P(v_i,u_r)\geq 2+2=4$, but $d_G(v_j^-,u_r)=2$. This contradicts the fact that G is almost distance-hereditary. Hence $r=1$. $\square$

Next, we consider the subgraph $H=\langle C[v_i,v_j)\cup\{u_1\}\rangle$. Since $C(v_i,v_j)\cap\{u_1\}=\emptyset$ and $v_iv_j^-\notin E(G)$ by Lemma 3.9(h), we obtain that $d_H(v_j^-,u_1)=3$ from the facts that G is almost distance-hereditary and $d_G(v_j^-,u_1)=2$. Thus there exists a vertex $w\in C[v_i^+,v_j^-]$ such that $v_iw\in E(G)$ and $v_j^-w\in E(G)$. We consider the following two subcases, depending on whether $v_j^-v_j^+\in E(G)$ or not.

Case 1. $v_j^-v_j^+\in E(G)$.

We prove three new claims in order to complete this case.

Claim 2. $v_j^+w\in E(G)$.

Proof. If $v_j^+w \notin E(G)$, then the induced subgraph $H = \langle\{v_j^+, v_j^-, w, v_i, u_1\}\rangle$ is isomorphic to P_5 and $d_H(v_j^+, u_1) = 4$. This contradicts the facts that G is almost distance-hereditary and $d_G(v_j^+, u_1) = 2$. □

By Claim 2 and Lemma 3.9(b), we conclude that $w \neq v_i^+$.

Claim 3. $v_j^+w^+ \notin E^*(G)$.

Proof. Consider the path $Q = w^+Cv_j\overline{P}v_iw\overline{C}v_i^+v_i^-\overline{C}v_j^+$. By using that $V(C)$ is a proper subset of $V(Q)$, applying Lemma 3.8, we obtain that $v_j^+w^+ \notin E^*(G)$. □

Claim 4. $v_iw^+ \notin E^*(G)$.

Proof. Consider the path $Q = w^+Cv_j^-w\overline{C}v_i^+v_i^-\overline{C}v_j\overline{P}v_i$. By using that $V(C)$ is a proper subset of $V(Q)$, applying Lemma 3.8, we obtain that $v_iw^+ \notin E^*(G)$. □

Using Claims 2-4 and Lemma 3.9(h), we conclude that $\{w, v_i, w^+, v_j^+\}$ induces a claw, and that G is not implicit claw-heavy, a contradiction. This completes the proof for Case 1.

Case 2. $v_j^-v_j^+ \notin E(G)$.

We prove four new claims in order to complete this case.

Claim 5. $w \neq v_i^+$.

Proof. Suppose that $w = v_i^+$. Consider $H = \langle\{v_i^-, v_i^+, v_j^-, v_j, u_1\}\rangle$. Using Lemma 3.9(b), (h), and (a), we conclude that H is isomorphic to P_5 and that $d_H(v_j^+, u_1) = 4$. This contradicts the facts that G is almost distance-hereditary and $d_G(v_j^+, u_1) = 2$. Hence, $w \neq v_i^+$. □

Claim 6. $w^-v_j \notin E(G)$.

Proof. Suppose $w^-v_j \in E(G)$ and consider $Q = v_j^-\overline{C}wv_iPv_jw^-\overline{C}v_i^+v_i^-\overline{C}v_j^+$. Observing that $V(C)$ is a proper subset of $V(Q)$, applying Lemma 3.8, we obtain that $v_j^-v_j^+ \notin E^*(G)$, but this contradicts Lemma 3.9(f). □

Claim 7. $w^-v_j^- \notin E^*(G)$.

Proof. Consider the path $Q = v_j^-\overleftarrow{C}wv_iPv_jCv_i^-v_i^+Cw^-$. Observing that $V(C)$ is a proper subset of $V(Q)$, applying Lemma 3.8, we obtain that $w^-v_j^- \notin E^*(G)$. □

Claim 8. $w^-v_i \notin E(G)$.

Proof. If $w^-v_i \in E(G)$, then $wv_j \notin E(G)$ (otherwise $v_j^-\overleftarrow{C}wv_j\overleftarrow{P}v_iw^-\overleftarrow{C}v_i^+v_i^-\overleftarrow{C}v_j^+$ is a path containing all the vertices of C. Then, using Lemma 3.8, we obtain that $v_j^-v_j^+ \notin E^*(G)$, but this contradicts Lemma 3(f)). By Claims 6 and 7, in combination with Lemma 3.9(a), we conclude that the induced subgraph $H = \langle\{u_1, v_j, v_j^-, w, w^-\}\rangle$ is isomorphic to P_5, and that $d_H(w^-, u_1) = 4$. This contradicts the facts that G is almost distance-hereditary and $d_G(w^-, u_1) = 2$. Hence, $w^-v_i \notin E(G)$. □

Using Claims 7 and 8, in combination with Lemma 3.9(h), we conclude that $\{w, v_i, w^-, v_j^-\}$ induces a claw, and $v_iv_j^- \notin E^*(G)$ and $w^-v_j^- \notin E^*(G)$. Since G is implicit claw-heavy, we obtain that $id(v_i) + id(w^-) \geq n$. Combining this with the fact that $id(v_j^-) + id(v_j^+) \geq n$, we deduce that $id(v_i) + id(v_j^+) \geq n$ or $id(w^-) + id(v_j^-) \geq n$, but in both cases we reach a contradiction. This completes the proof of Theorem 3.5. ■

3.4 Proof of Theorem 3.6

Suppose, to the contrary, that G is a graph satisfying the conditions of Theorem 3.6, but that G is nonhamiltonian. Since G is a 3-connected implicit 1-heavy graph, by Lemma 3.7, G contains an implicit heavy cycle. Let C be a longest implicit heavy cycle, and give C a cyclic orientation. Since G is nonhamiltonian, $V(G) \setminus V(C) \neq \emptyset$. Let R be a component of $G - V(C)$. Since G is 3-connected, for any vertex $u \in V(R)$ there exists a (u, C)-fan F such that $F = \{u; Q_1, Q_2, Q_3\}$, where $Q_1 = ux_1 \ldots x_{r_1}w_i$, $Q_2 = uy_1 \ldots y_{r_2}w_j$ and $Q_3 = uz_1 \ldots z_{r_3}w_k$ are three internally-disjoint paths of length at least 1, with $V(Q_1) \cap V(C) = w_i$, $V(Q_2) \cap V(C) = w_j$ and $V(Q_3) \cap V(C) = w_k$, and w_i, w_j, w_k are in this order according to the chosen orientation of C (so $w_j \in C(w_i, w_k)$).

By the choice of C, all internal vertices of F are not implicit heavy. We call such vertices light throughout the proof, and similarly we call an induced claw light if all its end vertices are light. By Lemma 3.9(b), there is at most one implicit heavy vertex in $N_C^+(R)$ and at most one implicit heavy vertex in $N_C^-(R)$. So, for at least one $\ell \in \{i, j, k\}$, w_ℓ^- and w_ℓ^+ are light. Hence, $w_\ell^- w_\ell^+ \in E(G)$; otherwise a light claw is immediate, contradicting that G is implicit 1-heavy. We may assume without loss of generality that $\ell = i$. We may also assume that $u = x_{r_1}$, since for every $u \in V(R)$, in particular $u = x_{r_1}$, a (u, C)-fan with (at least) three internally-disjoint paths exists, and we can choose one of these paths to consist of the edge $x_{r_1} w_i$. Our first claim shows that we can in fact assume that all three paths of the (u, C)-fan F consist of just one edge.

Claim 1. There exists a $u \in V(R)$ and a (u, C)-fan F such that $V(F) = \{u, w_i, w_j, w_k\}$.

Proof. Suppose that there is no such fan F with $V(F) = \{u, w_i, w_j, w_k\}$. By the above discussion, we may assume that the fan F in chosen in such a way that:

(1) $V(Q_1) = \{u, w_i\}$ and $w_i^- w_i^+ \in E(G)$;

(2) $|C(w_i, w_j)|$ is as small as possible subject to (1);

(3) $|V(Q_2)|$ is as small as possible subject to (1) and (2);

(4) $|C(w_k, w_i)|$ is as small as possible subject to (1), (2), and (3);

(5) $|V(Q_3)|$ is as small as possible subject to (1), (2), (3), and (4).

To complete the proof of Claim 1, we prove two subclaims.

Claim 1.1. $V(Q_2) = \{u, w_j\}$.

Proof. Suppose there exists a vertex in $V(Q_2) \setminus \{u, w_j\}$. Let $x = y_{r_2}$, and let $H = \langle V(Q_1) \cup V(Q_2) \cup C[w_i, w_j] \rangle - \{w_j\}$. By Lemma 3.9(c), we have that $w_i w_j^- \notin E(G)$, so $d_H(w_j^-, w_i) \geq 2$. Condition (2) in the choice of F implies that no vertex of $V(Q_2) \setminus \{w_j\}$ has a neighbor in $C(w_i, w_j)$. This implies that

$d_H(w_i,x) = d_F(w_i,x)$. If $xw_i \notin E(G)$, then $d_F(w_i,x) \geq 2$. Since $|V(Q_2)| > 2$, we have $d_H(w_j^-,x) = d_H(w_j^-,w_i) + d_F(w_i,x) \geq 4$. This yields a contradiction with the facts that G is almost distance-hereditary and $d_G(w_j^-,x) = 2$. Thus, $xw_i \in E(G)$. Let $F = (x; xw_i, xw_j, Q_2[x,u]Q_3[u,w_k])$. Then F is an (x,C)-fan satisfying (1), (2), and $|\{x,w_j\}| = 2$, contradicting condition (3) in the choice of F. Hence $V(Q_2) = \{u,w_j\}$. □

Claim 1.2. $V(Q_3) = \{u,w_k\}$.

Proof. Suppose there exists a vertex of $V(Q_3) \setminus \{u,w_k\}$. Let $x = z_{r_3}$. We first show that $xw_i \in E(G)$. Supposing that $xw_i \notin E(G)$, let $H = \langle V(Q_1) \cup V(Q_3) \cup C[w_k,w_i] \rangle - \{w_k\}$. By Lemma 3.9(c), we have that $w_iw_k^+ \notin E(G)$. This means $d_H(w_k^+,w_i) \geq 2$. By conditions (4) and (5) in the choice of F, no vertex of $V(Q_3) \setminus \{w_k\}$ has a neighbor in $C(w_k,w_i)$. Since $|V(Q_3)| > 2$, we have $d_H(w_i,x) \geq 2$, and we have $d_H(w_k^+,x) = d_H(w_k^+,w_i) + d_F(w_i,x) \geq 4$. This yields a contradiction with the facts that G is almost distance-hereditary and $d_G(w_k^+,x) = 2$. Thus, $xw_i \in E(G)$. If $xw_j \in E(G)$, then $(x; xw_i, xw_j, xw_k)$ is the required fan. Next suppose $xw_j \notin E(G)$.

Let $H = \langle C[w_i,w_j] \cup Q_3[u,x] \rangle - \{w_i\}$. By condition (2) in the choice of F, no vertex of $V(Q_3) \setminus \{w_k\}$ has a neighbor in $C(w_i,w_j)$. Since G is almost distance-hereditary and $d_G(w_i^+,x) = 2$, we conclude that $d_H(w_i^+,x) = 3$. This implies that $w_i^+w_j \in E(G)$. Consider the subgraph induced by $\{w_j, w_i^+, w_j^+, u\}$. This is clearly an induced claw. Recalling that w_i^+ and u are light, we conclude that w_j^+ is implicit heavy. We show in a similar way that w_j^- is implicit heavy. For this purpose, let $H = \langle \{w_i^-\} \cup C[w_i,w_j] \cup Q_3[u,x] \rangle - \{w_i\}$. Since G is almost distance-hereditary and $d_G(w_i^-,x) = 2$, we conclude that $d_H(w_i^-,x) = 3$, implying that $w_i^-w_j \in E(G)$. Now consider the claw induced by $\{w_j, w_i^-, w_j^-, u\}$, and recall that w_i^- and u are light. We obtain that w_j^- is implicit heavy.

Since both w_j^- and w_j^+ are implicit heavy, $w_j^-w_j^+ \in E^*(G)$. This contradicts Lemma 3.8, since there exists a path $P = w_j^-\overline{C}w_i^+w_juw_i\overline{C}w_j^+$ such that $V(C) \subset V(P)$. Hence $V(Q_3) = \{u,w_k\}$. □

This completes the proof of Claim 1. □

By Claim 1, there exists a (u,C)-fan F such that $V(F)\backslash V(C)=\{u\}$. Suppose that $N_C(u)=\{v_1,v_2,\ldots,v_r\}$ $(r\geq 3)$ and $v_1,v_2,\ldots,v_r$ are in the order of the orientation on C, meaning that u has no neighbors on C between v_i and v_{i+1} $(1\leq i\leq r-1)$ or between v_r and v_1. In the following, all subscripts of v are taken modulo r, with $v_0=v_r$. We continue with a simple claim.

Claim 2. For any vertex $v_i\in N_C(u)$ such that $v_i^-v_i^+\in E(G)$, there exists a vertex $l_i\in C[v_i^+,v_{i+1}^-)$ such that $v_{i+1}^-l_i\in E(G)$ and $v_il_i\in E(G)$; by symmetry, there exists a vertex $s_i\in C(v_{i-1}^+,v_i^-]$ such that $v_{i-1}^+s_i\in E(G)$ and $v_is_i\in E(G)$.

Proof. Let $H=\langle\{u\}\cup C[v_i,v_{i+1})\rangle$. Since $d_G(v_{i+1}^-,u)=2$ and G is almost distance-hereditary, we have $d_H(v_{i+1}^-,u)\leq 3$. By Lemma 3.9(c), we have $v_iv_{i+1}^-\notin E(G)$. Thus $d_H(v_{i+1}^-,u)=3$. It follows that $d_H(v_{i+1}^-,v_i)=2$. So there exists a vertex $l_i\in C[v_i^+,v_{i+1}^-)$ such that $v_{i+1}^-l_i\in E(G)$ and $v_il_i\in E(G)$. The other assertion can be proved similarly. □

By Lemma 3.9(b), there is at most one implicit heavy vertex in $N_C^+(u)$ and at most one implicit heavy vertex in $N_C^-(u)$. As before, this implies there exists a vertex $v_j\in N_C(u)$ such that v_j^-,v_j^+ are light and $v_j^-v_j^+\in E(G)$. Without loss of generality, we assume that v_1^-,v_1^+ are light, hence $v_1^-v_1^+\in E(G)$. In the remainder of the proof, we distinguish two cases, depending on whether one of v_2 and v_r has a similar property or both have not. We start with the latter case.

Case 1. $v_2^-v_2^+\notin E(G)$ and $v_r^-v_r^+\notin E(G)$.

In this case, both $\{v_2,v_2^-,v_2^+,u\}$ and $\{v_r,v_r^-,v_r^+,u\}$ induce claws. Using the fact that G is implicit 1-heavy and Lemma 3.9(b), we conclude that both v_2^- and v_r^+ are implicit heavy or both v_2^+ and v_r^- are implicit heavy. Before we distinguish these two subcases, we first prove the following claim.

Claim 3.

(1) $v_1,v_1^-,v_1^+,l_1^-,l_1^+,s_1^-,s_1^+$ are light;

(2) $v_1l_1^-\in E(G)$ and $v_1s_1^+\in E(G)$;

(3) l_1 and s_1 are light;

(4) $v_1^- v_2 \notin E(G)$;

(5) $v_1^- l_1 \in E(G)$ and $l_1 v_2 \in E(G)$.

Proof.

(1) We already know that v_1^- and v_1^+ are light by our assumptions. Since v_2^-, v_r^+ are implicit heavy or v_2^+, v_r^- are implicit heavy, by Lemma 3.9(a), v_1 is light. To prove that $l_1^-, l_1^+, s_1^-, s_1^+$ are light, we consider two subcases.

Case a. v_2^-, v_r^+ are implicit heavy.

Then l_1^- is light; otherwise $P = l_1^- \overline{C} v_1^+ v_1^- \overline{C} v_2 u v_1 l_1 C v_2^-$ is a path such that $V(C) \subset V(P)$ and $l_1^- v_2^- \in E^*(G)$, contradicting Lemma 3.8. Similarly, by symmetry we can prove that s_1^+ is light. Next, we claim that l_1^+ is light; otherwise $P = l_1^+ C v_r u v_1 l_1 \overline{C} v_1^+ v_1^- \overline{C} v_r^+$ is a path such that $V(C) \subset V(P)$ and $l_1^+ v_r^+ \in E^*(G)$, contradicting Lemma 3.8. Similarly, by symmetry we can prove that s_1^- is light.

Case b. v_2^+, v_r^- are implicit heavy.

Considering the path $P = l_1^+ C v_2 u v_1 l_1 \overline{C} v_1^+ v_1^- \overline{C} v_2^+$ and Lemma 3.8, we obtain that l_1^+ is light. Similarly, by symmetry we obtain that s_1^- is light. Considering the path $P = l_1^- \overline{C} v_1^+ v_1^- \overline{C} v_r u v_1 l_1 C v_r^-$ and Lemma 3.8, we obtain that l_1^- is light. Similarly, by symmetry we obtain that s_1^+ is light.

(2) Suppose that $v_1 l_1^- \notin E(G)$. Note that $v_1 l_1^+ \notin E^*(G)$ and $l_1^- l_1^+ \notin E^*(G)$ by Lemma 3.9(d). Since l_1^+, v_1, l_1^- are all light, in this case $\{l_1, l_1^+, v_1, l_1^-\}$ induces a light claw, a contradiction. The other assertion follows by symmetry.

(3) Since $v_1 l_1^- \in E(G)$, using Lemma 3.9(e) we get $v_2^+ l_1 \notin E^*(G)$ and $v_r^+ l_1 \notin E^*(G)$. Note that either v_r^+ or v_2^+ is implicit heavy. This implies that l_1 is light. Similarly, we can prove that s_1 is light.

(4) Suppose that $v_1^- v_2 \in E(G)$ and v_2^-, v_r^+ are implicit heavy. Then $P = v_2^- \overline{C} v_1 u v_r \overline{C} v_2 v_1^- \overline{C} v_r^+$ is a path such that $V(C) \subset V(P)$ and $v_2^- v_r^+ \in E^*(G)$, contradicting Lemma 3.8. Suppose that $v_1^- v_2 \in E(G)$ and v_2^+, v_r^- are implicit heavy. Then $\{v_2, v_2^-, u, v_1^-\}$ induces a light claw, a contradiction.

(5) Suppose that $v_1^- l_1 \notin E(G)$. Note that $uv_1^- \notin E^*(G)$ by Lemma 3.9(a), and that $ul_1 \notin E(G)$ by assumption. Now $\{v_1, l_1, u, v_1^-\}$ induces a light claw, a contradiction. Hence $v_1^- l_1 \in E(G)$. Suppose that $l_1 v_2 \notin E(G)$. Then $H = \langle\{v_1^-, l_1, v_2^-, v_2, u\}\rangle$ is an induced path of length 4 in G. It follows that $d_H(v_1^-, u) = 4$, contradicting the facts that $d_G(v_1^-, u) = 2$ and G is almost distance-hereditary. Hence $l_1 v_2 \in E(G)$.

This completes the proof of Claim 3. □

We continue by considering the aforementioned two subcases.

Case 1.1. v_2^-, v_r^+ are implicit heavy.

By Lemma 3.9(c) and $v_1 l_1^- \in E(G)$, $ul_1 \notin E^*(G)$. By Lemma 3.9(a) and 3.9(e), we have $uv_2^+ \notin E^*(G)$ and $l_1 v_2^+ \notin E^*(G)$. Using Claim 3(3), we conclude that $\{v_2, l_1, u, v_2^+\}$ induces a light claw, a contradiction.

Case 1.2. v_2^+, v_r^- are implicit heavy.

We first prove another claim.

Claim 4. $\{v_r^+, s_1, l_1, v_2, u\}$ induces a path in G.

Proof. It is easy to check that $v_r^+ s_1 \in E(G)$, $l_1 v_2 \in E(G)$ and $v_2 u \in E(G)$. By Lemma 3.9(a), $v_r^+ u \notin E(G)$. By using Lemma 3.8 with respect to the path $P = v_r^+ C v_1^- v_1^+ C l_1^- v_1 u v_r \overline{C} l_1$, we obtain that $v_r^+ l_1 \notin E(G)$. We also have that $ul_1 \notin E(G)$ and $us_1 \notin E(G)$. It remains to show that $s_1 l_1 \in E(G)$, $v_r^+ v_2 \notin E(G)$ and $s_1 v_2 \notin E(G)$. This is done by proving the next two subclaims.

Claim 4.1. $s_1 l_1 \in E(G)$.

Proof. Otherwise, $\{v_1, l_1, s_1, u\}$ induces a light claw, a contradiction. □

Claim 4.2. $v_r^+ v_2 \notin E(G)$ and $s_1 v_2 \notin E(G)$.

Proof. Suppose that $v_r^+ v_2 \in E(G)$. Consider the subgraph $\langle\{v_2, v_2^-, v_r^+, u\}\rangle$. By Lemma 3.9(a), we have $uv_2^- \notin E(G)$ and $uv_r^+ \notin E(G)$. Since v_2^-, v_r^+, u are light and G is implicit 1-heavy, we conclude that $v_r^+ v_2^- \in E(G)$. Now the cycle $C' = uv_1 l_1^- \overline{C} v_1^+ v_1^- \overline{C} v_r^+ v_2^- \overline{C} l_1 v_2 C v_r u$ contradicts the assumption that

C is a longest implicit heavy cycle. Suppose that $s_1v_2 \in E(G)$. Consider the subgraph induced by $\{v_2, v_2^-, s_1, u\}$. Since $us_1 \notin E(G)$, $uv_2^- \notin E(G)$ and G is implicit 1-heavy, we conclude that $s_1v_2^- \in E(G)$. Now the cycle $C'' = uv_2Cs_1v_2^-\overline{C}v_1^+v_1^-\overline{C}s_1^+v_1u$ contradicts the assumption that C is a longest implicit heavy cycle. □

This completes the proof of Claim 4, hence that $\{v_r^+, s_1, l_1, v_2, u\}$ induces a path in G. □

Let $H = \langle\{v_r^+, s_1, l_1, v_2, u\}\rangle$. By Claim 4, $d_H(v_r^+, u) = 4$, contradicting the facts that $d_G(v_r^+, u) = 2$ and G is almost distance-hereditary. This completes the proof for Case 1.

Case 2. $v_2^-v_2^+ \in E(G)$ or $v_r^-v_r^+ \in E(G)$.

In both cases, we have that $v_i^-v_i^+ \in E(G)$ and $v_{i+1}^-v_{i+1}^+ \in E(G)$ (for $i = 1$ or $i = r$). By Claim 2, for any vertex $v_i \in N_C(u)$ such that $v_i^-v_i^+ \in E(G)$, there exists a vertex $l_i \in C[v_i^+, v_{i+1}^-)$ such that $v_{i+1}^-l_i \in E(G)$ and $v_il_i \in E(G)$. Before we continue with distinguishing two subcases, we first prove the following claim.

Claim 5. If $v_i^-v_i^+ \in E(G)$ and $v_{i+1}^-v_{i+1}^+ \in E(G)$, then $v_{i+1}^+l_i \in E(G)$ and $\{l_i, l_i^-, v_i, v_{i+1}^-\}$ induces a claw.

Proof. Suppose that $v_{i+1}^+l_i \notin E(G)$. Let $H = \langle\{v_{i+1}^+, v_{i+1}^-, l_i, v_i, u\}\rangle$. We have $v_iv_{i+1}^- \notin E(G)$ and $v_iv_{i+1}^+ \notin E(G)$ by Lemma 3.9(c). We observe that H is an induced path of length 4 in G. Hence $d_H(v_{i+1}^+, u) = 4$, contradicting the facts that $d_G(v_{i+1}^+, u) = 2$ and G is almost distance-hereditary. We conclude that $v_{i+1}^+l_i \in E(G)$. By Lemma 3.9(c), we have $v_iv_{i+1}^- \notin E^*(G)$. By Lemma 3.9(e), we have $l_i^-v_{i+1}^- \notin E^*(G)$. Supposing that $v_il_i^- \in E(G)$, the cycle $C' = v_{i+1}^+Cv_i^-v_i^+Cl_i^-v_iuv_{i+1}\overline{C}l_iv_{i+1}$ contradicts the choice of C. Hence, $v_il_i^- \notin E(G)$, and $\{l_i, l_i^-, v_i, v_{i+1}^-\}$ induces a claw. □

By Claim 5 and Lemma 3.9(b), we have $l_i \neq v_i^+$. Without loss of generality, by symmetry, assume that $v_2^-v_2^+ \in E(G)$. We distinguish two subcases.

Case 2.1. $v_r^-v_r^+ \notin E(G)$.

By Lemma 3.9(a), we have $uv_r^- \notin E(G)$ and $uv_r^+ \notin E(G)$. Now $\{v_r, v_r^-, v_r^+, u\}$ induces a claw. Since G is implicit 1-heavy and u is light, v_r^- is implicit heavy or v_r^+ is implicit heavy.

Claim 6. v_r^+ is implicit heavy.

Proof. Suppose that v_r^- is implicit heavy. By Lemma 3.9(b) and (c), v_2^-, v_1 are light. Using the path $P = l_1^- \overline{C} v_1^+ v_1^- \overline{C} v_r u v_1 l_1 C v_r^-$ and Lemma 3.8, we conclude that l_1^- is light. By Claim 5, $\{l_1, l_1^-, v_1, v_2^-\}$ induces a light claw, a contradiction. □

Claim 7. v_1^+, v_2^+, v_1, l_1^+ are light.

Proof. By Claim 6, v_r^+ is implicit heavy. By Lemma 3.9(b) and (c), we have that v_1^+, v_2^+, v_1 are light. Using Lemma 3.8 and $P = l_1^+ C v_r u v_1 l_1 \overline{C} v_1^+ v_1^- \overline{C} v_r^+$, we conclude that l_1^+ is light. □

Claim 8. $v_1 l_1^+ \notin E(G)$.

Proof. Suppose that $v_1 l_1^+ \in E(G)$. Then the cycle $uv_1 l_1^+ C v_2^- l_1 \overline{C} v_1^+ v_1^- \overline{C} v_2 u$ contradicts the choice of C. □

Now consider the subgraph $\langle \{l_1, l_1^+, v_1, v_2^+\} \rangle$. Using Claim 6, Lemma 3.9(c) and (e), we have $v_1 v_2^+ \notin E^*(G)$ and $l_1^+ v_2^+ \notin E^*(G)$. By Claims 7 and 8, we have $v_1 l_1^+ \notin E^*(G)$. Hence $\{l_1, l_1^+, v_1, v_2^+\}$ induces a light claw, a contradiction. This completes the proof for Subcase 2.1.

Case 2.2. $v_r^- v_r^+ \in E(G)$.

We again proceed by proving a number of claims.

Claim 9. v_1 is implicit heavy.

Proof. By Claim 5, $\{l_r, l_r^-, v_r, v_1^-\}$ and $\{l_1, l_1^-, v_1, v_2^-\}$ both induce claws. Suppose first that v_2^- is implicit heavy. By Lemma 3.9(b) and (c), we conclude that v_1^- and v_r are light. Using Lemma 3.8 with respect to the path $P = l_r^- \overline{C} v_r^+ v_r^- \overline{C} v_2 u v_r l_r C v_2^-$, we conclude that l_r^- is light. Now $\{l_r, l_r^-, v_r, v_1^-\}$ induces a light claw, a contradiction. Hence, v_2^- is light. Suppose next that l_1^- is implicit heavy. Using Lemma 3.8 with the paths $P = l_1^-\overleftarrow{C}v_r^+v_r^-\overleftarrow{C}v_2^+l_1Cv_2uv_r$, $P' = l_1^-\overleftarrow{C}v_1uv_2\overleftarrow{C}l_1v_2^+Cv_1^-$, and $P'' = l_1^-\overleftarrow{C}l_rv_ruv_2\overleftarrow{C}l_1v_2^+Cv_r^-v_r^+Cl_r^-$, we conclude that $l_1^-v_r \notin E^*(G)$, $l_1^-v_1^- \notin E^*(G)$ and $l_1^-l_r^- \notin E^*(G)$. This implies that v_r, v_1^-, l_r^- are light. Now $\{l_r, l_r^-, v_r, v_1^-\}$ induces a light claw, a contradiction. Hence, l_1^- is light. Noting that $\{l_1, l_1^-, v_1, v_2^-\}$ induces a claw and recalling that v_2^-, l_1^- are both light, we conclude that v_1 is implicit heavy. □

By Claim 2, there exists a vertex $s_2 \in C(v_1^+, v_2^-]$ such that $v_1^+s_2 \in E(G)$ and $v_2s_2 \in E(G)$.

Claim 10. $l_r^-, l_r^+, s_2^-, s_2^+$ are light.

Proof. By Claim 5, $l_rv_1^+ \in E(G)$. Using Lemma 3.8 with respect to the paths $P = l_r^+Cv_1^-v_1^+Cv_r^-v_r^+Cl_rv_ruv_1$ and $P' = l_r^-\overleftarrow{C}v_r^+v_r^-\overleftarrow{C}v_1^+v_1^-\overleftarrow{C}l_rv_ruv_1$, we conclude that $l_r^+v_1 \notin E^*(G)$ and $l_r^-v_1 \notin E^*(G)$. Since v_1 is implicit heavy (by Claim 9), l_r^+ and l_r^- are light. Similarly, by symmetry, we can prove that s_2^-, s_2^+ are light. □

Claim 11. v_r is implicit heavy.

Proof. Consider the subgraph $\langle\{l_r, l_r^-, l_r^+, v_r\}\rangle$. By Claim 5 and Lemma 3.9(e), we have $v_rl_r^- \notin E(G)$. By Lemma 3.9(d), we have $l_r^-l_r^+ \notin E^*(G)$ and $v_rl_r^+ \notin E(G)$. Since G is implicit 1-heavy, using Claim 10, we conclude that v_r is implicit heavy. □

Claim 12. $\{v_1^-, v_1^+, l_r^+, s_2^+\}$ induces a light claw.

Proof. By Claim 10, l_r^+, s_2^+ are light. Recall that v_1^+ is light by assumption. Now it is sufficient to prove the following facts: $\{v_1^-l_r^+, v_1^-s_2^+\} \subset E(G)$ and $\{v_1^+l_r^+, v_1^+s_2^+, l_r^+s_2^+\} \cap E(G) = \emptyset$. This will be done in the form of the following two subclaims.

Claim 12.1. $\{v_1^-l_r^+, v_1^-s_2^+\} \subset E(G)$.

Proof. Suppose that $v_1^-l_r^+ \notin E(G)$. By Lemma 3.9(d) and (e), we have $l_r^-l_r^+ \notin E^*(G)$ and $v_1^-l_r^- \notin E^*(G)$. Now $\{l_r, l_r^-, l_r^+, v_1^-\}$ induces a light claw, a contradiction. Suppose that $v_1^-s_2^+ \notin E(G)$. By Claim 5 and symmetry, we have $v_1^-s_2 \in$

$E(G)$. Consider the subgraph induced by $\{s_2, s_2^-, s_2^+, v_1^-\}$. By Lemma 3.9(d) and (e), we have $s_2^- s_2^+ \notin E^*(G)$ and $v_1^- s_2^- \notin E^*(G)$. Now $\{s_2, s_2^-, s_2^+, v_1^-\}$ induces a light claw, a contradiction. □

Claim 12.2. $\{v_1^+ l_r^+, v_1^+ s_2^+, l_r^+ s_2^+\} \cap E(G) = \emptyset$.

Proof. By Lemma 3.9(e), since $v_r l_r \in E(G)$ and $v_2 s_2 \in E(G)$, we have $v_1^+ l_r^+ \notin E^*(G)$ and $v_1^+ s_2^+ \notin E^*(G)$. Using Lemma 3.8 with respect to the path $P = l_r^+ C v_1^- s_2 \overline{C} v_1 u v_r l_r \overline{C} v_r^+ v_r^- \overline{C} s_2^+$, we conclude that $l_r^+ s_2^+ \notin E^*(G)$. □

This completes the proof of Claim 12, hence $\{v_1^-, v_1^+, l_r^+, s_2^+\}$ induces a light claw. □

Since Claim 12 contradicts the assumptions, this completes the proof of Theorem 3.6. ■

Chapter 4

Implicit degree for hamiltonicity of claw-free graphs

Degree conditions are among the most studied sufficient conditions for hamiltonicity, inspired by the classical result due to Dirac that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \frac{n}{2}$ has a Hamilton cycle. The lower bound on $\delta(G)$ in this result is best-possible, but Matthews and Sumner proved that it can be improved to $\delta(G) \geq \frac{n-2}{3}$ if G is claw-free and 2-connected. In another direction, Zhu, Li and Deng introduced the implicit degree and generalized Dirac's result by replacing $\delta(G)$ in the condition by the minimum implicit degree. All of the above results have been generalized to conditions involving the minimum degree sum of sets of independent vertices. In this chapter, we discuss a combination of the two approaches, i.e., we aim for best-possible implicit degree conditions that guarantee the hamiltonicity of claw-free graphs. In contrast to our earlier expectations, we cannot prove straightforward analogues of the above results, and we explain why this does not work.

4.1 Introduction

The research on sufficient degree conditions for hamiltonicity of graphs originates with Dirac [33], one of the leading graph theorists of the 1950s. He developed methods of great originality and made many fundamental discoveries. He showed in a paper of 1952 [33] that a graph G on $n \geq 3$ vertices is hamiltonian if $\delta(G) \geq \frac{n}{2}$. This result is best-possible in the sense that the lower bound $\frac{n}{2}$ in this result cannot be relaxed, i.e., there exist infinitely many nonhamiltonian graphs in which all vertices have degree at least $\frac{n-1}{2}$. Nevertheless, one can try to improve, extend or generalize Dirac's result in many other different ways. And, in fact this has been done to a great extent! Instead of listing a large number of related results here, we refer the reader to the excellent survey paper due to Gould [45] and the references therein, including references to earlier surveys, and we stick here to the most relevant ones for our exposition.

Dirac's result has inspired multiple lines of research aimed at finding milder sufficient conditions for hamiltonicity of graphs that can either be applied to a larger class of general graphs, or to a specific class of graphs. We focus mainly on two of these lines of research here, namely the restriction to claw-free graphs and the introduction of the notion of the implicit degree. Our aim is to combine the two approaches, as we will explain shortly. Apart from Dirac's classical result, our starting point is the following analogue of Dirac's result for claw-free graphs.

Theorem 4.1 (Matthews and Sumner [64]). *Let G be a 2-connected claw-free graph of order n such that $\delta(G) \geq \frac{n-2}{3}$. Then G is hamiltonian.*

Note that the lower bound on the minimum degree in Dirac's result can be improved considerably from $\frac{n}{2}$ to $\frac{n-2}{3}$, by restricting the statement to claw-free graphs. Like Dirac's result, the above theorem is best-possible, in the sense that there are infinitely many nonhamiltonian 2-connected claw-free graphs in which all vertices have degree at least $\frac{n-1}{3}$. Unlike Dirac's result however, it is known that the lower bound in the above result can be relaxed considerably further by imposing a larger connectivity, or by excluding specific infinite families of claw-free graphs. The degree condition can even be omitted completely if the imposed connectivity is sufficiently large. We will refrain from listing such results explicitly here, since they are not relevant for our purpose. For a discussion on such results and several open problems on claw-free graphs we refer the interested reader to Section 4 in [14] and to Section 7 in [18]. There is one result due to Zhang that we do want to state explicitly for our discussion.

Theorem 4.2 (Zhang [77])**.** *Let G be a k-connected claw-free graph of order n such that $k \geq 2$ and $\Sigma_{v \in S} d(v) \geq n-k$ for any independent set S of $k+1$ vertices. Then G is hamiltonian.*

Note that for $k=2$, the above result is more general than Theorem 4.1. We will use an analogue of this result (Lemma 4.6 in the next section) for preparing the set-up for our proofs.

We use $\delta_1(G)$ to denote the minimum implicit degree of a graph G. In [79], Zhu, Li and Deng presented generalizations of such degree condition results, implying the following analogue of Dirac's result involving the minimum implicit degree.

Theorem 4.3 (Zhu, Li and Deng [79])**.** *Let G be a graph on n vertices with $\delta_1(G) \geq \frac{n}{2}$. Then G is hamiltonian.*

For more results involving implicit degrees, we refer the reader to [30, 62, 76, 78]. Motivated by the above results, it is natural to expect that the following counterpart of Theorem 4.1 holds: any 2-connected claw-free graph G of order n with $\delta_1(G) \geq \frac{n-2}{3}$ is hamiltonian. However, unfortunately this is not the case. The following infinite class of nonhamiltonian claw-free graphs shows that this statement is false. Suppose n is a sufficiently large integer that is divisible by 3. Let G be the graph of order n consisting of three vertex-disjoint complete graphs on $\frac{n}{3}$ vertices each, with the additional edges of two vertex-disjoint triangles, each containing one vertex from each of the three complete graphs. Then one readily checks that G is claw-free, nonhamiltonian and that $\delta_1(G) \geq \frac{n+3}{3}$. So, there does not exist a straightforward counterpart of Theorem 4.1 in which $\delta(G)$ is simply replaced by $\delta_1(G)$. Exploring the structure of the above examples in more detail, note that for every 2-cut

$\{u, v\}$ of G, $G - \{u, v\}$ has a complete component of order $\frac{n-6}{3}$. If we exclude this by imposing an additional condition, we can prove the following result.

Theorem 4.4. *Let G be a 2-connected claw-free graph on n vertices with $\delta_1(G) \geq \frac{n+3}{3}$. If G contains no 2-cut $\{u, v\}$ such that $G - \{u, v\}$ has a complete component of order less than $\frac{n-5}{3}$, then G is hamiltonian.*

We postpone the proof of Theorem 4.4 to Section 4.3. Note that the additional condition is easy to check, since any 2-cut in a 2-connected claw-free graph leaves precisely two components. The examples show that the condition on the 2-cuts in the above result cannot be omitted and is sharp. We clearly get rid of this condition if we restrict ourselves to 3-connected claw-free graphs. However, in this case we can lower the degree bound by a factor if we add a similar condition on the 3-cuts, and obtain the following result.

Theorem 4.5. *Let G be a 3-connected claw-free graph on n vertices with $\delta_1(G) \geq \frac{n+6}{4}$. If G contains no 3-cut $\{u, v, w\}$ such that $G - \{u, v, w\}$ has a complete component of order less than $\frac{n-7}{4}$, then G is hamiltonian.*

We continue with some standard, but useful observations and some preliminary results for setting up the proofs of Theorems 4.4 and 4.5.

4.2 Some useful observations and preliminary results

We start with some notational conventions that are used in the previous chapters, but we repeat them here for convenience. For a cycle C in a graph G with a chosen fixed orientation and a vertex x on C, we use x^+ and x^- to denote the (immediate) successor and the (immediate) predecessor of x on C, respectively. Similarly, for any $I \subseteq V(C)$, we let $I^- = \{x^- \mid x \in I\}$ and $I^+ = \{x^+ \mid x \in I\}$. We use x^{+2} for $(x^+)^+$, et cetera. For two vertices $x, y \in V(C)$, we use xCy to denote the subpath (or segment) of C from x to y in the direction of the orientation of C. We use $y\overline{C}x$ for the same path in the reverse direction, and we use $|xCy|$ for the number of vertices of xCy (including x and y). Similar notations are used for paths. Suppose that H is a component of $G - V(C)$. Then for any two distinct vertices

$x, y \in N_C(H) \cup V(H)$, there exists an (x, y)-path in G whose internal vertices are all in $V(H)$. Sometimes we denote such a path by xHy without specifying its internal vertices.

We will use the following variation for claw-free graphs on a cyclability result due to Shi [72] for general graphs. It is the cyclability analogue of Theorem 4.2, i.e., Zhang's result on hamiltonicity of claw-free graphs [77]. The result is an immediate consequence of Theorem 5 in [15].

Lemma 4.6 (Broersma and Lu [15]). *Let G be a k-connected claw-free graph on n vertices, and let $S = \{v \in V(G) \mid d(v) \geq \frac{n-k}{k+1}\}$. Then G has a cycle containing all vertices of S.*

Recall that a cycle C is called (non)extendable if there is a (there is no) longer cycle in G containing all vertices of C. We continue with listing the following useful observations about nonextendable cycles.

Lemma 4.7. *Let G be a 2-connected claw-free nonhamiltonian graph of order n, and let C be a nonextendable cycle of G. Define $R = G - V(C)$, let H be a component of R, and let $A = \{v_1, v_2, \ldots, v_k\}$ denote the set of neighbors of H on C in the order of a chosen fixed orientation of C. For any distinct $v_i, v_j \in A$ (assuming $1 \leq i < j \leq k$ without loss of generality), we have:*

(a) $A \cap A^+ = \emptyset$, $A \cap A^- = \emptyset$;

(b) $v_i^- v_i^+ \in E(G)$;

(c) $|v_i^+ C v_{i+1}^-| \geq 3$;

(d) $\{v_j^{-2}, v_j^-, v_j^+, v_j^{+2}\} \cap N(v_i) = \emptyset$;

(e) $\{v_j^{-2}, v_j^-\} \cap N(v_i^-) = \emptyset$, $\{v_j^+, v_j^{+2}\} \cap N(v_i^+) = \emptyset$;

(f) $N^-_{v_i C v_j}(v_i^+) \cap N_{v_i C v_j}(v_j^+) = \emptyset$, $N_{v_j C v_i}(v_i^+) \cap N^-_{v_j C v_i}(v_j^+) = \emptyset$;

(g) $|N_{v_i C v_j}(v_i^+) \cap N_{v_i C v_j}(v_j^+)| \leq 1$, $|N_{v_j C v_i}(v_i^+) \cap N_{v_j C v_i}(v_j^+)| \leq 1$;

(h) $N_R(v_i^+) \cap N_R(v_j^+) = \emptyset$, $N_R(v_i^-) \cap N_R(v_j^-) = \emptyset$.

Proof. (a) If $A \cap A^+ \neq \emptyset$ or $A \cap A^- \neq \emptyset$, it means that there are two consecutive vertices on C that are both neighbors of H. Then C is clearly extendable, a contradiction.

(b) Suppose the neighbor of v_i in H is x_i. By (a), v_i^-, v_i^+ are not neighbors of x_i. If $v_i^- v_i^+ \notin E(G)$, then $\{v_i, v_i^-, v_i^+, x_i\}$ induces a claw, a contradiction.

(c) In case $|v_i^+ C v_{i+1}^-| = 1$ or $|v_i^+ C v_{i+1}^-| = 2$, it is easy to show that C is extendable, a contradiction.

(d) Suppose that $v_i v_j^{-2} \in E(G)$. Using (c), we know $v_j^{-2} \neq v_i^+$. Now the cycle $C' = v_i H v_j v_j^- v_j^+ C v_i^- v_i^+ C v_j^{-2} v_i$ clearly contradicts the choice of C. Suppose that $v_i v_j^- \in E(G)$. Then the cycle $C'' = v_i H v_j C v_i^- v_i^+ C v_j^- v_i$ contradicts the choice of C. By symmetry, $v_i v_j^+ \notin E(G)$ and $v_i v_j^{+2} \notin E(G)$.

(e) Suppose that $v_i^- v_j^{-2} \in E(G)$. Then the cycle $C' = v_i H v_j v_j^- v_j^+ C v_i^- v_j^{-2} \overline{C} v_i$ contradicts the choice of C. Suppose that $v_i^- v_j^- \in E(G)$. Then the cycle $C'' = v_i H v_j C v_i^- v_j^- \overline{C} v_i$ contradicts the choice of C. By symmetry, $v_i^+ v_j^+ \notin E(G)$ and $v_i^+ v_j^{+2} \notin E(G)$.

(f) To the contrary, suppose there is a vertex $x \in N^-_{v_i C v_j}(v_i^+) \cap N_{v_i C v_j}(v_j^+)$. Using (d) and (e), $x \notin \{v_i, v_i^+, v_i^{+2}, v_j^-, v_j\}$. Now $C' = v_i H v_j \overline{C} x^+ v_i^+ C x v_j^+ C v_i$ contradicts the choice of C. By symmetry, $N_{v_j C v_i}(v_i^+) \cap N^-_{v_j C v_i}(v_j^+) = \emptyset$.

(g) Suppose that $|N_{v_i C v_j}(v_i^+) \cap N_{v_i C v_j}(v_j^+)| \geq 2$. Let x and y be two distinct vertices in $N_{v_i C v_j}(v_i^+) \cap N_{v_i C v_j}(v_j^+)$, with x before y in the order of the orientation. Using (d), (e) and (f), we know $x, y \in v_i^{+3} C v_j^{-3}$, and $x^-, y^- \notin N(v_j^+)$, $x^+, y^+ \notin N(v_i^+)$. In order to avoid an induced claw with center x or y, we have $x^-, y^- \in N(v_i^+)$, $x^+, y^+ \in N(v_j^+)$, and, using (f), we know $|x^+ C y^-| \geq 3$.

Next, we first prove that $\{x^-, x, y^-, y\}$ induces a K_4. For any vertex $p \in N_{v_i C v_j}(v_j^+)$, we have $v_i^- p^- \notin E(G)$; otherwise $C' = v_i H v_j \overline{C} p v_j^+ C v_i^- p^- \overline{C} v_i$ contradicts the choice of C. So, $x^-, x, y^-, y \notin N(v_i^-)$. To avoid induced claws with center v_i^+ and v_i^- as one of the end vertices, we conclude that $\{x^-, x, y^-, y\}$ induces a K_4. Now, using that $|x^+ C y^-| \geq 3$, we observe that $C' = v_i H v_j \overline{C} y x^- \overline{C} v_i^+ x y^- \overline{C} x^+ v_j^+ C v_i$ contradicts the choice of C. By symmetry, $|N_{v_j C v_i}(v_i^+) \cap N_{v_j C v_i}(v_j^+)| \leq 1$.

(h) From (a), we know that v_i^+ and v_j^+ have no neighbors in H. If $N_R(v_i^+) \cap N_R(v_j^+) \neq \emptyset$, then there is another component H' of R such that $v_i^+, v_j^+ \in N_C(H')$. Now there is a cycle $C' = v_i H v_j \overline{C} v_i^+ H' v_j^+ C v_i$ contradicting the choice of C. By symmetry, $N_R(v_i^-) \cap N_R(v_j^-) = \emptyset$.

□

4.3 The proofs of our two results

We start with a rough sketch of the common ingredients of the proofs, and provide the details for the two different results in the subsections that follow. The main idea of the proof for k-connected graphs ($k = 2, 3$) is that we start with a longest nonextendable cycle C among all cycles that contain all vertices of degree at least $\frac{n-k}{k+1}$. Such a cycle exists by Lemma 4.6. Assuming that C is not a Hamilton cycle, we consider a component W of $G-V(C)$. Clearly, using the lower bounds $\frac{n+3}{3}$ and $\frac{n+6}{4}$ on $\delta_1(G)$ for $k = 2$ and $k = 3$, respectively, for a vertex $w \in W$, we have that $id(w) \geq \delta_1(G)$ and $d(w) < \delta_1(G)$, implying that $d^*(w) \geq \delta_1(G)$, and hence that $d_{\ell+1}^w \geq \delta_1(G)$, where $d(w) = \ell + 1$. The latter inequality can be strengthened to $d_\ell^w \geq \delta_1(G)$ if $M_2(w) \geq d_{\ell+1}^w$.

Using the hypothesis of the theorem and the definition of implicit degree we are able to find at least $k+1$ neighbors of W on the cycle. Using the lower bound on the implicit degrees, among these neighbors of W on C and their immediate predecessors and successors on C, we can find suitable vertices with sufficiently high degree at least $\delta_1(G)$. The statements of the technical Lemma 4.7 and additional counting arguments are then applied to reach a contradiction.

4.3.1 The proof of Theorem 4.4

Suppose G is a graph that satisfies the conditions of Theorem 4.4. Let C be a longest nonextendable cycle among all cycles in G that contain all vertices with degree at least $\frac{n-2}{3}$. Such a cycle exists by Lemma 4.6. If C is a Hamilton cycle, there is nothing to prove. Assuming that C is not a Hamilton cycle, consider a component W of $G-V(C)$. Clearly, W only contains vertices whose degree is smaller than their implicit degree. Since G is 2-connected, W has at least two neighbors on C. We first prove three claims before we start the counting arguments.

Claim 1. W has at least three neighbors on C.

Proof. Suppose to the contrary that $|N(W) \cap V(C)| = 2$. We first deal with the case that W is not a complete subgraph of G. In this case, let w_1 and w_2 denote two vertices at distance 2 in W, such that $|N(w_1) \cap V(C)| \leq |N(w_2) \cap V(C)|$. Clearly, w_1 and w_2 cannot have a common neighbor on C, since G is claw-free and by the choice of C. This implies that $d(w_1) \leq |N(w_1) \cap V(W)| + 1$. Obviously, since $d(w_1, w_2) = 2$, $|(N(w_1) \cup N_2(w_1)) \cap V(W)| \geq |N(w_1) \cap V(W)| + 1$. Applying the definition of implicit degree with $d(w_1) = \ell + 1$, and recalling that all vertices of W have degree smaller than $\frac{n-2}{3}$, we conclude that $d^{w_1}_{\ell+1} < \frac{n-2}{3}$, a contradiction. The remaining case is that W is a complete subgraph of G. If one of the vertices of W has two neighbors on C, then this vertex has degree $|W| + 1$, which is at least $\frac{n-2}{3}$ by the hypothesis on 2-cuts of the theorem, a contradiction. In the other case all vertices of W have at most one neighbor on C. Then there exist two vertices z_1, z_2 in W such that $z_1 v_1 \in E(G)$ and $z_2 v_2 \in E(G)$ for two distinct vertices v_1, v_2 on C. Then $d(z_1, v_2) = d(z_2, v_1) = 2$. Let $d(z_1) = \ell + 1$. Then $d(z_1) = d_W(z_1) + d_C(z_1) = |W| = \ell + 1$. Since all ℓ neighbors of z_1 in W have degree less than $\delta_1(G)$, the implicit degree of z_1 is equal to $d^{z_1}_{\ell+1}$. From the definition of implicit degree, we conclude that z_1 satisfies the condition $M_2(z_1) < d^{z_1}_{\ell+1}$. This implies that $id(z_1) = d(v_1)$, and hence that $d(v_1) > d(v_2)$, since $v_2 \in N_2(z_1)$. On the other hand, by analyzing the vertex z_2, and using symmetry, we obtain that $d(v_2) > d(v_1)$, a contradiction. We conclude that $|N(W) \cap V(C)| \geq 3$. □

Claim 2. For any neighbor w of C in W, at least $2d_C(w) + 2$ of the neighbors of w on C and their immediate successors and predecessors on C have degree at least $\delta_1(G)$.

Proof. For a fixed neighbor w of C in W, let $d(w) = \ell + 1$. By Lemma 4.7(c), it is clear that $|N(w) \cup N_2(w)| \geq d_W(w) + 3d_C(w)$. Since $d^w_{\ell+1} = d^w_{d_W(w)+d_C(w)} \geq \delta_1(G)$, we get that $d^w_i \geq \delta_1(G)$ for any $i \geq d_W(w) + d_C(w)$, so for at least $|N(w) \cup N_2(w)| - (d_W(w) + d_C(w)) + 1 \geq 2d_C(w) + 1$ vertices. By the definition of implicit degree, if we can prove that $M_2(w) \geq d^w_{\ell+1}$, then the lower bound can be increased by one, establishing the claim.

By observing that $|(N(w) \cap V(W)) \cup (N(w) \cap V(C))^+| = d(w) = \ell + 1$, we know that at least one vertex of $(N(w) \cap V(C))^+$ has degree at least $\delta_1(G)$, so the vertex with the largest degree in $(N(w) \cap V(C))^+$ has degree at least $d^w_{\ell+1}$. This implies that $M_2(w) = \max\{d(v) \mid v \in N_2(w)\} \geq d^w_{\ell+1}$, hence the proof of Claim 2 is complete. □

Let $u_1, u_2, \ldots, u_p$ denote the vertices of $N(W) \cap V(C)$ in this cyclic order around C. Denote with $x_1, x_2, \ldots, x_p$ and with $y_1, y_2, \ldots, y_p$ the immediate successors and predecessors of $u_1, u_2, \ldots, u_p$, respectively. We call any combination of $\{u_i, x_i, y_i\}$, for $i = 1, 2, \ldots, p$, a $\{u, x, y\}$-triple. If we say that such a triple has degree at least $\delta_1(G)$ we mean that every vertex of the triple has degree at least $\delta_1(G)$. We use the same convention for subpairs of such triples. We next show that without loss of generality (possibly reversing the orientation of C), we may assume the following.

Claim 3. There exist three distinct indices i, j and m such that all the vertices of $\{x_i, x_j, x_m\}$ have degree at least $\delta_1(G)$.

Proof. From Claim 2, we can infer that any neighbor w of C in W has at least two neighbors on C, and if w has precisely two neighbors on C, then all six indicated vertices have degree at least $\delta_1(G)$. If no vertex of W has more than two neighbors on C, then by Claim 1 and Claim 2, we can find at least three neighbors of W on C with the property that these three and all their immediate successors and predecessors on C have degree at least $\delta_1(G)$. Then the statement of the claim holds. Finally, assume that a vertex w of W has $t \geq 3$ neighbors on C. Then, by Claim 2, at least $2t + 2$ of the neighbors of w on C and their immediate successors and predecessors on C have degree at least $\delta_1(G)$. Using straightforward counting arguments, this implies that at least two distinct $\{u, x, y\}$-triples have degree at least $\delta_1(G)$, and if this number is exactly 2, then additionally at least one of the other $\{u, x\}$, $\{u, y\}$ or $\{x, y\}$-pairs has degree at least $\delta_1(G)$. In all cases (after possibly reversing the orientation of C) we can find three distinct indices i, j and m such that all the vertices of $\{x_i, x_j, x_m\}$ have degree at least $\delta_1(G)$. □

We will reach a contradiction with the statement of Claim 3 by showing that $d(x_i)+d(x_j)+d(x_m) \leq n+2$. Since we will only focus on these three successors and ignore the other possible neighbors of W on C, for convenience we use the indices 1, 2 and 3 instead of i, j and m, and consider the neighborhoods of x_1, x_2 and x_3 in the three segments x_1Cu_2, x_2Cu_3 and x_3Cu_1 separately. We use C_1, C_2 and C_3 to denote these three segments, respectively (so C_3 does not indicate a cycle on three vertices).

We start with the segment C_1 and the following claim.

Claim 4. $N^-_{C_1}(x_1) \cup N_{C_1}(x_2) \cup N_{C_1}(x_3) \subseteq V(C_1)$, and the following intersections are empty: $N^-_{C_1}(x_1) \cap N_{C_1}(x_2)$, $N^-_{C_1}(x_1) \cap N_{C_1}(x_3)$, and $N^-_{C_1}(x_1) \cap N_{C_1}(x_2) \cap N_{C_1}(x_3)$.

Proof. Using Lemma 4.7, we obtain $N_{C_1}(x_1) \subseteq V(C_1) \setminus \{x_1, u_2\}$, $N^-_{C_1}(x_1) \subseteq V(C_1) \setminus \{y_2, u_2\}$, $N_{C_1}(x_2) \subseteq V(C_1) \setminus \{x_1, x_1^+\}$, and $N_{C_1}(x_3) \subseteq V(C_1) \setminus \{x_1, x_1^+, u_2\}$. Thus $N^-_{C_1}(x_1) \cup N_{C_1}(x_2) \cup N_{C_1}(x_3) \subseteq V(C_1)$.

Suppose that $q \in N^-_{C_1}(x_1) \cap N_{C_1}(x_2)$. By Lemma 4.7 we know that $q \notin \{x_1, x_1^+, y_2, u_2\}$, hence $q \in x_1^{+2}Cy_2^-$. Then the cycle $C' = u_1Wu_2\overline{C}q^+x_1Cqx_2Cu_1$ contradicts the choice of C. Similarly, we obtain that $N^-_{C_1}(x_1) \cap N_{C_1}(x_3) = \emptyset$. Then it is obvious that $N^-_{C_1}(x_1) \cap N_{C_1}(x_2) \cap N_{C_1}(x_3) = \emptyset$. □

Using Claim 4, Lemma 4.7(g), and the Inclusion-Exclusion Principle we obtain:

$|N_{C_1}(x_1)| + |N_{C_1}(x_2)| + |N_{C_1}(x_3)| = |N^-_{C_1}(x_1)| + |N_{C_1}(x_2)| + |N_{C_1}(x_3)| = |N^-_{C_1}(x_1) \cup N_{C_1}(x_2) \cup N_{C_1}(x_3)| + |N^-_{C_1}(x_1) \cap N_{C_1}(x_2)| + |N^-_{C_1}(x_1) \cap N_{C_1}(x_3)| + |N_{C_1}(x_2) \cap N_{C_1}(x_3)| - |N^-_{C_1}(x_1) \cap N_{C_1}(x_2) \cap N_{C_1}(x_3)| \leq |V(C_1)| + 1.$

By symmetry, for the segments C_2 and C_3, we obtain in a completely analogous way:

$|N_{C_2}(x_1)| + |N_{C_2}(x_2)| + |N_{C_2}(x_3)| \leq |V(C_2)| + 1$ and

$|N_{C_3}(x_1)| + |N_{C_3}(x_2)| + |N_{C_3}(x_3)| \leq |V(C_3)| + 1.$

From Lemma 4.7 we know that x_1, x_2 and x_3 have no neighbor in W, and that none of the pairs has a common neighbor in $V(G - V(C)) \setminus W$. So their degree sum outside the cycle C is at most $n - |V(C)| - |V(W)|$. So we get $|N_{G-V(C)}(x_1)| + |N_{G-V(C)}(x_2)| + |N_{G-V(C)}(x_3)| \leq n - |V(C)| - 1$.

Combining the above inequalities, and using Claim 3 (recalling the relabeling of the indices), we obtain that

$$3\delta_1(G) \le d(x_1) + d(x_2) + d(x_3) \le n+2,$$

contrary to the assumption that $\delta_1(G) \ge \frac{n+3}{3}$. ■

4.3.2 The proof of Theorem 4.5

Our proof of Theorem 4.5 is very similar to the above proof of Theorem 4.4, with several arguments that are almost exactly the same, but we present it here for completeness. However, we allow ourselves to shorten very similar parts of the argumentation, and leave some of the details to the reader.

Suppose the graph G satisfies the conditions of Theorem 4.5. Let C be a longest nonextendable cycle among all cycles in G that contain all vertices with degree at least $\frac{n-3}{4}$. Such a cycle exists by Lemma 4.6. If C is a Hamilton cycle, there is nothing to prove. Assuming that C is not a Hamilton cycle, consider a component W of $G—V(C)$. Clearly, W only contains vertices whose degree is smaller than their implicit degree. Since G is 3-connected, W has at least three neighbors on C. Just as in the above proof of Theorem 4.4, we can get one more, as follows.

Claim 5. *W has at least four neighbors on C.*

Proof. Suppose to the contrary that $|N(W) \cap V(C)| = 3$. If W is not a complete subgraph of G, let w_1 and w_2 denote two vertices at distance 2 in W such that $|N(w_1) \cap V(C)| \le |N(w_2) \cap V(C)|$. Clearly, w_1 and w_2 cannot have a common neighbor on C since G is claw-free and by the choice of C. This implies that $d(w_1) \le |N(w_1) \cap V(W)| + 1$ and $|(N(w_1) \cup N_2(w_1)) \cap V(W)| \ge |N(w_1) \cap V(W)| + 1$. With $d(w_1) = \ell + 1$ we conclude that $d^{w_1}_{\ell+1} < \frac{n-3}{4}$, a contradiction. The remaining case is that W is a complete subgraph of G. If one of the vertices of W has two neighbors on C, then the vertex has degree $|W| + 1$, which is at least $\frac{n-3}{4}$ by the hypothesis on 3-cuts of the theorem, a contradiction. In the other case all vertices of W have at most one neighbor on C. Then there are two vertices z_1, z_2 in W such that $z_1 v_1 \in E(G)$ and

$z_2v_2 \in E(G)$ for two distinct vertices v_1, v_2 on C. Then $d(z_1, v_2) = d(z_2, v_1) = 2$. Let $d(z_1) = \ell + 1$. Then $d(z_1) = d_W(z_1) + d_C(z_1) = |W| = \ell + 1$. Since all ℓ neighbors of z_1 on W have degree less than $\delta_1(G)$, the implicit degree of z_1 is equal to $d^{z_1}_{\ell+1}$. From the definition of implicit degree, we conclude that z_1 satisfies the condition $M_2(z_1) < d^{z_1}_{\ell+1}$. This implies that $id(z_1) = d(v_1)$, and hence that $d(v_1) > d(v_2)$, since $v_2 \in N_2(z_1)$. On the other hand, by analyzing the vertex z_2, and using symmetry, we obtain that $d(v_2) > d(v_1)$, a contradiction. We conclude that $|N(W) \cap V(C)| \geq 4$. □

The next claim and its proof are identical to Claim 2 and its proof. We recall the statement for convenience.

Claim 6. For every neighbor w of C in W, at least $2d_C(w)+2$ of the neighbors of w on C and their immediate successors and predecessors on C have degree at least $\delta_1(G)$.

We adopt the notation, and use $u_1, u_2, \ldots, u_p$, $x_1, x_2, \ldots, x_p$ and $y_1, y_2, \ldots, y_p$ to denote the vertices of $N(W) \cap V(C)$ and their immediate successors and predecessors, in this cyclic order around C, respectively. We next prove the following slightly different analogue of Claim 3. It shows that without loss of generality (possibly reversing the orientation of C), we may assume the following.

Claim 7. There exist four distinct indices i, j, p and q such that all the vertices of $\{u_i, x_j, x_p, x_q\}$ have degree at least $\delta_1(G)$.

Proof. From Claim 6, we can infer that any neighbor w of C in W has at least two neighbors on C. If w has precisely two neighbors on C, then all six indicated vertices have degree at least $\delta_1(G)$. Next we distinguish the following three cases to complete the proof of the claim.

1. If no vertex of W has more than two neighbors on C, then by Claim 5 and Claim 6, we can find at least four neighbors of W on C with the property that these four and all their immediate successors and predecessors on C have degree at least $\delta_1(G)$, and the statement of the claim clearly holds.

2. If some vertex w of W has $t \geq 4$ neighbors on C, then using Claim 6, among these t neighbors there are at least two that together with their immediate successors and predecessors constitute triples with degree at least $\delta_1(G)$. Among the other $t-2$ neighbors and their immediate successors and predecessors on C, there are at least $2(t-2) \geq 4$ vertices with degree at least $\delta_1(G)$. This implies that there are either a triple and a single u_i, x_i or y_i vertex with degree at least $\delta_1(G)$, or two pairs of $\{u_i, x_i\}$, $\{u_i, y_i\}$ and $\{x_i, y_i\}$ with degree at least $\delta_1(G)$. In all cases (after possibly reversing the orientation of C) we can find four distinct indices i, j, p and q such that all the vertices of $\{u_i, x_j, x_p, x_q\}$ have degree at least $\delta_1(G)$.

3. In the remaining case, all neighbors of C in W have two or three neighbors on C, and some neighbor w of C in W has three neighbors on C. Then assume that the neighbors of w on C are u_1, u_2, u_3, in this order around C. Using Claim 6, at least two distinct $\{u, x, y\}$-triples have degree at least $\delta_1(G)$ among these three neighbors of w on C. Without loss of generality, we assume $\{u_1, x_1, y_1\}$ and $\{u_2, x_2, y_2\}$ are such triples. In addition, either the other triple or one of the other pairs $\{u_3, x_3\}$, $\{u_3, y_3\}$ and $\{x_3, y_3\}$ has degree at least $\delta_1(G)$. By Claim 5, there is another neighbor u_i of a vertex $w' \neq w$ in W on C, and w' has two or three neighbors on C. Using the same arguments as before, we either find another triple or another pair with degree at least $\delta_1(G)$. In all cases (after possibly reversing the orientation of C) we can find four distinct indices i, j, p and q such that all the vertices of $\{u_i, x_j, x_p, x_q\}$ have degree at least $\delta_1(G)$.

This completes the proof of Claim 7. □

We proceed in a similar way as in the proof of Theorem 4.4, so we forget about the earlier indices and assume that u_1, x_2, x_3, x_4 are of degree at least $\delta_1(G)$, and in this order around C. We divide the cycle C into four segments and give bounds for the degree sums of u_1, x_2, x_3, x_4 on every segment. Due to the inclusion of u_1, unfortunately we have less symmetry than in the proof of Theorem 4.4, but the argumentation is very similar.

The segment $C_1 = x_1Cu_2$

For the segment C_1, we obtain the following result.

Claim 8. $N^+_{C_1}(u_1) \cup N_{C_1}(x_2) \cup N_{C_1}(x_3) \cup N_{C_1}(x_4) \subseteq (V(C_1) \cup \{x_2\}) \setminus \{x_1\}$, and the following intersections are empty: $N^+_{C_1}(u_1) \cap N_{C_1}(x_2)$, $N^+_{C_1}(u_1) \cap N_{C_1}(x_3)$, $N^+_{C_1}(u_1) \cap N_{C_1}(x_4)$, and $N_{C_1}(x_2) \cap N_{C_1}(x_3) \cap N_{C_1}(x_4)$.

Proof. Using Lemma 4.7, we obtain $N_{C_1}(u_1) \subseteq V(C_1) \setminus \{y_2^-, y_2\}$, $N^+_{C_1}(u_1) \subseteq (V(C_1) \cup \{x_2\}) \setminus \{x_1, y_2, u_2\}$, $N_{C_1}(x_2) \subseteq V(C_1) \setminus \{x_1, x_1^+\}$, $N_{C_1}(x_3) \subseteq V(C_1) \setminus \{x_1, x_1^+, u_2\}$ and $N_{C_1}(x_4) \subseteq V(C_1) \setminus \{x_1, x_1^+, u_2\}$. Thus $N^+_{C_1}(u_1) \cup N_{C_1}(x_2) \cup N_{C_1}(x_3) \cup N_{C_1}(x_4) \subseteq (V(C_1) \cup \{x_2\}) \setminus \{x_1\}$.

Now suppose that $q \in N^+_{C_1}(u_1) \cap N_{C_1}(x_2)$. Using Lemma 4.7, we know that $q \notin \{x_1, x_1^+, y_2, u_2\}$, hence $q \in x_1^{+2}Cy_2^-$. Now $C' = u_1Wu_2\overline{C}qx_2Cy_1x_1Cq^-u_1$ contradicts the choice of C. Similarly, we can obtain that $N^+_{C_1}(u_1) \cap N_{C_1}(x_3) = N^+_{C_1}(u_1) \cap N_{C_1}(x_4) = \emptyset$. If $N_{C_1}(x_2) \cap N_{C_1}(x_3) \cap N_{C_1}(x_4) \neq \emptyset$, then G contains an induced claw, a contradiction. □

Using Claim 8 and the Inclusion-Exclusion Principle we obtain:

$$
\begin{aligned}
&|N_{C_1}(u_1)| + |N_{C_1}(x_2)| + |N_{C_1}(x_3)| + |N_{C_1}(x_4)| \\
=&|N^+_{C_1}(u_1)| + |N_{C_1}(x_2)| + |N_{C_1}(x_3)| + |N_{C_1}(x_4)| \\
=&|N^+_{C_1}(u_1) \cup \bigcup_{i=2}^{4} N_{C_1}(x_i)| + \sum_{i=2}^{4} |N^+_{C_1}(u_1) \cap N_{C_1}(x_i)| \\
+&|N_{C_1}(x_2) \cap N_{C_1}(x_3)| + |N_{C_1}(x_2) \cap N_{C_1}(x_4)| + |N_{C_1}(x_3) \cap N_{C_1}(x_4)| \\
-&|N^+_{C_1}(u_1) \cap N_{C_1}(x_2) \cap N_{C_1}(x_3)| - |N^+_{C_1}(u_1) \cap N_{C_1}(x_2) \cap N_{C_1}(x_4)| \\
-&|N^+_{C_1}(u_1) \cap N_{C_1}(x_3) \cap N_{C_1}(x_4)| - |N_{C_1}(x_2) \cap N_{C_1}(x_3) \cap N_{C_1}(x_4)| \\
+&|N^+_{C_1}(u_1) \cap N_{C_1}(x_2) \cap N_{C_1}(x_3) \cap N_{C_1}(x_4)| \\
\leq&|V(C_1)| + |N_{C_1}(x_2) \cap N_{C_1}(x_3)| + |N_{C_1}(x_2) \cap N_{C_1}(x_4)| + |N_{C_1}(x_3) \cap N_{C_1}(x_4)|.
\end{aligned}
\tag{4.1}
$$

The segments $C_2 = x_2Cu_3$ and $C_3 = x_3Cu_4$

For the segments C_2 and C_3, using completely analogous arguments, we obtain the following inequalities:

$$\begin{aligned}&|N_{C_2}(u_1)|+|N_{C_2}(x_2)|+|N_{C_2}(x_3)|+|N_{C_2}(x_4)|\\ \leq&|V(C_2)|+|N_{C_2}(x_2)\cap N_{C_2}(x_3)|+|N_{C_2}(x_2)\cap N_{C_2}(x_4)|+|N_{C_2}(x_3)\cap N_{C_2}(x_4)|.\end{aligned} \tag{4.2}$$

$$\begin{aligned}&|N_{C_3}(u_1)|+|N_{C_3}(x_2)|+|N_{C_3}(x_3)|+|N_{C_3}(x_4)|\\ \leq&|V(C_3)|+|N_{C_3}(x_2)\cap N_{C_3}(x_3)|+|N_{C_3}(x_2)\cap N_{C_3}(x_4)|+|N_{C_3}(x_3)\cap N_{C_3}(x_4)|.\end{aligned} \tag{4.3}$$

The segment $C_4 = x_4Cu_1$

For the segment C_4, using slightly different but analogous arguments, we get the following result.

Claim 9. $N^+_{C_4}(u_1)\cup N_{C_4}(x_2)\cup N_{C_4}(x_3)\cup N_{C_4}(x_4) \subseteq V(C_4)\setminus\{x_4\}$, and the following intersections are empty: $N^+_{C_4}(u_1)\cap N_{C_4}(x_2)$, $N^+_{C_4}(u_1)\cap N_{C_4}(x_3)$, $N^+_{C_4}(u_1)\cap N_{C_4}(x_4)$, and $N_{C_4}(x_2)\cap N_{C_4}(x_3)\cap N_{C_4}(x_4)$.

Proof. Using Lemma 4.7, we obtain $N_{C_4}(u_1) \subseteq V(C_4)\setminus\{x_4, x_4^+, u_1\}$, $N^+_{C_4}(u_1) \subseteq V(C_4) \setminus \{x_4, x_4^+, x_4^{+2}\}$, $N_{C_4}(x_2) \subseteq V(C_4) \setminus \{x_4, x_4^+, u_1\}$, $N_{C_4}(x_3) \subseteq V(C_4) \setminus \{x_4, x_4^+, u_1\}$, and $N_{C_4}(x_4) \subseteq V(C_4)\setminus\{x_4, u_1\}$. Thus $N^+_{C_4}(u_1)\cup N_{C_4}(x_2)\cup N_{C_4}(x_3)\cup N_{C_4}(x_4) \subseteq V(C_4) \setminus \{x_4\}$.

Now suppose that $q \in N^+_{C_4}(u_1)\cap N_{C_4}(x_2)$. Using Lemma 4.7, we know that $q \notin \{x_4, x_4^+, x_4^{+2}, u_1\}$, hence $q \in x_4^{+3}Cy_1$. Now $C' = u_1Wu_2\overline{C}x_1y_1\overline{C}qx_2Cq^-u_1$ contradicts the choice of C. Similarly, we obtain that $N^+_{C_4}(u_1) \cap N_{C_4}(x_3) = N^+_{C_4}(u_1) \cap N_{C_4}(x_4) = \emptyset$. If $N_{C_4}(x_2) \cap N_{C_4}(x_3) \cap N_{C_4}(x_4) \neq \emptyset$, then G contains an induced claw, a contradiction. □

Using Claim 9 and the Inclusion-Exclusion Principle we obtain:

$$\begin{aligned}
&|N_{C_4}(u_1)|+|N_{C_4}(x_2)|+|N_{C_4}(x_3)|+|N_{C_4}(x_4)|\\
=&|N^+_{C_4}(u_1)|+|N_{C_4}(x_2)|+|N_{C_4}(x_3)|+|N_{C_4}(x_4)|\\
=&|N^+_{C_4}(u_1)\cup\bigcup_{i=2}^{4}N_{C_4}(x_i)|+\sum_{i=2}^{4}|N^+_{C_4}(u_1)\cap N_{C_4}(x_i)|\\
&+|N_{C_4}(x_2)\cap N_{C_4}(x_3)|+|N_{C_4}(x_2)\cap N_{C_4}(x_4)|+|N_{C_4}(x_3)\cap N_{C_4}(x_4)|\\
&-|N^+_{C_4}(u_1)\cap N_{C_4}(x_2)\cap N_{C_4}(x_3)|-|N^+_{C_4}(u_1)\cap N_{C_4}(x_2)\cap N_{C_4}(x_4)|\\
&-|N^+_{C_4}(u_1)\cap N_{C_4}(x_3)\cap N_{C_4}(x_4)|-|N_{C_4}(x_2)\cap N_{C_4}(x_3)\cap N_{C_4}(x_4)|\\
&+|N^+_{C_4}(u_1)\cap N_{C_4}(x_2)\cap N_{C_4}(x_3)\cap N_{C_4}(x_4)|\\
\le&|V(C_4)|-1+|N_{C_4}(x_2)\cap N_{C_4}(x_3)|+|N_{C_4}(x_2)\cap N_{C_4}(x_4)|\\
&+|N_{C_4}(x_3)\cap N_{C_4}(x_4)|.
\end{aligned}\tag{4.4}$$

Using Lemma 4.7(g), we deduce that

$$\sum_{i=1}^{4}|N_{C_i}(x_2)\cap N_{C_i}(x_3)|\le 2.$$

Similarly, we have

$$\sum_{i=1}^{4}|N_{C_i}(x_2)\cap N_{C_i}(x_4)|\le 2,\quad \sum_{i=1}^{4}|N_{C_i}(x_3)\cap N_{C_i}(x_4)|\le 2.$$

Hence, we conclude that

$$\sum_{i=1}^{4}(|N_{C_i}(x_2)\cap N_{C_i}(x_3)|+|N_{C_i}(x_2)\cap N_{C_i}(x_4)|+|N_{C_i}(x_3)\cap N_{C_i}(x_4)|)\le 6. \tag{4.5}$$

From Lemma 4.7 we know that x_i ($i\in\{2,3,4\}$) has no neighbor in W, u_1 has no neighbor in $V(G-V(C))\setminus W$, and x_i,x_j ($i,j\in\{2,3,4\}$) have no common neighbor in $V(G-V(C))\setminus W$. So we have the following inequality

$$|N_{G-C}(u_1)|+\sum_{i=2}^{4}|N_{G-C}(x_i)|\leq n-|V(C)|. \tag{4.6}$$

Combining the above inequalities (4.1)—(4.6), and using Claim 9 (recalling the relabeling of the indices), we obtain that

$$4\delta_1(G)\leq d(u_1)+d(x_2)+d(x_3)+d(x_4)\leq n+5,$$

contrary to the assumption that $\delta_1(G)\geq \frac{n+6}{4}$. ■

Chapter 5

Toughness, forbidden subgraphs, hamiltonian-connectivity

In a recent paper due to Li et al. [54], the aim was to characterize all possible graphs H such that every 1-tough H-free graph is hamiltonian. The almost complete answer was given there by the conclusion that every proper induced subgraph H of $K_1 \cup P_4$ can act as a forbidden subgraph to ensure that every 1-tough H-free graph is hamiltonian, and that there is no other forbidden subgraph with this property, except possibly for the graph $K_1 \cup P_4$ itself. This was left as an open case for hamiltonicity, and it seems to be a very hard case. Instead of researching this open case, we consider the stronger property of being hamiltonian-connected under the same additional forbidden subgraph conditions, assuming the toughness to be strictly larger than one. We find that the results are completely analogous to the hamiltonian case: every graph H such that any 1-tough H-free graph is hamiltonian also ensures that every H-free graph with toughness larger than one is hamiltonian-connected. And similarly, there is no other forbidden subgraph having this property, except possibly for the graph $K_1 \cup P_4$ itself. We leave this as an open case.

5.1 Introduction

It is easy to verify and a well-known fact that a hamiltonian graph is 1-tough, and that a hamiltonian-connected graph has toughness strictly larger than one. It is also known that the reverse statements do not hold, i.e., there exist infinitely many nonhamiltonian 1-tough graphs, and there exist infinitely many graphs with toughness strictly larger than one that are not hamiltonian-connected. More specifically, to answer Chvátal's Conjecture [32] which states that there exists a constant t_0 such that every t_0-tough graph on $n \geq 3$ vertices is hamiltonian, the authors in [4] proved that $t_0 \geq 9/4$ by constructing an infinite family of nonhamiltonian graphs with toughness arbitrarily close to 9/4 from below. It is natural and interesting to investigate under which additional conditions the reverse statements do hold. In other words, under which additional conditions are the properties of being 1-tough and being hamiltonian equivalent, and similarly for the stronger properties of having toughness strictly larger than one and being hamiltonian-connected. The type of additional conditions we focus on in this chapter are forbidden subgraph conditions. For hamiltonicity this type of problem was recently addressed by the authors of [54]. More relations between different hamiltonian properties and toughness conditions have been studied in [3], leading to several equivalent conjectures, some seemingly stronger and some seemingly weaker than Chvátal's Conjecture. The survey paper [5] deals with a large number of results that have been established until more than ten years ago. A more recent survey of results and open problems appeared a few years ago [13].

Recall that a path on k vertices is denoted by P_k, a complete graph on k vertices by K_k, and that we use $G \cup H$ to denote the disjoint union of two disjoint graphs G and H, and we use kG to denote the graph consisting of k disjoint copies of the graph G. In [54], the following sufficient condition for hamiltonicity of H-free 1-tough graphs was established.

Theorem 5.1 (B. Li et al. [54]). *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

Note that every induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$ is also an induced subgraph of $K_1 \cup P_4$, and that $K_1 \cup P_4$ is the only induced subgraph of $K_1 \cup P_4$ that is not an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. The following complementary result in [54] shows that there is no graph H other than the induced subgraphs of $K_1 \cup P_4$ that can ensure every 1-tough H-free graph is hamiltonian.

Theorem 5.2 (B. Li et al. [54]). *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

The two theorems together clearly leave $K_1 \cup P_4$ as the only open case in characterizing all the graphs H such that every H-free 1-tough graph is hamiltonian, and it seems to be a very hard case. In fact, this was the conjecture of Nikoghosyan in [66] that motivated the work in [54]. To date it is even unknown whether there exists some constant t such that every t-tough $K_1 \cup P_4$-free graph is hamiltonian.

A hamiltonian graph is 1-tough, and hence 2-connected, so a hamiltonian-connected graph G on at least three vertices is also 2-connected. It is even clearly 3-connected: if there exists a cut set $\{u, v\}$ in G, then u and v cannot be connected by a Hamilton path in G, because only the vertices of one component of $G - \{u, v\}$ can be picked up between u and v. It is almost equally easy to show that a hamiltonian-connected graph has toughness strictly larger than one. This can be seen by considering an arbitrary cut set S in a hamiltonian-connected graph G, and a Hamilton path P between two distinct vertices u and v of S (noting that $|S| \geq 3$ since G is 3-connected). Now, obviously $\omega(G - S) \leq \omega(P - S) \leq |S| - 1$, hence $\tau(G) > 1$.

In 1978, Jung [51] obtained the following result, in which he showed that for P_4-free graphs, the necessary condition $\tau(G) > 1$ is also a sufficient condition for hamiltonian-connectivity.

Theorem 5.3 (Jung [51]). *Let G be a P_4-free graph. Then G is hamiltonian-connected if and only if $\tau(G) > 1$.*

In a paper of 2000 [28], Chen and Gould concluded that if $\{S, T\}$ is a pair of graphs such that every 2-connected $\{S, T\}$-free graph is hamiltonian, then every 3-connected $\{S, T\}$-free graph is hamiltonian-connected. Following up on this idea, we considered the following question. Suppose R is a graph such that every 1-tough R-free graph is hamiltonian. Is then every R-free graph G with $\tau(G) > 1$ hamiltonian-connected? For the purpose of answering this question, we tried to prove each of the forbidden subgraph cases analogous to the statement in Theorem 5.1. Of course Theorem 5.3 has already given us a partial positive answer. And indeed, we get a positive answer for each of these cases, as indicated in the following result.

Theorem 5.4. *Let R be an induced subgraph of $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free graph G with $\tau(G) > 1$ on at least three vertices is hamiltonian-connected.*

We note here that from the proof of this result, it can be observed that the toughness condition $\tau(G) > 1$ in the above result cannot be weakened to the condition that the graph is 3-connected. We also proved the following analogue of Theorem 5.2, showing that except for the induced subgraphs of $K_1 \cup P_4$, there are no other forbidden induced subgraphs that can ensure every graph with toughness larger than one is hamiltonian-connected.

Theorem 5.5. *Let R be a graph on at least three vertices. If every R-free graph G with $\tau(G) > 1$ on at least three vertices is hamiltonian-connected, then R is an induced subgraph of $K_1 \cup P_4$.*

We conclude this section with the left unknown case as an open problem.

Problem 5.1. Is every $K_1 \cup P_4$-free graph G with $\tau(G) > 1$ on at least three vertices hamiltonian-connected?

As remarked earlier, we do not even know whether such graphs are hamiltonian, even if the condition on the toughness is replaced by $\tau(G) > t$ for any constant $t \geq 1$.

The next two closing sections of this chapter are devoted to the proofs of Theorem 5.4 and Theorem 5.5, respectively.

5.2 Proof of Theorem 5.4

Similar to the conventions in the other chapters, for a path P in G with a given orientation and a vertex x on P, x^+ and x^- denote the immediate successor and the immediate predecessor of x on P (if they exist), respectively. For any subset $I \subseteq V(P)$, let $I^- = \{x^- \mid x \in I\}$ and $I^+ = \{x^+ \mid x \in I\}$. For two vertices $x, y \in V(P)$, xPy denotes the subpath of P from x to y, and $y\overline{P}x$ denotes the path from y to x in the opposite direction. For a subgraph H disjoint from P in G, we use xHy to denote a path in G from x to y with all internal vertices in H.

Now, let P be a longest (u, v)-path in a graph G, and let H be a component of $G - V(P)$. Furthermore, let $I = N_P(H) = \{x_1, x_2, \dots, x_s\}$, and let w be a vertex of H. Then we start with the following lemma.

Lemma 5.6. *Both $\{w\} \cup I^+$ and $\{w\} \cup I^-$ are independent sets.*

Proof. Suppose, to the contrary, that there is an edge in $\{w\} \cup I^+$. If the edge appears between w and a vertex of I^+, say $wx_i^+ \in E(G)$ with $x_i^+ \in I^+$, then $uPx_iHx_i^+Pv$ is a (u, v)-path longer than P, contradicting the choice of P. If the edge appears between two vertices of the set I^+, say $x_i^+x_j^+ \in E(G)$ with $x_i^+, x_j^+ \in I^+$, then $uPx_iHx_j\overline{P}x_i^+x_j^+Pv$ is a (u, v)-path longer than P, a contradiction. Hence $\{w\} \cup I^+$ is an independent set. Similarly, by symmetry $\{w\} \cup I^-$ is also an independent set. □

Next we complete the proof for the two choices of R in Theorem 5.4, respectively. Note that we do not have to consider proper induced subgraphs of R, since a graph is R-free if it is S-free for an induced subgraph S of R.

The case $R = K_1 \cup P_3$.

Assume that G is a $K_1 \cup P_3$-free graph with $\tau(G) > 1$. Suppose to the contrary that G is not hamiltonian-connected, and that u, v is a pair of distinct vertices of G that is not connected by a Hamilton path in G. Let P be a longest (u, v)-path in G. Since P is not a Hamilton path, $V(G) \setminus V(P) \neq \emptyset$. Assume that H is a component of $G - V(P)$. Then $|N_P(H)| \geq 3$ since $\tau(G) > 1$. Assume that $N_P(H) = \{v_1, v_2, \dots, v_s\}$ with $s \geq 3$, in this order according to the fixed chosen orientation of P. We denote the segment of P from v_i^+ to v_{i+1}^- by Q_i for all i with $1 \leq i \leq s-1$. If $v_1 \neq u$, then let $Q_0 = uPv_1^-$. If $v_s \neq v$, then let $Q_s = v_s^+Pv$.

Before completing the proof for this case, we first prove the following two claims.

Claim 1. At least two of the segments of P are connected by a path (possibly an edge) that is internally-disjoint with P.

Proof. By Lemma 5.6, the neighbors of H on P are not consecutive vertices on P. If none of the segments of P is connected to another segment of P by a path (or edge) internally-disjoint with P, then every segment is in a separate component after removal of the vertices of $N_P(H)$. Then there will be at least s components after deleting the s vertices of $N_P(H)$, contradicting the fact that $\tau(G) > 1$. □

Using Claim 1, we assume that Q_i and Q_j $(0 \leq i < j \leq s)$ are connected by a path (edge) that is internally-disjoint with P. In fact, the next claim shows that we may assume that this path is actually an edge.

Claim 2. Q_i and Q_j are connected by an edge.

Proof. Supposing the statement is false, we consider a shortest path that connects Q_i and Q_j and is internally-disjoint with P, and denote it as $Q = q_1q_2\ldots q_r$ (with $q_1 \in V(Q_i)$ and $q_r \in V(Q_j)$). Obviously, Q is an induced path, and $N(Q) \cap V(H) = \emptyset$. If $r \geq 3$, then $\{w, q_1, q_2, q_3\}$ induces a copy of $K_1 \cup P_3$, where w is a vertex of $V(H)$, a contradiction. Hence, the shortest path connecting Q_i and Q_j is an edge, and the claim holds. □

We use Claim 2 and distinguish two cases, depending on the value of the indices i and j, as follows.

Case A. Q_i and Q_j are connected by an edge, for some i and j with $1 \leq i < j \leq s-1$.

Suppose that xy is an edge with $x \in V(Q_i)$ and $y \in V(Q_j)$, and chosen such that $|v_i^+Pxy\overline{P}v_j^+|$ is as small as possible. Using Lemma 5.6, we know that either $x \neq v_i^+$ or $y \neq v_j^+$. Without loss of generality, say $x \neq v_i^+$. By the choice of xy, we have that $x^-y \notin E(G)$. Then an arbitrary vertex w of $V(H)$ together with the three vertices of $\{x^-, x, y\}$ induces a copy of $K_1 \cup P_3$, a contradiction.

Case B. All edges connecting two different segments of P have at least one end vertex in Q_0 or Q_s.

By Claim 2, the assumption of this case implies that any two of the $s-1$ segments Q_i ($i \in \{1,2,\ldots,s-1\}$) of P are not connected by a path internally-disjoint with P. Then there must be a segment Q_i ($i \in \{1,2,\ldots,s-1\}$) that has a neighbor in Q_0 or Q_s. Otherwise, P has $s+1$ segments and only Q_0 and Q_s among all these segments are connected by such a path. Then, by deleting the s neighbors of H on P, we obtain $s+1$ components, contradicting the fact that $\tau(G) > 1$. Without loss of generality, we assume that Q_i ($i \in \{1,2,\ldots,s-1\}$) is connected to Q_0 by an edge. We use xy to denote an edge between $V(Q_0)$ and $V(Q_i)$, chosen in such a way that $|v_1^-\overline{P}xyPv_{i+1}^-|$ is as small as possible. Using Lemma 5.6, we know that either $x \neq v_1^-$ or $y \neq v_{i+1}^-$. Without loss of generality, say $x \neq v_1^-$. By the choice of xy, we have that $x^+y \notin E(G)$. Then an arbitrary vertex w of $V(H)$ together with the three vertices of $\{x^+, x, y\}$ induces a copy of $K_1 \cup P_3$, a contradiction.

This completes the proof for the case $R = K_1 \cup P_3$. We now turn to the remaining case that $R = 2K_1 \cup K_2$.

The case $R = 2K_1 \cup K_2$.

Suppose that G is a $2K_1 \cup K_2$-free graph with $\tau(G) > 1$, and assume that G is not hamiltonian-connected. Let u, v be a pair of distinct vertices of G that is not connected by a Hamilton path in G, and let P be a longest (u,v)-path in G. Then $V(G) \setminus V(P) \neq \emptyset$. Assume that H is a component of $G - V(P)$. Since $\tau(G) > 1$, we have $|N_P(H)| \geq 3$. Similarly as in the case $R = K_1 \cup P_3$, we use $N_P(H) = \{v_1, v_2, \ldots, v_s\}$ to denote all neighbors of H on P, so with $s \geq 3$ and in this order according to the chosen orientation of P.

We continue with first proving three useful claims.

Claim 3. *H is trivial, i.e., $|V(H)| = 1$.*

Proof. Suppose H contains an edge w_1w_2. Using Lemma 5.6, we get that $\{w_1, v_1^+, v_2^+\}$ and $\{w_2, v_1^+, v_2^+\}$ are independent sets. Then $\{v_1^+, v_2^+, w_1, w_2\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. □

Let $H = \{w\}$. Then $N_P(w) = \{v_1, v_2, \ldots, v_s\}$, with $s \geq 3$. Let Q_i be the segment of P from v_i^+ to v_{i+1}^- for $1 \leq i \leq s-1$, denoted as $Q_i = x_{i_1}x_{i_2}\ldots x_{i_{r_i}}$, with $x_{i_1} = v_i^+$ and $x_{i_{r_i}} = v_{i+1}^-$. If $v_1 \neq u$, then let $Q_0 = ux_{0_1}x_{0_2}\ldots x_{0_{r_0}}$, with

$x_{0_{r_0}} = v_1^-$. If $v_s \neq v$, then let $Q_s = x_{s_1}x_{s_2}\dots x_{s_{r_s}}v$, with $x_{s_1} = v_s^+$. Now we prove the following useful facts.

Claim 4. For all $i \in \{1,2,\dots,s-1\}$, we have $v_1^+x_{i_j} \notin E(G)$ for every odd j, $v_1^+x_{i_j} \in E(G)$ for every even j, and r_i is odd. In addition, if Q_0 and Q_s exist, then v_1^+ is alternately adjacent and nonadjacent to the vertices of the segments Q_0 and Q_s with $v_1^+x_{0_{r_0}} \notin E(G)$ and $v_1^+x_{s_1} \notin E(G)$.

Proof. We divide the proof into two cases according to the length of the segment Q_1.

Case A. $|Q_1| = 1$, i.e., $v_2 = v_1^{++}$.
In this case, the claim holds for the segment Q_1 itself. For the segments Q_i $(i = 0,2,3,\dots,s)$, we first prove that if $v_1^+x_{i_j} \in E(G)$, then $v_1^+x_{i_{j+1}} \notin E(G)$, and if $v_1^+x_{i_j} \notin E(G)$, then $v_1^+x_{i_{j+1}} \in E(G)$ for all $j \in \{1,2,...,i_{r_i}-1\}$. Suppose that there is a segment Q_i with $i = 0,2,3,\dots,s$ such that $v_1^+x_{i_j} \in E(G)$ and $v_1^+x_{i_{j+1}} \in E(G)$ for some $j \in \{1,2,...,i_{r_i}-1\}$. Then there exists a longer (u,v)-path $P' = uPv_1wv_2Px_{i_j}v_1^+x_{i_{j+1}}Pv$ (if $i \neq 0$), or $P' = uPx_{i_j}v_1^+x_{i_{j+1}}Pv_1wv_2Pv$ (if $i = 0$), a contradiction. Suppose that Q_i $(i = 0,2,3,\dots,s)$ is a segment with $v_1^+x_{i_j} \notin E(G)$ and $v_1^+x_{i_{j+1}} \notin E(G)$ for some $j \in \{1,2,...,i_{r_i}-1\}$. Then $\{w, v_1^+, x_{i_j}, x_{i_{j+1}}\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Thus the neighbors of v_1^+ occur on every segment Q_i alternately along the path. By Lemma 5.6 we have $v_1^+x_{i_1} \notin E(G)$ for $i = 2,3,\dots,s$ and $v_1^+x_{i_{r_i}} \notin E(G)$ for $i = 0,2,3,\dots,s-1$. Hence r_i is odd for $i \in \{1,2,\dots,s-1\}$, and the claim holds.

Case B. $|Q_1| \geq 2$, i.e., $v_1^{++} \notin N_P(w)$.
Firstly, we consider the case that $i \in \{0,2,3,\dots,s\}$. By Lemma 5.6, $v_1^+x_{i_1} \notin E(G)$ for $i \in \{2,3,\dots,s\}$. To avoid that $\{w, x_{i_1}, v_1^+, v_1^{++}\}$ induces a copy of $2K_1 \cup K_2$, we have $v_1^{++}x_{i_1} \in E(G)$. If there exists an index $j \in \{1,2,...,i_{r_i}-1\}$ such that $v_1^+x_{i_j} \in E(G)$ and $v_1^+x_{i_{j+1}} \in E(G)$, then we have a longer (u,v)-path $P' = uPv_1wv_i\overline{P}v_1^{++}x_{i_1}Px_{i_j}v_1^+x_{i_{j+1}}Pv$ (if $i \neq 0$), or a longer (u,v)-path $P' = uPx_{i_j}v_1^+x_{i_{j+1}}Pv_1wv_2\overline{P}v_1^{++}x_{2_1}Pv$ (if $i = 0$), a contradiction. If $v_1^+x_{i_j} \notin E(G)$ and $v_1^+x_{i_{j+1}} \notin E(G)$ for some $j \in \{1,2,...,i_{r_i}-1\}$, then $\{w, v_1^+, x_{i_j}, x_{i_{j+1}}\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Thus the neighbors of v_1^+ occur on every segment Q_i $(i = 0,2,3,\dots,s)$ alternately along the path. We

know $v_1^+ x_{i_1} \notin E(G)$ for $i = 2, 3, \ldots, s$, by Lemma 5.6. Now $v_1^+ x_{i_{r_i}} \notin E(G)$ for $i = 0, 2, 3, \ldots, s-1$; otherwise $P' = uPv_1^+ x_{i_{r_i}} \overline{P} x_{i_1} v_1^{++} P v_i w v_{i+1} P v$ is a longer (u, v)-path (if $i \neq 0$), or $P' = uPx_{i_{r_i}} v_1^+ v_1 w v_2 \overline{P} v_1^{++} x_{2_1} P v$ is a longer (u, v)-path (if $i = 0$). Hence r_i is odd for $i \in \{2, \ldots, s-1\}$.

Secondly, we consider the remaining case that $i = 1$. If $v_1^+ x_{1_j} \in E(G)$ and $v_1^+ x_{1_{j+1}} \in E(G)$ for some $j \in \{2, 3, ..., i_{r_i} - 1\}$, then we have a longer (u, v)-path $P' = uPv_1 w v_2 \overline{P} x_{1_{j+1}} v_1^+ x_{1_j} \overline{P} v_1^{++} x_{2_1} P v$ (if $x_{1_j} \neq v_1^{++}$), or $P' = uPv_1 w v_2 \overline{P} x_{1_{j+1}} v_1^+ x_{1_j} x_{2_1} P v$ (if $x_{1_j} = v_1^{++}$), a contradiction. If $v_1^+ x_{1_j} \notin E(G)$ and $v_1^+ x_{1_{j+1}} \notin E(G)$ for some $j \in \{2, 3, ..., i_{r_i} - 1\}$, then $\{w, v_1^+, x_{1_j}, x_{1_{j+1}}\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Thus the neighbors of v_1^+ occur on the segment Q_1 alternately along the path. Suppose that $v_1^+ x_{1_{r_1}} \in E(G)$. If $wx_{2_2} \in E(G)$, then $uPv_1^+ x_{1_{r_1}} \overline{P} v_1^{++} x_{2_1} v_2 w v_3 P v$ is a longer (u, v)-path. If $wx_{2_2} \notin E(G)$, then $v_1^+ x_{2_2} \in E(G)$ and $uPv_1 w v_2 x_{2_1} v_1^{++} P x_{1_{r_1}} v_1^+ x_{2_2} P v$ is a longer (u, v)-path. Hence, $v_1^+ x_{1_{r_1}} \notin E(G)$ and r_1 is odd. Therefore the claim holds for all cases. □

We need one more claim which is easy to prove.

Claim 5. $N(v_1^+) \subseteq V(P)$.

Proof. If there is a vertex $z \in V(G) \setminus V(P)$ such that $v_1^+ z \in E(G)$, then the vertex set $\{w, x_{2_1}, v_1^+, z\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Therefore, $N(v_1^+) \subseteq V(P)$. □

Let $S = N(v_1^+) \cup N_P(H)$ and $|S| = s'$. By Claim 4, the vertices of S occur on the path P alternately. If $|V(P)|$ is odd, then $s' = \lceil \frac{|V(P)|}{2} \rceil$; if $|V(P)|$ is even, then $s' = \frac{|V(P)|}{2}$. Moreover, S is a cut set whose deletion yields at least three components, including the two trivial ones with vertices w and v_1^+. If one of the other components contains an edge $z_1 z_2$, then $\{w, v_1^+, z_1, z_2\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Thus all components of $G - S$ are trivial, meaning that every vertex of $V(P) \setminus S$ is a component. Hence, $\omega(G - S) \geq s' - 1 + 1 = |S|$, contradicting the fact that $\tau(G) > 1$.

This completes the proof of Theorem 5.4. ■

5.3 Proof of Theorem 5.5

For our proof that there is no graph H, apart from the induced subgraphs of $K_1 \cup P_4$, that can ensure every H-free graph with toughness larger than one is hamiltonian-connected, we make use of the following lemma.

Lemma 5.7 (B. Li et al. [54]). *Let R be a graph on at least three vertices. If R is not an induced subgraph of $K_1 \cup P_4$, then R contains one of the graphs in $\mathscr{H} = \{C_3, C_4, C_5, K_{1,3}, 2K_2, 4K_1\}$ as an induced subgraph.*

Using Lemma 5.7, we can complete our proof of Theorem 5.5 by showing that not every R-free graph with toughness larger than one is hamiltonian-connected, for each of the graphs $R \in \mathscr{H}$. To show this, we continue by giving suitable counterexamples; some of these graphs are even not hamiltonian. The only class for which we cannot refer to known results, is the class of $4K_1$-free graphs. It is not difficult to check that the graphs sketched in Figure 5.1 are examples of $4K_1$-free graphs that are not hamiltonian-connected but have toughness larger than one. In this sketch, the middle three vertices in the figure are supposed to be joined to all the vertices of the complete graph on the left, and u and v are also joined to all vertices of the complete graphs on the right; the other middle vertex is only joined to the two indicated vertices on the right; these indicated vertices are not joined to u or v. Note that between u and v there is no Hamilton path (even if $m = s = t = 1$), since the deletion of $\{u, v\}$ leaves a graph with a cut vertex z (the other vertex in the middle), and one cannot pick up all the vertices in both components that result from deleting z.

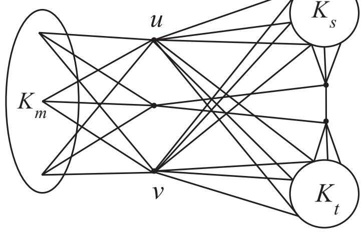

FIGURE 5.1: $4K_1$-free non-hamiltonian-connected graphs

For $R = C_3$, the well-known nonhamiltonian Petersen graph is a suitable counterexample, since it is C_3-free and has toughness $4/3$.

For $R = C_4$, C_5 or $2K_2$, we can find suitable split graphs as counterexamples. Split graphs consist of a clique C and an independent set I with some (or possibly all or none) of the edges joining a vertex of C and a vertex of I (but no edges joining pairs of vertices of I). Split graphs are known to be $\{C_4, C_5, 2K_2\}$-free. It was proved in [52] that every $\frac{3}{2}$-tough split graph is hamiltonian, and that there is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor (a 2-regular spanning subgraph, not necessarily connected) and $\tau(G_n) \to 3/2$. The latter graphs clearly serve as suitable examples for our purposes.

For $R = K_{1,3}$, we use the known fact that for a claw-free noncomplete graph G, $2\tau(G) = \kappa(G)$, where $\kappa(G)$ denotes the (vertex) connectivity of G. In [63], the authors conjectured that every 4-connected claw-free graph is hamiltonian, and they showed examples of 3-connected claw-free graphs that are not hamiltonian. These examples have toughness $3/2$ and clearly serve our purposes.

This completes our proof of Theorem 5.5. ■

Chapter 6

Toughness, forbidden subgraphs and pancyclicity

Various sufficient conditions for a graph to be hamiltonian are so strong that they imply considerably more about the cycle structure of the graph. Based on this observation, Bondy [9] presented a metaconjecture in 1971 in which he stated that almost any nontrivial condition on a graph which implies that the graph is hamiltonian also implies that it is pancyclic (except for maybe a simple family of exceptional graphs). Inspired by Bondy's metaconjecture, in this chapter we examine whether the conditions on toughness and forbidden subgraphs for hamiltonicity in [54] in fact imply pancyclicity, and we get a positive answer except for a few specific classes of graphs.

6.1 Introduction

For hamiltonicity, Chvátal's Conjecture states that there exists a constant t_0 such that every t_0-tough graph on $n \geq 3$ vertices is hamiltonian, and it is proved in [4] that $t_0 \geq 9/4$. For pancyclicity, the following theorem shows that there exists no such constant.

Theorem 6.1 (Brandt [12]). *There are t-tough graphs with t arbitrarily large which are not weakly pancyclic.*

A graph is called *weakly pancyclic* if it contains cycles of every length between the girth (the length of a shortest cycle) and the circumference (the length of a longest cycle). Hence, if a graph is not weakly pancyclic, then it will also not be pancyclic.

As introduced in previous chapters, forbidden subgraph conditions are an important type of sufficient conditions for the existence of Hamilton cycles in graphs. Over the years, researchers have established full characterizations of all possible single forbidden graphs and pairs of forbidden subgraphs ensuring that every 2-connected graph is hamiltonian. Some of these forbidden subgraph results give support for Bondy's metaconjecture, as shown by the following theorems.

Theorem 6.2 (Bedrossian [6]; Faudree and Gould [38]). *Let R and S be connected graphs with $R, S \neq P_3$, and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W (see Figure 2.1).*

Theorem 6.3 (Bedrossian [6]). *Let R and S be connected graphs with $R, S \neq P_3$, and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

It can be observed that many of the nonhamiltonian graph families that show the necessity of forbidding certain subgraphs are not 1-tough. This fact caused researchers to consider using the necessary condition of being 1-tough instead of 2-connected. In [66], Nikoghosyan posed several conjectures relating toughness and forbidden subgraph conditions to hamiltonicity. Motivated by one of these conjectures, as we introduced in the previous chapter, Li et al. [54] considered single forbidden subgraphs under the condition of 1-toughness, and came up with the following results.

Theorem 6.4 (Li et al. [54]). *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

Theorem 6.5 (Li et al. [54]). *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

Inspired by Bondy's metaconjecture, we examined whether the condition in Theorem 6.4 in fact implies pancyclicity, and we obtained the following three results. We postpone the proofs of these result to Sections 6.3, 6.4 and 6.5, respectively.

Theorem 6.6. *Let G be a $K_1 \cup P_3$-free 1-tough graph on $n \geq 3$ vertices. Then G is pancyclic or $G \in \{C_5, K_{\frac{n}{2},\frac{n}{2}}\}$.*

Clearly, the latter case can only occur when n is even. For the next result we first define the graph C_6^+ and the class of graphs $\mathscr{K}^-$. The graph C_6^+ is obtained from C_6 by adding an edge between two vertices at distance 2 in C_6. The class $\mathscr{K}^-$ consists of all balanced bipartite graphs $K_{s,s} - M$ ($s \geq 2$), where M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s$.

Theorem 6.7. *Let G be a $2K_1 \cup K_2$-free 1-tough graph on at least three vertices. Then G is pancyclic or $G \in \mathscr{K}^- \cup \{C_5, C_6^+\}$.*

Theorem 6.8. *Let G be a P_4-free 1-tough graph on $n \geq 3$ vertices. Then G is pancyclic or $G = K_{\frac{n}{2},\frac{n}{2}}$.*

By Theorem 6.5, there is no graph H other than the induced subgraphs of $K_1 \cup P_4$ that can ensure every 1-tough H-free graph is hamiltonian. Hence, we obtain the following conclusion.

Theorem 6.9. *Let R be a graph on at least three vertices. If every R-free 1-tough graph G on at least three vertices is pancyclic, then R is an induced subgraph of $K_1 \cup P_4$.*

For the open case with $K_1 \cup P_4$, it is natural to ask whether every $K_1 \cup P_4$-free 1-tough graph G on at least three vertices is pancylic (except possibly for some well-defined classes of graphs). Note that the small cycles C_4, C_5 and C_6, as well as the graph C_6^+ are exceptional graphs for this statement. In fact, there are infinite classes of exceptional graphs for this statement. One of these classes is illustrated in Figure 6.1. This class $\mathscr{C}_5^s$ consists of graphs that are obtained from a C_5 by replacing each vertex v of the C_5 by an independent set I_v of cardinality $s \geq 1$, and adding all edges between I_u and I_v whenever uv is an edge of the C_5. These graphs clearly contain no C_3, so they are not pancyclic. They are hamiltonian (even if the sets I_v have different cardinalities, as long as the graphs are 1-tough. We refer to [16], where these graphs are called C_5^*-type graphs and treated as special cases of triangle-free $2K_2$-free graphs). Using that C_5 is $K_1 \cup P_4$ -free, it is easy to check that all these graphs are $K_1 \cup P_4$ -free. There are basically two choices for cut sets that should be considered for determining the toughness. One option is to delete two nonconsecutive sets I_v and I_w , resulting in $s+1$ components; the other option is to delete an additional set I_z, resulting in $2s$ components. The latter option determines the toughness if $s \geq 3$, i.e., the toughness of these graphs is $\frac{3}{2}$ if $s \geq 3$. Hence, the class of graphs $\mathscr{C}_5^s$ shows that even with a toughness strictly larger than one, there exist infinitely many exceptional graphs to the above statement. In a similar way, one can define the classes $\mathscr{C}_4^s$ (balanced complete bipartite graphs $K_{2s,2s}$) and $\mathscr{C}_6^s$, based on a C_4 and C_6, respectively. Graphs from these classes are also not pancyclic, $K_1 \cup P_4$-free, and they have toughness equal to one. Here, we pose the following problem.

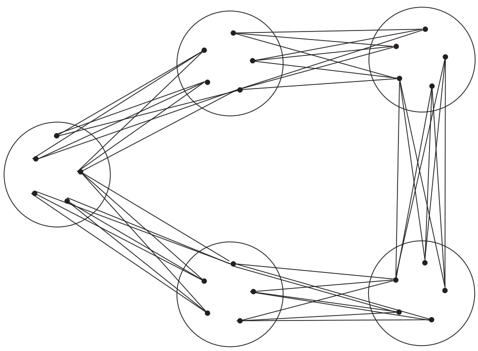

FIGURE 6.1: Graphs $\mathscr{C}_5^s$

Problem 6.1. Except for C_6^+ and the graphs from $\mathscr{C}_4^s$,$\mathscr{C}_5^s$, and $\mathscr{C}_6^s$, are there any other 1-tough $K_1 \cup P_4$-free graphs that are not pancylic?

In Theorems 6.6, 6.7 and 6.8, all the exceptional graphs have toughness exactly one. Hence, recalling the above remarks on the class $\mathscr{C}_5^s$, we obtain the following corollary.

Corollary 6.10. *Let R be a graph, and let G be a graph with $\tau(G) > 1$. Then G is R-free implies G is pancyclic if and only if R is a proper induced subgraph of $K_1 \cup P_4$.*

The remainder of this chapter is devoted to the proofs of our main results, but we start with a short section containing some preliminaries.

6.2 Preliminaries

We call a cycle with m vertices an *m-cycle*. Let C be an m-cycle of G with a given orientation, and denoted as $C = x_1x_2\dots x_mx_1$. For a vertex $x_i \in V(C)$ $(1 \le i \le m)$, let x_i^{-l}, x_i^{+l} $(1 \le i-l < i+l \le m)$ denote the vertices x_{i-l} and x_{i+l} on C, respectively. Instead of x_i^{-1} and x_i^{+1}, we simply use x_i^- and x_i^+ to denote the immediate predecessor and successor of x_i on C, respectively. For two vertices $x_i, x_j \in V(C)$, x_iCx_j denotes the subpath of C from x_i to x_j, and $x_j\overline{C}x_i$ denotes the path from x_j to x_i in the reverse direction. For any $I \subseteq V(C)$, let $I^- = \{x_i^- \mid x_i \in I\}$ and $I^+ = \{x_i^+ \mid x_i \in I\}$. A similar notation is used for paths. In the proofs, we often use $\{u,v,w,x\} \cong H$ as shorthand for $\{u,v,w,x\}$ induces a copy of H in G.

The main idea of our proofs of Theorems 6.6–6.8 is as follows. First we consider a shortest cycle of the graph G. In case G does not contain some specific short cycles, we characterize G as one of the exceptional graphs. For the other case, we prove by contradiction that if G contains a k-cycle, then it also contains a $(k+1)$-cycle for any integer $k \in \{3,4,\dots,n-1\}$. By induction, this is sufficient to show that G is pancyclic. All our proofs are modelled along these lines and look similar, but contain different argumentations. In particular, our proofs considerably differ in length.

6.3 Proof of Theorem 6.6

Suppose that G is a graph satisfying the conditions of Theorem 6.6. Since G is 1-tough, G contains a cycle. Clearly, the shortest (induced) cycle of G is C_3, C_4 or C_5; otherwise, G has an induced subgraph isomorphic to $K_1 \cup P_3$. We first prove the following claim to characterize the exceptional graphs.

Claim 1. If the shortest cycle of G is C_5, then G is C_5; if the shortest cycle of G is C_4, then G is $K_{s,s}$ ($s \geq 2$).

Proof. Suppose that the shortest cycle of G is C_5. Let $C = v_1 v_2 \dots v_5 v_1$ be a shortest cycle. Then C is an induced cycle. If $|V(G)| = 5$, then $G = C_5$, and the claim holds. Now we assume that $|V(G)| \geq 6$ and x is a vertex of $G - V(C)$. To avoid $\{x, v_1, v_2, v_3\}$ inducing a $K_1 \cup P_3$, we have that x is adjacent to v_1, v_2 or v_3. Without loss of generality, we assume $xv_1 \in E(G)$. If $N(x) \cap \{v_2, v_3, v_4\} \neq \emptyset$, then G has a C_3 or C_4, contradicting the fact that C_5 is a shortest cycle. If $N(x) \cap \{v_2, v_3, v_4\} = \emptyset$, then $\{x, v_2, v_3, v_4\}$ induces a copy of $K_1 \cup P_3$, a contradiction. Hence, G has 5 vertices and G is C_5.

Suppose that the shortest cycle of G is C_4. Let $C = v_1 v_2 \dots v_4 v_1$ be a shortest cycle. Hence C is an induced cycle. If G has 4 vertices, then $G = C_4 = K_{2,2}$, and the claim holds. If G is not C_4, then there is a vertex x_0 in $G - V(C)$. Similar to the above case, without loss of generality, we assume $x_0 v_1 \in E(G)$. To avoid inducing $K_1 \cup P_3$ and C_3, we have that $N(x_0) \cap \{v_2, v_3, v_4\} = \{v_3\}$. Using the same arguments, we have that every vertex x_i of $G - V(C)$ has two neighbors on C, which are $\{v_1, v_3\}$ or $\{v_2, v_4\}$. For two vertices $x_i, x_i \in G - V(C)$, if x_i and x_j have the same neighbors on C, then $x_i x_j \notin E(G)$; otherwise x_i, x_j and one of their neighbors on C induce a copy of C_3, a contradiction. If x_i and x_j have different neighbors on C, without loss of generality assume that $N_C(x_i) = \{v_1, v_3\}$ and $N_C(x_j) = \{v_2, v_4\}$. Then $x_i x_j \in E(G)$; otherwise $\{x_i, x_j, v_2, v_4\} \cong K_1 \cup P_3$, a contradiction. Now denote $A = N_G(v_1, v_3)$, $B = N_G(v_2, v_4)$. Then $V(G) = A \cup B$. Moreover, A, B are two independent vertex sets and every vertex of A is adjacent to every vertex of B. Hence, G is a complete bipartite graph, and according to the toughness, G is a balanced complete bipartite graph $K_{s,s}$. $\square$

By Claim 1, we see that if G has no C_3, then G is either C_5 or $K_{s,s}$. Next, we suppose that G is neither C_5 nor $K_{s,s}$. This implies that G contains a C_3. We show that G is pancyclic by proving the following fact.

Fact: If G has a k-cycle ($k = 3, 4, \ldots, n-1$), then G has a $(k+1)$-cycle.

Proof. Suppose, by contradiction, that G has a k-cycle, but no $(k+1)$-cycle, for some $k \in \{3, 4, \ldots, k-1\}$. Let $C = v_1 v_2 \ldots v_k v_1$ be a k-cycle, and let H be a component of $G - V(C)$. Since G is 1-tough, H has at least two neighbors on C. We distinguish two cases.

Case 1. *H has two neighbors that are consecutive on C.*
Clearly, H is not trivial in this case. We prove a number of claims before we complete the proof for this case.

Claim 2. *H contains an edge ab such that $av_i \in E(G)$ and $bv_{i+1} \in E(G)$, $i \in \{1, 2, ..., k\}$.*

Proof. Choose two consecutive neighbors of H on C such that their distance in H is shortest. Without loss of generality, assume that v_1, v_2 are such vertices, and $a, b \in V(H)$ are neighbors of v_1 and v_2, respectively. We have that $a \neq b$; otherwise G has a $(k+1)$-cycle $v_1 a v_2 C v_1$, contradicting the assumption that G has no $(k+1)$-cycle. Let $P = a x_1 x_2 \ldots x_s b$ be a shortest (a, b)-path in H. Clearly, $V(P) \cap N(v_1) = \{a\}$ and $V(P) \cap N(v_2) = \{b\}$. Then we have $s \leq 1$; otherwise $\{v_2, a, x_1, x_2\} \cong K_1 \cup P_3$, a contradiction. Suppose that $s = 1$ and $P = a x_1 b$. By the choice of v_1, v_2 and the assumption that G has no $(k+1)$-cycle, we have that $\{x_1, b\} \cap N(v_3) = \emptyset$ and $\{a, x_1\} \cap N(v_k) = \emptyset$. If $k = 3$, i.e., $v_k = v_3$, then $\{v_k, a, x_1, b\} \cong K_1 \cup P_3$, a contradiction. Assume that $k \geq 4$. To avoid $\{v_k, a, x_1, b\}$ and $\{v_3, a, x_1, b\}$ inducing $K_1 \cup P_3$, we have $bv_k \in E(G)$ and $av_3 \in E(G)$. To avoid $\{v_3, x_1, b, v_k\}$ inducing $K_1 \cup P_3$, we have $v_3 v_k \in E(G)$. If $k = 4$, then $v_1 a x_1 b v_2 v_1$ is a $(k+1)$-cycle, a contradiction. If $k \geq 5$, then according to the choice of v_1, v_2 and the assumption that G has no $(k+1)$-cycle we have that $v_4 a, v_4 x_1 \notin E(G)$. To avoid $\{v_4, a, x_1, b\}$ inducing $K_1 \cup P_3$, we have $v_4 b \in E(G)$, but then $v_3 a x_1 b v_4 C v_k v_3$ is a $(k+1)$-cycle, a contradiction. Hence, $s = 0$ and the claim holds. □

By Claim 2 and the assumptions, we have that $k \geq 4$. Without loss of generality, assume that $ab \in E(H)$ and $av_1 \in E(G)$, $bv_2 \in E(G)$. We next prove the following claim.

Claim 3. $av_3 \in E(G)$.

Proof. Suppose that $av_3 \notin E(G)$. To avoid $\{v_3, b, a, v_1\}$ inducing $K_1 \cup P_3$, we have $v_1v_3 \in E(G)$. We also have that $av_4 \notin E(G)$ and $bv_4 \notin E(G)$; otherwise, $v_2bav_4Cv_2$ or $v_2bv_4Cv_1v_3v_2$ is a $(k+1)$-cycle, respectively. To avoid $\{v_4, a, b, v_2\}$ inducing $K_1 \cup P_3$, we have $v_2v_4 \in E(G)$. Then $v_1abv_2v_4Cv_1$ is a $(k+1)$-cycle, a contradiction. □

We now divide $V(C)$ into two sets. Let $A = \{v_i \mid i \text{ is odd}, 1 \leq i \leq k\}$, and $B = \{v_i \mid i \text{ is even}, 1 \leq i \leq k\}$. Then clearly $V(C) = A \cup B$. We prove the following claim on the structure of A and B.

Claim 4. k is even, and $A \subseteq N(a) \setminus N(b)$, $B \subseteq N(b) \setminus N(a)$. Moreover, A and B are independent sets, and each vertex of A is adjacent to each vertex of B.

Proof. We use induction to prove that $A \subseteq N(a) \setminus N(b)$ and $B \subseteq N(b) \setminus N(a)$. First, we show that $v_4 \in N(b) \setminus N(a)$. Since $av_3 \in E(G)$, $av_4 \notin E(G)$. We also have $v_2v_4 \notin E(G)$; otherwise, $v_1abv_2v_4Cv_1$ is a $(k+1)$-cycle, a contradiction. Hence $v_4 \in N(b)$; otherwise $\{v_4, a, b, v_2\} \cong K_1 \cup P_3$, a contradiction. Next, we show that if $v_{s-1} \in N(a), v_s \in N(b)$ $(s \geq 4)$, then $v_{s+1} \in N(a) \setminus N(b)$. First, $v_{s+1} \notin N(b)$ since $v_s \in N(b)$. We have $v_{s-1}v_{s+1} \notin E(G)$; otherwise, $v_1abv_2Cv_{s-1}v_{s+1}Cv_1$ is a $(k+1)$-cycle, a contradiction. To avoid $\{v_{s+1}, b, a, v_{s-1}\}$ inducing a $K_1 \cup P_3$, we have $v_{s+1} \in N(a)$. By a similar analysis, we get that if $v_{s-1} \in N(b), v_s \in N(a)$ $(s \geq 4)$, then $v_{s+1} \in N(b) \setminus N(a)$. Thus, $V(C) \subseteq N(a) \cup N(b)$, and $N_C(a)$ and $N_C(b)$ occur alternately on C. Since $v_1 \in N(a)$, $v_k \in N(b)$. Hence, k is even, and $A \subseteq N(a) \setminus N(b)$, $B \subseteq N(b) \setminus N(a)$.

Suppose that $v_i, v_j \in A$ and $v_iv_j \in E(G)$. Then $v_{i+1}, v_{j+1} \in B \subseteq N(b)$, and there is a $(k+1)$-cycle $v_{i+1}bv_{j+1}Cv_iv_j\overline{C}v_{i+1}$, a contradiction. Hence, A is an independent set. Similarly, B is also an independent set. Suppose that there is a pair of vertices $v_i \in A, v_j \in B$ such that $v_iv_j \notin E(G)$. We have that $v_j \neq v_{i+1}$, and since $\{v_j, v_{i+1}\} \in B$, we have $v_jv_{i+1} \notin E(G)$. Then

$\{v_j, a, v_i, v_{i+1}\} \cong K_1 \cup P_3$, a contradiction. Hence, $v_i v_j \in E(G)$ for any $v_i \in A$, $v_j \in B$, and the claim holds. □

We need two more claims before we can complete our proof for this case.

Claim 5. $G - V(C) = H$.

Proof. Suppose, by contradiction, that H' is another component distinct from H of $G - V(C)$. Then there is a vertex $y \in V(H')$ such that y has a neighbor on C. By symmetry, assume $yv_1 \in E(G)$. If y has another neighbor v_i on C distinct from v_1, then $v_i \notin A$; otherwise, $\{b, v_1, y, v_i\} \cong K_1 \cup P_3$, a contradiction. Thus, $v_i \in B \setminus \{v_2, v_k\}$ and $k \neq 4$. By Claim 4, $v_{i+1}v_2 \in E(G)$. Then $v_i y v_1 \overline{C} v_{i+1} v_2 C v_i$ is a $(k+1)$-cycle, a contradiction. Hence y has only one neighbor on C. Since G is 1-tough, there is another vertex y' in $V(H')$, and y' has a neighbor on C distinct from v_1. By the same arguments, y' has only one neighbor on C, say v_i. If $yy' \in E(G)$, then $\{b, y', y, v_1\} \cong K_1 \cup P_3$, a contradiction. If $yy' \notin E(G)$, then there is an induced P_3 in H', and this induced P_3 together with the vertex a in H will induce a $K_1 \cup P_3$, a contradiction. Hence, $G - V(C) = H$. □

By Claim 5, we know that H is the only component of $G - V(C)$, and ab is an edge of H. We denote $A' = N_H(a)$ and $B' = N_H(b)$.

Claim 6. The following properties hold:

(1) $V(H) = A' \cup B'$ and $A' \cap B' = \emptyset$.

(2) each vertex of A' is adjacent to each vertex of B, and each vertex of B' is adjacent to each vertex of A.

(3) $A \cup A'$ and $B \cup B'$ are independent sets.

(4) each vertex of A' is adjacent to each vertex of B'.

Proof. We prove the properties in the same order.

(1) If $V(H) = \{a, b\}$, then the claim holds. Now we suppose that $V(H) \neq \{a, b\}$ and by contradiction, we suppose that $V(H) \neq A' \cup B'$. There is a vertex $x \in V(H) \setminus (A' \cup B')$ such that $xa_1 \in E(G)$ or $xb_1 \in E(G)$, where

$a_1 \in A'$ and $b_1 \in B'$. Without loss of generality, assume that $xa_1 \in E(G)$. If x is adjacent to a vertex v_i of B, then by Claim 4 we have that $v_iv_{i+3} \in E(G)$, and there is a $(k+1)$-cycle $v_{i-1}aa_1xv_iv_{i+3}Cv_{i-1}$ (if $k=4$, $v_{i+3} = v_{i-1}$), a contradiction. Thus, x has no neighbor in B. For a vertex v_i of B, $\{x, a, b, v_i\}$ induces a $K_1 \cup P_3$, a contradiction. Hence, $V(H) = A' \cup B'$. Suppose that $x' \in A' \cap B'$. Then there is a $(k+1)$-cycle $v_1ax'bv_2v_5Cv_1$ (possible $v_5 = v_1$), a contradiction. Hence, $A' \cap B' = \emptyset$.

(2) Let a_1 be a vertex of A'. From (1), we have $ba_1 \notin E(G)$. Since $k \geq 4$, $|A| = |B| \geq 2$. If there are two vertices $v_i, v_j \in B$ such that $a_1v_i \notin E(G)$ and $a_1v_j \notin E(G)$, then $\{a_1, v_i, b, v_j\} \cong K_1 \cup P_3$, a contradiction. Thus, there is at most one vertex of B that is not adjacent to a_1. Suppose that $a_1v_i \notin E(G)$ and $a_1v_j \in E(G)$ ($v_i, v_j \in B$). Then $\{v_i, a, a_1, v_j\} \cong K_1 \cup P_3$, a contradiction. Hence, a_1 is adjacent to each vertex of B. By the arbitrary selection of a_1, each vertex of A' is adjacent to each vertex of B. By symmetry, each vertex of B' is adjacent to all the vertices of A.

(3) First, A' is independent set; otherwise, suppose $a_1, a_2 \in A'$ and $a_1a_2 \in E(G)$. Then by (2), there is a $(k+1)$-cycle $v_1aa_1a_2v_4Cv_1$, a contradiction. Similarly, B' is also an independent set. Next, $N(A') \cap A = \emptyset$; otherwise, suppose that $a_1 \in A'$ and $v_i \in A$ such that $a_1v_i \in E(G)$. Then there is a $(k+1)$-cycle $v_{i-2}aa_1v_iCv_{i-2}$, a contradiction. Hence, $A \cup A'$ is an independent set. Similarly, $B \cup B'$ is also an independent set.

(4) Suppose that $a_1 \in A', b_1 \in B'$ and $a_1b_1 \notin E(G)$. By (2) and (3), for any pair of vertices $v_i, v_j \in A$, $\{a_1, v_i, b_1, v_j\} \cong K_1 \cup P_3$, a contradiction. Hence, each vertex of A' is adjacent to each vertex of B'.

□

From Claims 4–6, we have that G is a complete bipartite graph with two independent sets $A \cup A'$ and $B \cup B'$. Since G is 1-tough, G is a balanced complete bipartite graph $K_{s,s}$, contradicting the assumption. This completes the proof for Case 1.

Case 2. For every component H of $G - V(C)$, any two neighbors of H on C are not consecutive.

Let $N_C(H) = \{u_1, u_2, \ldots, u_s\}$ ($s \geq 2$, and with all u_i chosen in this order according to the orientation of C). The s neighbors of H on C divide the cycle C into s segments, denoted by $S_i = u_i^+ C u_{i+1}$ ($i = 1, 2, \ldots, s$, and with $u_{s+1} = u_1$), so with $|S_i| \geq 2$ for any $i \in \{1, 2, \ldots, s\}$. Since G is 1-tough, there are at least two segments that are connected by a path internally-disjoint with $V(C) \cup V(H)$. As we will see, the choice of these two segments is irrelevant for the remainder of the proof. So we ignore the indices, and assume without loss of generality that S_1 and S_2 are connected by such a path. We choose a path P with two end vertices $y \in S_1, y' \in S_2$ such that the path $u_1^+ C y P y' \overline{C} u_2^+$ is as short as possible. Then we have that $|V(P)| = 2$ and $y = u_1^+, y' = u_2^+$; otherwise, the path $u_1^+ C y P y' \overline{C} u_2^+$ contains an induced P_3. Combining that induced P_3 with one vertex of H we get an induced $K_1 \cup P_3$, a contradiction. Thus, $u_1^+ u_2^+ \in E(G)$. Suppose that $au_1 \in E(G), bu_2 \in E(G)$ for $a, b \in V(H)$. We have $a \neq b$; otherwise, $u_1 a u_2 \overline{C} u_1^+ u_2^+ C u_1$ is a $(k+1)$-cycle. We also have that $ab \in E(G)$; otherwise, H contains an induced P_3, and combining that induced P_3 with u_1^+ we get an induced $K_1 \cup P_3$, a contradiction. If $u_1^+ = u_2^-$, then $u_1 a b u_2 C u_1$ is a $(k+1)$-cycle, a contradiction. Hence, $u_1^+ \neq u_2^-$. If $u_1^{++} u_2^+ \in E(G)$, then $u_1 a b u_2 \overline{C} u_1^{++} u_2^+ C u_1$ is a $(k+1)$-cycle, a contradiction. Hence, $u_1^{++} u_2^+ \notin E(G)$. Then $\{a, u_1^{++}, u_1^+, u_2^+\} \cong K_1 \cup P_3$, our final contradiction. □

This completes the proof of Theorem 6.6. ■

6.4 Proof of Theorem 6.7

Suppose that G is a graph satisfying the conditions of Theorem 6.7. Since G is 1-tough, G contains a cycle. The shortest (induced) cycle of G is C_3, C_4, C_5 or C_6; otherwise, G clearly has an induced subgraph isomorphic to $2K_1 \cup K_2$. We again start by proving a number of claims.

Claim 1. If the shortest cycle of G is C_5 or C_6, then G is C_5 or C_6, respectively.

Proof. Suppose that the shortest cycle of G is C_5. Let $C = v_1 v_2 \ldots v_5 v_1$ be an induced 5-cycle. If G is not C_5, then there is a vertex $u \in G - V(C)$ such that u is adjacent to a vertex of C, say v_1. Since G has neither a C_3 nor a C_4, u has no other neighbor on C distinct from v_1. Now $\{u, v_2, v_4, v_5\} \cong 2K_1 \cup K_2$, a contradiction. In the same way, we can prove that if the shortest cycle of G is C_6, then G is C_6. □

Claim 2. If the shortest cycle of G is C_4, then G is C_4 or $K_{s,s}-M$, where $s \geq 3$ and M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s$.

Proof. Suppose that w is a vertex of G with the maximum degree. If $d(w) = 2$, then G is C_4, and the claim holds. Assume that $d(w) \geq 3$. Now we draw the graph G arranged as a rooted tree: w is the root and denoted as the first layer L_1, all the neighbors of w are arranged as the second layer L_2, all the neighbors of vertices of L_2 that are new (where new means that the vertices do not appear in existing layers) are arranged as the third layer L_3, etc., until all layers together cover $V(G)$. By this labeling method, we know that there is no edge between L_i and L_j if $i \neq j-1$ and $i \neq j+1$. Since G has no C_3, L_2 is an independent set. Next we prove three subclaims on the structure of the layers.

Claim 2.1. G has at most 4 layers.

Proof. Suppose that G has 5 or more layers. Let u_1, u_2 be two vertices of L_2, and let x and y be two vertices of L_4 and L_5, respectively, such that $xy \in E(G)$. Then $\{u_1, u_2, x, y\} \cong 2K_1 \cup K_2$, a contradiction. □

Let $l_i = |L_i|$ for $i = 1, 2, 3, 4$. Then we denote $L_1 = \{w\}$, $L_2 = \{u_1, u_2, \ldots, u_{l_2}\}$, $L_3 = \{v_1, v_2, \ldots, v_{l_3}\}$, $L_4 = \{z_1, z_2, \ldots, z_{l_4}\}$.

Claim 2.2. $L_4 = \{z_1\}$ or $L_4 = \emptyset$.

Proof. If L_4 has 2 or more vertices, then $\{z_1, z_2, w, u_1\} \cong 2K_1 \cup K_2$ (if $z_1 z_2 \notin E(G)$) or $\{u_1, u_2, z_1, z_2\} \cong 2K_1 \cup K_2$ (if $z_1 z_2 \in E(G)$), a contradiction. □

Claim 2.3. L_3 is independent set.

Proof. By contradiction, suppose that $v_i v_j$ is an edge of L_3. Since G has no C_3, v_i and v_j have no common neighbor in L_2. Assume that $u_i v_i, u_j v_j \in E(G)$ and $u_i \neq u_j$. We have that u_i, u_j have no common neighbor in L_3. Otherwise, suppose that v_k is a common neighbor of u_i and u_j in L_3. Obviously,

$v_iv_k, v_jv_k \notin E(G)$; otherwise G contains a C_3. Then $\{w, v_k, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction. Moreover, $N_{L_3}(u_i) = \{v_i\}$ and $N_{L_3}(u_j) = \{v_j\}$. Otherwise, suppose that $v_k \in N_{L_3}(u_i) \setminus \{v_i\}$. Clearly, $v_iv_k \notin E(G)$; otherwise G contains a C_3. Then $\{v_i, v_k, w, u_j\} \cong 2K_1 \cup K_2$, a contradiction. For any vertex $u_k \in L_2 \backslash \{u_i, u_j\}$, we have that $N_{L_3}(u_k) = \{v_i\}$ or $N_{L_3}(u_k) = \{v_j\}$. Otherwise, suppose that v_k is a vertex of L_3 different from v_i, v_j that is adjacent to u_k. Then $\{u_i, u_j, u_k, v_k\} \cong 2K_1 \cup K_2$, a contradiction. Thus, $L_3 = \{v_1, v_2\}$ and every vertex of L_2 has degree 2. Since G is 1-tough and L_2 is independent, $l_2 \leq 3$. Since $l_2 = d(w) \geq 3$, $l_2 = 3$. Without loss of generality, assume that $u_1v_1, u_2v_2, u_3v_2 \in E(G)$. Then $\{u_2, u_3, u_1, v_1\} \cong 2K_1 \cup K_2$, a contradiction. □

To complete the proof of Claim 2, we consider two cases.

Case A. $L_4 = \{z_1\}$. Suppose that v_1 is a neighbor of z_1 in L_3. Since G is 1-tough, the two independent sets L_2 and L_3 obviously have the same order, i.e., $l_2 = l_3 \geq 3$. If there is a vertex $v_i \in L_3 \setminus \{v_1\}$ such that $v_iz_1 \notin E(G)$, then $\{w, v_i, v_1, z_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $N(z_1) = L_3$. If there are two vertices $u_i, u_j \in L_2$ that are not adjacent to a vertex $v_k \in L_3$, then $\{u_i, u_j, v_k, z_1\} \cong 2K_1 \cup K_2$, a contradiction. If all the vertices of L_2 are neighbors of v_k, then $d(v_k) = l_2 + 1 > d(w)$, contradicting the assumption that w is a vertex with maximum degree. Thus, every vertex of L_3 has exactly $l_2 - 1$ neighbors in L_2. Similarly, every vertex of L_2 has exactly $l_3 - 1$ neighbors in L_3. Hence, G is a balanced bipartite graph $K_{s,s} - M^*$ with two independent vertex set $L_2 \cup \{z_1\}$ and $L_3 \cup \{w\}$, where $s = l_2 + 1 = l_3 + 1$ and M^* is a perfect matching of $K_{s,s}$.

Case B. $L_4 = \emptyset$. Since G is 1-tough, $l_2 = l_3 + 1$. For a vertex $u_k \in L_2$, if there are two vertices $v_i, v_j \in L_3$ that are not neighbors of u_k, then $\{v_i, v_j, u_k, w\} \cong 2K_1 \cup K_2$, a contradiction. Thus, every vertex of L_2 has at least $l_3 - 1$ neighbors in L_3, and has degree at least l_3 in the graph G. For a vertex $v_k \in L_3$, suppose that $v_ku_k \in E(G)$. If there are two vertices $u_i, u_j \in L_2$ that are not neighbors of v_k, then $\{u_i, u_j, v_k, u_k\} \cong 2K_1 \cup K_2$, a contradiction. Thus, every vertex of L_3 has at least $l_2 - 1 = l_3$ neighbors in L_3. Hence, G is a balanced bipartite graph $K_{s,s} - M$ with two independent vertex sets L_2 and $\{w\} \cup L_3$, where $s = l_2 = l_3 + 1$ and M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s - 1$.

In both cases, we conclude that G is either C_4 or $K_{s,s} - M$, where $s \geq 3$ and M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s$. This completes the proof of Claim 2. □

Claim 3. If the minimum cycle of G is C_3 and $G \neq C_3$, then G contains C_4, unless $G = C_6^+$, where C_6^+ is the graph obtained by adding an edge to C_6 between two vertices at distance 2 in C_6.

Proof. Suppose that $C = abca$ is a 3-cycle of G. For a component H of $G - V(C)$, H has at least two neighbors on C. If one vertex of H has two neighbors on C, then G contains C_4, and the claim holds. Assume that each vertex of H is adjacent to at most one vertex of C. Suppose that $a_1, b_1 \in V(H)$ and $aa_1, bb_1 \in E(G)$. If $a_1b_1 \in E(G)$, then G contains C_4, and the claim holds. Assume that $a_1b_1 \notin E(G)$. Let $P = a_1x_1x_2 \ldots x_sb_1$ be a shortest (a_1, b_1)-path in H. We prove the following claims.

Claim 3.1. $s = 1$, i.e., $P = a_1x_1b_1$.

Proof. Suppose that $s \geq 2$. Since P is a shortest path, $x_1b_1 \notin E(G)$. If $x_1c \in E(G)$, then G contains C_4, and the claim holds. Assume $x_1c \notin E(G)$. Then $\{b_1, c, a_1, x_1\} \cong 2K_1 \cup K_2$, a contradiction. □

Claim 3.2. $H = P$.

Proof. Suppose, by contradiction, that $w \in V(H) \setminus \{a_1, x_1, b_1\}$. If $wa_1 \in E(G)$ and $wb_1 \in E(G)$, then $a_1wb_1x_1a_1$ is a 4-cycle, and the claim holds. Without loss of generality, we assume that $wb_1 \notin E(G)$. If $wa_1 \in E(G)$ and $wc \in E(G)$, then a_1wcaa_1 is a 4-cycle, and the claim holds. If $wa_1 \in E(G)$ and $wc \notin E(G)$, then $\{c, b_1, a_1, w\} \cong 2K_1 \cup K_2$, a contradiction. Thus, $wa_1 \notin E(G)$, and $N_H(a) \cup N_H(b) = \{x_1\}$. There must be a vertex $w' \in V(H) \setminus \{a_1, x_1, b_1\}$ such that $w'x_1 \in E(G)$ (possibly $w' = w$). If $w'b \in E(G)$, then $w'bb_1x_1w'$ is a 4-cycle, and the claim holds. Assume that $w'b \notin E(G)$. If $w'c \in E(G)$, then $\{a_1, b_1, w', c\} \cong 2K_1 \cup K_2$, a contradiction. If $w'c \notin E(G)$, then $\{a_1, w', b, c\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $H = P$. □

By Claim 3.2, $H = a_1x_1b_1$. If x_1 is adjacent to a vertex of C, then G contains C_4, and the claim holds. Assume that x_1 has no neighbor on C. If $G - V(C)$

has no other component than H, then G is C_6^+, and the claim holds. If H' is a component of $G-V(C)$ different from H, then H' is trivial; otherwise, an edge of H' with a_1, b_1 will induce a $2K_1 \cup K_2$, a contradiction. Since G is 1-tough, the vertex of H' has two neighbors on C. Thus, G contains C_4, and the claim holds. $\square$

By Claims 1 and 2, if G has no 3-cycle, then G is C_4, C_5, C_6 or $K_{s,s}-M$, where $s \geq 3$ and M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s$. By Claim 3, if G has a 3-cycle and G is not C_3, then G has a 4-cycle, unless G is C_6^+. Now we assume that $G \notin \{C_4, C_5, C_6, C_6^+, K_{s,s}-M\}$, hence that G has a C_3 and a C_4. Next we will show that G is pancyclic by proving the following fact.

Fact: If G has a k-cycle ($k = 4, 5, \ldots, n-1$), then G has a $(k+1)$-cycle.

Proof. Suppose, by contradiction, that G has a k-cycle but no $(k+1)$-cycle for some $k \in \{4, 5, \ldots, k-1\}$. Let $C = v_1 v_2 \ldots v_k v_1$ be a k-cycle, and let H be a component of $G-V(C)$. Since G is 1-tough, H has at least two neighbors on C. We distinguish the cases that H is trivial, i.e., $|V(H)| = 1$, and that all components of $G-V(C)$ contain at least one edge.

Case 1. H is trivial.
Suppose that $H = \{w\}$. Denote $N_C(w) = \{u_1, u_2, \ldots, u_s\}$, with $s \geq 2$, the vertices u_i chosen in this order according to the orientation of C, and taking $u_{s+1} = u_1$. Clearly, $u_{i+1} \neq u_i^+$ for any $i \in \{1, 2, \ldots, s\}$; otherwise G contains a $(k+1)$-cycle, a contradiction. Now, the s neighbors of H on C divide the cycle C into s segments. Let S_i be the segment of C from u_i^+ to u_{i+1}^-, denoted as $S_i = x_{i_1} x_{i_2} \ldots x_{i_{r_i}}$. We again prove a number of claims.

Claim 4. For any $i \in \{1, 2, \ldots, s\}$, r_i is odd, and $x_{1_1} x_{i_j} \notin E(G)$ for every odd j, and $x_{1_1} x_{i_j} \in E(G)$ for every even j.

Proof. We divide the proof into two cases according to the length of the segment S_1.

Case A. $|S_1| = 1$. In this case, $S_1 = x_{1_1}$, and the claim holds for segment S_1 itself. For any segment S_i ($i = 2, 3, \ldots, s$), we have that if $x_{1_1} x_{i_j} \in E(G)$, then $x_{1_1} x_{i_{j+1}} \notin E(G)$; otherwise, suppose that $x_{1_1} x_{i_j} \in E(G)$ and $x_{1_1} x_{i_{j+1}} \in$

$E(G)$. Then there is a $(k+1)$-cycle $u_1wu_2Cx_{i_j}x_{1_1}x_{i_{j+1}}Cu_1$, a contradiction. If $x_{1_1}x_{i_j} \notin E(G)$, then $x_{1_1}x_{i_{j+1}} \in E(G)$; otherwise, suppose that $x_{1_1}x_{i_j} \notin E(G)$ and $x_{1_1}x_{i_{j+1}} \notin E(G)$. Then $\{w, x_{1_1}, x_{i_j}, x_{i_{j+1}}\} \cong 2K_1 \cup K_2$, a contradiction. For the first vertex x_{i_1} and the last vertex $x_{i_{r_i}}$ of a segment, we have $x_{1_1}x_{i_1} \notin E(G)$ and $x_{1_1}x_{i_{r_i}} \notin E(G)$; otherwise, $u_1wu_i\overline{C}x_{1_1}x_{i_1}Cu_1$ or $u_2wu_{i+1}Cx_{1_1}x_{i_{r_i}}\overline{C}u_2$ is a $(k+1)$-cycle, a contradiction. Thus, the neighbors of x_{1_1} occur alternately along the cycle on every segment S_i, and the two end vertices of S_i are not its neighbor. Hence, r_i is odd, and $x_{1_1}x_{i_j} \notin E(G)$ for every odd j, and $x_{1_1}x_{i_j} \in E(G)$ for every even j.

Case B. $|S_1| \geq 2$. First, we deal with the segment S_1. We have that $x_{1_1}x_{2_1} \notin E(G)$; otherwise, $u_1wu_2\overline{C}x_{1_1}x_{2_1}Cu_1$ is a $(k+1)$-cycle, a contradiction. We also have that $x_{1_2}x_{2_1} \in E(G)$; otherwise, $\{w, x_{2_1}, x_{1_1}, x_{1_2}\} \cong 2K_1 \cup K_2$, a contradiction. If $x_{1_1}x_{1_j} \in E(G)$ and $x_{1_1}x_{1_{j+1}} \in E(G)$, then we have a $(k+1)$-cycle $u_1wu_2\overline{C}x_{1_{j+1}}x_{1_1}x_{1_j}\overline{C}x_{1_2}x_{2_1}Cu_1$, a contradiction. If $x_{1_1}x_{1_j} \notin E(G)$ and $x_{1_1}x_{1_{j+1}} \notin E(G)$, then $\{w, x_{1_1}, x_{1_j}, x_{1_{j+1}}\} \cong 2K_1 \cup K_2$, a contradiction. For the last vertex $x_{1_{r_1}}$ of S_1, we will show that it is not a neighbor of x_{1_1}. Suppose that $|S_2| \geq 2$. We have that $x_{1_1}x_{2_2} \in E(G)$; otherwise, $\{w, x_{1_1}, x_{2_1}, x_{2_2}\} \cong 2K_1 \cup K_2$, a contradiction. If $x_{1_1}x_{1_{r_1}} \in E(G)$, then $u_1wu_2x_{2_1}x_{1_2}Cx_{1_{r_1}}x_{1_1}x_{2_2}Cu_1$ is a $(k+1)$-cycle, a contradiction. Suppose that $|S_2| = 1$. If $x_{1_1}x_{1_{r_1}} \in E(G)$, then $u_2wu_3Cx_{1_1}x_{1_{r_1}}\overline{C}x_{1_2}x_{2_1}u_2$ is a $(k+1)$-cycle, a contradiction. Hence, $x_{1_1}x_{1_{r_1}} \notin E(G)$. Then the neighbors of x_{1_1} on segment S_1 occur alternately along the cycle, and the first vertex and the last vertex of S_1 are not its neighbor. Therefor, r_1 is odd and the claim holds for segment S_1.

Next, we consider the other segments S_i $(i = 2, 3, \ldots, s)$. Similar with x_{2_1}, for x_{i_1} we have that $x_{1_1}x_{i_1} \notin E(G)$. Also, we have that $x_{1_2}x_{i_1} \in E(G)$; otherwise, $\{w, x_{i_1}, x_{1_1}, x_{1_2}\} \cong 2K_1 \cup K_2$, a contradiction. If $x_{1_1}x_{i_j} \in E(G)$ and $x_{1_1}x_{i_{j+1}} \in E(G)$, then we have a $(k+1)$-cycle $u_1wu_i\overline{C}x_{1_2}x_{i_1}Cx_{i_j}x_{1_1}x_{i_{j+1}}Cu_1$, a contradiction. If $x_{1_1}x_{i_j} \notin E(G)$ and $x_{1_1}x_{i_{j+1}} \notin E(G)$, then $\{w, x_{1_1}, x_{i_j}, x_{i_{j+1}}\} \cong 2K_1 \cup K_2$, a contradiction. Thus the neighbors of x_{1_1} occur alternately on every segment S_i along the cycle. Moreover, $x_{1_1}x_{i_{r_i}} \notin E(G)$; otherwise, we have a $(k+1)$-cycle $u_iwu_{i+1}Cx_{1_1}x_{i_{r_i}}\overline{C}x_{i_1}x_{1_2}Cu_i$, a contradiction. Hence, r_i is odd, and $x_{1_1}x_{i_j} \notin E(G)$ for every odd j, and $x_{1_1}x_{i_j} \in E(G)$ for every even j. □

Denote $W = N_C(w)$ and $A = N_C(x_{1_1}) \setminus W$. By Claim 4, $|W| + |A| = \frac{|V(C)|}{2}$. If $N(x_{1_1}) = N_C(x_{1_1})$, then $A \cup W$ is a cut set and $G - (A \cup W)$ generates at least three components, including two trivial components with vertex sets $\{w\}$ and $\{x_{1_1}\}$. Since G is $2K_1 \cup K_2$-free, all the other components are also trivial. Then we get $\frac{|V(C)|}{2} + 1$ components by deleting $\frac{|V(C)|}{2}$ vertices, contradicting the fact that G is 1-tough. Hence, we next assume that x_{1_1} has a neighbor outside the cycle C. Suppose that y is a neighbor of x_{1_1} in $G - V(C)$. Then $y \neq w$ and $yw \notin E(G)$. Denote $B = V(C) \setminus (W \cup A)$. Before continuing the proof for Case 1, we first prove three more claims.

Claim 5. $B \subseteq N(y)$, $A \cap N(y) = \emptyset$, $W \cap N(y) = \emptyset$ and A, B, and W are independent sets.

Proof. By the definition of B, we have $B \cap N(x_{1_1}) = \emptyset$. If there is a vertex $v_i \in B$ such that $by \notin E(G)$, then $\{w, v_i, x_{1_1}, y\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $B \subseteq N(y)$. Suppose that $v_j \in A$ and $v_j y \in E(G)$. Since $v_j^+ \in B$, $v_j^+ y \in E(G)$, and there is a $(k+1)$-cycle, a contradiction. For any vertex $v_i \in W$, $v_i^-, v_i^+ \in B$, hence $v_i^- y, v_i^+ y \in E(G)$. If $v_i y \in E(G)$, then there is a $(k+1)$-cycle, a contradiction. Hence, $A \cap N(y) = \emptyset$ and $W \cap N(y) = \emptyset$.

If there is an edge $v_i v_j$ in A, then $\{w, y, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction. If there is an edge $v_i v_j$ in B, then $\{w, x_{1_1}, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction. Suppose $v_i, v_j \in W$ and $v_i v_j \in E(G)$. Since $v_i^-, v_j^- \in B$, $v_i^- y, v_j^- y \in E(G)$. Then $v_i^- y v_j^- \overline{C} v_i v_j C v_i^-$ is a $(k+1)$-cycle, a contradiction. Hence, A, B, and W are independent sets. □

Claim 6. $A = \emptyset$.

Proof. Suppose, by contradiction, that $A \neq \emptyset$. We claim that every vertex of A is adjacent to every vertex of W; otherwise, suppose that $v_i \in A, v_j \in W$ and $v_i v_j \notin E(G)$. Then, using Claim 5 we have that $\{y, v_i, v_j, w\} \cong 2K_1 \cup K_2$, a contradiction. Suppose that $v_i \in A$ and $v_j \in W$, with $v_i v_j \in E(G)$. Since $v_i^+, v_j^+ \in B$, by Claim 5, $v_i^+ y \in E(G), v_j^+ y \in E(G)$. Then there is a $(k+1)$-cycle $v_i^+ y v_j^+ C v_i v_j \overline{C} v_i^+$, a contradiction. □

By Claim 6, $S_i = x_{i_1}$ for every $i \in \{1, 2, \ldots, s\}$, and $V(C) = B \cup W$, $|B| = |W|$. By Claim 5, we have that B and W are independent sets and $B \subseteq N(y), W = N(w), W \cap N(y) = \emptyset$.

Claim 7. $N_{G-V(C)}(C) = \{w, y\}$.

Proof. Suppose, by contradiction, that $z \in N_{G-V(C)}(C) \setminus \{w, y\}$. If $zv_i \in E(G)$ and $v_i \in B$, then $yz \notin E(G)$; otherwise, $v_i z y v_i^{+2} C v_i$ is a $(k+1)$-cycle, a contradiction. Then, $\{y, z, v_i^+, w\} \cong 2K_1 \cup K_2$, a contradiction. Now suppose that $zv_i \in E(G)$ and $v_i \in W$. We have that $yz \in E(G)$; otherwise, $\{w, z, y, v_i^+\} \cong 2K_1 \cup K_2$, a contradiction. If there is another vertex $z' \in N_{G-V(C)}(C) \setminus \{w, y, z\}$, using the same arguments, we have that $N_C(z') \subseteq W$ and $z'y \in E(G), z'w \notin E(G)$. That means, if $N_{G-V(C)}(C) \neq \{w, y\}$, then every vertex $a \in N_{G-V(C)}(C) \setminus \{w, y\}$ has the properties: $ay \in E(G), aw \notin E(G)$ and $N_C(a) \subseteq W$. By deleting the vertices of $W \cup \{y\}$ we obtain at least $|B| + 2$ components, contradicting the fact that G is 1-tough. □

Now, we are ready to complete the proof for Case 1. By Claim 7, the component that contains y is trivial; otherwise y is a cut vertex. Thus, $G - V(C)$ has precisely two trivial components $\{w\}$ and $\{y\}$. Since $k \geq 4$, $|B| = |W| \geq 2$. If there are two vertices $v_i, v_j \in W$ that are not adjacent to a vertex $v_k \in B$, then $\{v_i, v_j, v_k, y\} \cong 2K_1 \cup K_2$, a contradiction. If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_k \in W$, then $\{v_i, v_j, v_k, w\} \cong 2K_1 \cup K_2$, a contradiction. Thus, each vertex of W has at least $|B| - 1$ neighbors in B and each vertex of B has at least $|W| - 1$ neighbors in W. Hence, G is a balanced bipartite graph $K_{s,s} - M$ with two vertex set $\{w\} \cup B$ and $\{y\} \cup W$, where $s \geq 3$ and M is a matching of $K_{s,s}$. Since $wy \notin E(G)$, $1 \leq |M| \leq s$. This contradicts the assumption and completes the proof for Case 1.

Case 2. All the components of $G - V(C)$ contain at least one edge.

Suppose that H is a component of $G - V(C)$. We distinguish three subcases according to the distribution of $N_C(H)$, in particular whether there are nontrivial segments (containing at least one nonneighbor of H between two subsequent neighbors of H on C) or not. We start with the subcase that there are at least two such segments.

Case 2.1. There are at least two nontrivial segments on C.

For a vertex $v_i \in V(C)$, if $v_i \notin N_C(H)$ and $v_i^- \in N_C(H)$, then we say v_i is a *break vertex*. By the assumption, there are at least two break vertices on C. We call any two break vertices a *break pair*. Let S denote the set of all break vertices, and call S the *break set* (of C). We use the shorthand S is complete to indicate that S induces a complete graph in G. Suppose ab is an edge of H. We next prove four claims.

Claim 8. S is complete, $V(H) = \{a, b\}$, and $V(C) \setminus S \subseteq N(a) \cup N(b)$, hence $S = V(C)\setminus N_C(H)$. Moreover, for any two subsequent break vertices $v_i, v_j \in S$, $|v_i^+ C v_j^-|$ is even, and the vertices of $v_i^+ C v_j^-$ are alternately neighbors of a and b.

Proof. First, we have that $V(C) \setminus N_C(H)$ induces a complete graph; otherwise, suppose that v_i, v_j are two nonadjacent vertices of $V(C) \setminus N_C(H)$. Then $\{v_i, v_j, a, b\} \cong 2K_1 \cup K_2$, a contradiction. Hence, S is complete, since $S \subseteq V(C) \setminus N_C(H)$. Also, H is a complete graph; otherwise, suppose that w_1, w_2 are two nonadjacent vertices of H, and v_i, v_j are two vertices of $V(C)\setminus N_C(H)$. Then $\{w_1, w_2, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction.

Suppose that $\{v_i, v_j\}$ is a break pair and in the segment $v_i^+ C v_j^-$ there is no other break vertex. Since $v_i, v_j \in S$, $v_i v_j \in E(G)$. We have that $N_H(v_i^-) \cap N_H(v_j^-) = \emptyset$; otherwise, let $w \in N_H(v_i^-) \cap N_H(v_j^-)$. Then $v_i^- w v_j^- \overline{C} v_i v_j C v_i^-$ is a $(k+1)$-cycle, a contradiction. Since H is complete, without loss of generality, we assume that $v_i^- a \in E(G)$ and $v_j^- b \in E(G)$. Then $v_i^+ \neq v_j^-$; otherwise, $v_i^- a b v_j^- C v_i^-$ is a $(k+1)$-cycle, a contradiction. If $v_i^+ \notin N_C(H)$, then $v_i^+ v_j \in E(C)$, and $v_i^- a b v_j^- \overline{C} v_i^+ v_j C v_i^-$ is a $(k+1)$-cycle, a contradiction. Hence, $v_i^+ \in N_C(H)$. We have that $N_H(v_i^+) = \{a\}$; otherwise, suppose that $v_i^+ c \in E(G)$ and $c \in V(H) \setminus \{a\}$. Then $v_i^- a c v_i^+ C v_i^-$ is a $(k+1)$-cycle, a contradiction. Since v_i^{+2} is not a break vertex and $v_i^-, v_i^+ \in N(a)$, we have $v_i^{+2} \in N(b)$; otherwise, $\{v_i, v_i^{+2}, a, b\} \cong 2K_1 \cup K_2$, a contradiction. If $V(H) \neq \{a, b\}$, suppose $c \in V(H) \setminus \{a, b\}$. Then $v_i^- a c b v_i^{+2} C v_i^-$ is a $(k+1)$-cycle, a contradiction. Hence, $V(H) = \{a, b\}$.

Since $v_i^+ C v_j^-$ contains no other break vertex, $v_i^+ C v_j^- \subseteq N_C(H) = N(a) \cup N(b)$. Since G has no $(k+1)$-cycle, the vertices of $v_i^+ C v_j^-$ are alternately neighbors of a and b. And since the two end vertices of $v_i^+Cv_j^-$ belong to different neighbor sets of a and b, $|v_i^+Cv_j^-|$ is even.

By the definition of break vertex, any two break vertices are not consecutive vertices on C. Thus the break vertices divide the cycle into segments, and from the above analysis we know that each segment (between two break vertices, and not containing any break vertices) belongs to the union of the neighbor sets of a and b. Hence, $V(C)\setminus S \subseteq N(a)\cup N(b)$, and therefore, $S = V(C)\setminus N_C(H)$. □

Claim 9. $|S| = 2$.

Proof. Suppose, by contradiction, that $|S| \geq 3$. Assume that $v_i, v_j, v_k \in S$. By Claim 8, S is complete. Thus, $v_iv_j, v_iv_k, v_jv_k \in E(G)$. Since $V(H) = \{a,b\}$, two of the three vertices v_i^-, v_j^-, v_k^- have a common neighbor in $\{a,b\}$. Without loss of generality, assume that $v_i^-, v_j^- \in N(a)$. Then $v_i^- a v_j^- \overline{C} v_i v_j C v_i^-$ is a $(k+1)$-cycle, a contradiction. □

Suppose that $S = \{v_s, v_t\}$. By Claim 8, all vertices of $V(C)\setminus\{v_s,v_t\}$ are adjacent to a or b alternately, v_s^-, v_s^+ share the same neighbor in $\{a,b\}$, while v_s^+, v_t^- have different neighbors in $\{a,b\}$. Thus, we have that either $v_s^-, v_s^+ \in N(a)$ and $v_t^-, v_t^+ \in N(b)$, or $v_s^-, v_s^+ \in N(b)$ and $v_t^-, v_t^+ \in N(a)$. Without loss of generality, we assume that $v_s^-, v_s^+ \in N(a)$ and $v_t^-, v_t^+ \in N(b)$. Let $A = N_C(a)$, $B = N_C(b)$. Clearly, $A\cap B = \emptyset$, $|A| = |B|$ and $V(C) = A\cup B\cup S$.

Claim 10. $A\cup\{v_t\}$ and $B\cup\{v_s\}$ are independent sets.

Proof. First, we prove that A and B are independent sets. Suppose that $v_i, v_j \in A$ and $v_iv_j \in E(G)$. If $\{v_i, v_j\} = \{v_s^-, v_s^+\}$, then $v_s^+ab v_s^{+2} C v_s^- v_s^+$ is a $(k+1)$-cycle, a contradiction. If $\{v_i, v_j\} \neq \{v_s^-, v_s^+\}$, then either $v_i^+, v_j^+ \in B$ or $v_i^-, v_j^- \in B$. Without loss of generality, assume that $v_i^+, v_j^+ \in B$. Then $v_i^+ b v_j^+ C v_i v_j \overline{C} v_i^+$ is a $(k+1)$-cycle, a contradiction. Hence, A is independent. Similarly, B is also independent.

Next, we prove that $N(v_t)\cap A = \emptyset$ and $N(v_s)\cap B = \emptyset$. Suppose that $v_i \in A$ and $v_iv_t \in E(G)$. Clearly, either $v_i^- \in B$ or $v_i^+ \in B$. Without loss of generality, assume $v_i^- \in B$. Then we have a $(k+1)$-cycle $v_t^- b v_i^- \overline{C} v_t v_i C v_t^-$, a contradiction. Hence, $N(v_t) \cap A = \emptyset$. Similarly, $N(v_s) \cap B = \emptyset$. □

Claim 11. $G - V(C) = H$.

Proof. Suppose that $w \in V(G) \setminus (V(C) \cup V(H))$. We have that $wv_s, wv_t \in E(G)$; otherwise, suppose that $wv_s \notin E(G)$. Then $\{w, v_s, a, b\} \cong 2K_1 \cup K_2$, a contradiction. Since G has no $(k+1)$-cycle, $v_s^-, v_s^+, v_t^-, v_t^+ \notin N(w)$. Using Claim 10, $\{v_t^-, v_t^+, v_s, w\} \cong 2K_1 \cup K_2$, a contradiction. □

For a vertex $v_i \in A$, if $v_i v_s \notin E(G)$, then $\{v_i, b, v_s, v_t\} \cong 2K_1 \cup K_2$, a contradiction. Hence, for each vertex $v_i \in A$, $v_i v_s \in E(G)$. Similarly, for each vertex $v_j \in B$, $v_j v_t \in E(G)$. If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_k \in A$, then $\{v_i, v_j, v_k, a\} \cong 2K_1 \cup K_2$, a contradiction. Hence, every vertex of A has at least $|B|-1$ neighbors in B. Similarly, every vertex of B has at least $|A|-1$ neighbors in A. Let $A' = A \cup \{v_t\} \cup \{b\}$ and $B' = B \cup \{v_s\} \cup \{a\}$. From the above analysis, we conclude that $G = K_{p,p} - M$ for the two independent vertex set A' and B', and a matching M of $K_{p,p}$. Since $v_s b, v_t a \notin E(G)$, $2 \leq |M| \leq p$. This final contradiction to the assumption completes the proof for Case 2.1.

Case 2.2. There is precisely one nontrivial segment on C.

In this subcase, there exist two vertices v_i, v_j on C such that $v_i C v_j \subseteq N_C(H)$ and $v_j^+ C v_i^- \cap N_C(H) = \emptyset$. Let $P = v_j^+ C v_i^-$, and let ab be an edge of H. We again start by proving several claims and considering some subcases.

Claim 12. $|V(P)| \leq 2$.

Proof. Suppose, by contradiction, that $|V(P)| \geq 3$. We have that $\langle V(P) \rangle$ is complete; otherwise, any two nonadjacent vertices of P together with the edge ab will induce a $2K_1 \cup K_2$. Similarly, H is complete; otherwise, two nonadjacent vertices of H together with an edge of P will induce a $2K_1 \cup K_2$, a contradiction. Let v_k, v_k^+ be two consecutive vertices of $N_C(H)$, and w_1, w_2 be their neighbors in H, respectively. Then $w_1 w_2 \in E(G)$ and we get a $(k+1)$-cycle $v_k w_1 w_2 v_k^+ C v_j^+ v_j^{+3} C v_k$, a contradiction. □

We distinguish the subcases that $|V(P)| = 2$ and $|V(P)| = 1$.

Case 2.2.1. $|V(P)| = 2$, i.e., $P = v_j^+ v_i^-$.
For convenience, we denote $C = v_1 v_2 ... v_k v_1$, and $N_C(H) = v_1 C v_{k-2}$. Then $P = v_{k-1} v_k$. In this case, H is complete; otherwise, any two nonadjacent vertices of H together with the edge of P will induce a $2K_1 \cup K_2$, a contradiction.

Claim 13. $V(H) = \{a, b\}$ and k is even.

Proof. If v_1 and v_{k-2} have two different neighbors in H, without loss of generality, assume that $v_1 a, v_{k-2} b \in E(G)$. Then $V(H) = \{a, b\}$; otherwise, suppose $c \in V(H) \setminus \{a, b\}$. Then $v_1 a c b v_{k-2} \overline{C} v_1$ is a $(k+1)$-cycle, a contradiction. Suppose that $N_H(v_1) = N_H(v_{k-2}) = \{a\}$. If $v_1 v_{k-2} \in E(G)$, then $V(H) = \{a, b\}$; otherwise, suppose that $c \in V(H) \setminus \{a, b\}$, and without loss of generality, suppose $v_2 b \in E(G)$. Then $v_1 a c b v_2 C v_{k-2} v_1$ is a $(k+1)$-cycle, a contradiction. If $v_1 v_{k-2} \notin E(G)$, then $V(H) = \{a, b\}$; otherwise, for any vertex $c \in V(H) \setminus \{a, b\}$ we have $\{v_1, v_{k-2}, b, c\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $V(H) = \{a, b\}$ and $V(C) \setminus V(P) \subseteq N(a) \cup N(b)$. Since G has no $(k+1)$-cycle, each vertex of $V(C) \setminus V(P)$ is adjacent to one and only one of $\{a, b\}$, alternately. If we can prove that v_1, v_{k-2} are adjacent to different vertices of $\{a, b\}$, then we get that k is even. First, we have that $v_k v_2 \notin E(G)$ and $v_{k-3} v_{k-1} \notin E(G)$; otherwise, suppose $v_k v_2 \in E(G)$. Then $v_2 b a v_3 C v_k v_2$ (if $k \geq 5$) or $v_k v_2 b a v_1 v_k$ (if $k = 4$) is a $(k+1)$-cycle, a contradiction. Suppose that v_1, v_{k-2} are adjacent to the same vertex of $\{a, b\}$, say a without loss of generality. Clearly, $v_2 v_{k-1} \notin E(G)$. Then $\{a, v_2, v_{k-1}, v_k\} \cong 2K_1 \cup K_2$, a contradiction. Hence, v_1, v_{k-2} are adjacent to different vertices of $\{a, b\}$, and $|V(C) \setminus V(P)|$ is even, so k is even. □

Let $A = N_C(a) = \{v_1, v_3, \ldots, v_{k-3}\}$, $B = N_C(b) = \{v_2, v_4, \ldots, v_{k-2}\}$. Then $V(C) = A \cup B \cup V(P)$, $A \cap B = \emptyset$ and $|A| = |B|$.

Claim 14. $G - V(C) = H$.

Proof. Suppose, by contradiction, that $w \in V(G) \setminus (V(C) \cup V(H))$. To avoid $\{w, v_{k-1}, a, b\}$ or $\{w, v_k, a, b\}$ inducing $2K_1 \cup K_2$, we have that $w v_{k-1}, w v_k \in E(G)$. Then there clearly is a $(k+1)$-cycle, a contradiction. □

Claim 15. $A \cup \{v_{k-1}\}$ and $B \cup \{v_k\}$ are independent sets.

Proof. First, we prove that A and B are independent sets. If $|A| = |B| = 1$, then the claim holds. Now we suppose that $|A| = |B| \geq 2$. If $v_i, v_j \in A$ and $v_iv_j \in E(G)$, then either $v_i^-, v_j^- \in B$ or $v_i^+, v_j^+ \in B$. Without loss of generality, assume $v_i^-, v_j^- \in B$. Then there is a $(k+1)$-cycle $v_i^- b v_j^- \overline{C} v_i v_j C v_i^-$, a contradiction. Hence, A is independent set. Similarly, B is also independent set.

Next, we prove that $N(v_{k-1}) \cap A = \emptyset$ and $N(v_k) \cap B = \emptyset$. Suppose that $v_i \in A$ and $v_iv_{k-1} \in E(G)$. If $v_i = v_1$, then $v_{k-1}v_1abv_2Cv_{k-1}$ is a $(k+1)$-cycle. If $v_i \neq v_1$, then $v_i^- \in B$ and $v_i^- bv_{k-2}\overline{C}v_iv_{k-1}Cv_i^-$ is a $(k+1)$-cycle, a contradiction. Hence, $N(v_{k-1}) \cap A = \emptyset$. Similarly, $N(v_k) \cap B = \emptyset$. □

If there is a vertex $v_i \in A$ such that $v_iv_k \notin E(G)$, then $v_i \neq v_1$. Since $v_1 \in A$, $v_iv_1 \notin E(G)$. Then $\{b, v_i, v_k, v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $A \subseteq N(v_k)$. Similarly, $B \subseteq N(v_{k-1})$. If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_s \in A$, then $\{v_i, v_j, v_s, a\} \cong 2K_1 \cup K_2$, a contradiction. Hence, each vertex of A has at least $|B| - 1$ neighbors in B. Similarly, each vertex of A has at least $|B| - 1$ neighbors in B. Thus, $G = K_{s,s} - M$ for the two independent vertex sets $A \cup \{b\} \cup \{v_{k-1}\}$ and $B \cup \{a\} \cup \{v_k\}$, and a matching M of $K_{s,s}$. Since $av_{k-1} \notin E(G)$ and $bv_k \notin E(G)$, $2 \leq |M| \leq s$. This contradiction to the assumption completes the proof for Case 2.2.1.

Case 2.2.2. $|V(P)| = 1$.

Without loss of generality, we assume that v_k is the only vertex that has no neighbor in H.

Claim 16. $|V(H)| \geq 3$.

Proof. Suppose, by contradiction, that $V(H) = \{a, b\}$. All vertices except v_k of C are adjacent to a or b. Without loss of generality, assume $v_1 \in N(a)$. Since G has no $(k+1)$-cycle, k is even, and $N_C(a) = \{v_1, v_3, v_5, \ldots, k-1\}$, $N_C(b) = \{v_2, v_4, v_6, \ldots, k-2\}$. Denote $A = N_C(a)$ and $B = N_C(b)$. Then $V(C) = A \cup B \cup \{v_k\}$, and $A \cap B = \emptyset$.

Claim 16.1. $G - V(C) = H$.

Proof. Suppose that H' is another component of $G - V(C)$. We have that $V(H') \subseteq N(v_k)$; otherwise, suppose that $w \in V(H')$ and $wv_k \notin E(G)$. Then

$\{w, v_k, a, b\} \cong 2K_1 \cup K_2$, a contradiction. Now $v_1, v_{k-1} \notin N_C(H')$; otherwise there is a $(k+1)$-cycle. Since $k \geq 4$ and $|N_C(H')| \geq 2$, the neighbors of H' on C are not consecutive, and we are in Case 2.1, a contradiction. □

Claim 16.2. A and B are independent sets.

Proof. Suppose that $v_i, v_j \in A$ and $v_iv_j \in E(G)$. If $\{v_i, v_j\} = \{v_1, v_{k-1}\}$, then $v_1abv_2Cv_{k-1}v_1$ is a $(k+1)$-cycle, a contradiction. If $\{v_i, v_j\} \neq \{v_1, v_{k-1}\}$, then either $v_i^+, v_j^+ \in B$ or $v_i^-, v_j^- \in B$, say $v_i^+, v_j^+ \in B$ without loss of generality. Then we have a $(k+1)$-cycle $v_i^+bv_j^+Cv_iv_j\overline{C}v_i^+$, a contradiction. Hence, A is an independent set. If $v_i, v_j \in B$ and $v_iv_j \in E(G)$, then $v_i^+, v_j^+ \in A$ and we get a $(k+1)$-cycle $v_i^+av_j^+Cv_iv_j\overline{C}v_i^+$, a contradiction. Hence, B is an independent set. □

Claim 16.3. $A \subseteq N(v_k)$ and $N(v_k) \cap B = \emptyset$.

Proof. Suppose there is a vertex $v_i \in A$ such that $v_iv_k \notin E(G)$. Clearly, $v_i \neq v_1$, and $v_iv_1 \notin E(G)$ by Claim 16.2. Then $\{b, v_i, v_k, v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $A \subseteq N(v_k)$. If $v_j \in B$ such that $v_jv_k \in E(G)$, then $v_j^- \in A$ and there is a $(k+1)$-cycle $v_j^-av_{k-1}\overline{C}v_jv_kCv_j^-$, a contradiction. Hence, $N(v_k) \cap B = \emptyset$. □

If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_k \in A$, then $\{v_i, v_j, v_k, a\} \cong 2K_1 \cup K_2$, a contradiction. Similarly, if there are two vertices $v_i, v_j \in A$ that are not adjacent to a vertex $v_k \in B$, then $\{v_i, v_j, v_k, b\} \cong 2K_1 \cup K_2$, a contradiction. Hence, every vertex of A has at least $|B|-1$ neighbors in B, and every vertex of B has at least $|A|-1$ neighbors in A. Thus, $G = K_{s,s} - M$ for the two independent sets $A \cup \{b\}$ and $B \cup \{a\} \cup \{v_k\}$, and a matching M of $K_{s,s}$. Since $bv_k \notin E(G)$ and $A \cup \{b\} \subseteq N(a)$, $1 \leq |M| \leq s-1$. This contradicts the assumption. □

Claim 17. H is not complete.

Proof. Suppose, by contradiction, that H is complete. Using Claim 16, if $v_1w_1 \in E(G), v_{k-1}w_2 \in E(G)$ for two distinct vertices $w_1, w_2 \in V(H)$, then $v_1w_1w_2v_{k-1}\overline{C}v_1$ is a $(k+1)$-cycle, a contradiction. Hence v_1 and v_{k-1} have only one common neighbor, say w_1. We have that $v_k v_{k-2} \notin E(G)$; otherwise, $v_1 C v_{k-2} v_k v_{k-1} w_1 v_1$ is a $(k+1)$-cycle, a contradiction. Suppose that $w_1 w_2 \in E(H)$. To avoid $\{v_k, v_{k-2}, w_1, w_2\}$ inducing $2K_1 \cup K_2$, we have $w_2 v_{k-2} \in E(G)$. Since $|V(H)| \geq 3$ and H is complete, there is another vertex $w_3 \in V(H) \setminus \{w_1, w_2\}$ and $w_1 w_3, w_2 w_3 \in E(G)$. Then $v_1 C v_{k-2} w_2 w_3 w_1 v_1$ is a $(k+1)$-cycle, a contradiction. Hence, H is not complete. □

Denote $A = \{v_i \in V(C) \mid 1 \leq i < k \text{ and } i \text{ is odd}\}$, $B = \{v_i \in V(C) \mid 2 \leq i < k \text{ and } i \text{ is even}\}$, and denote $A_H = N_H(A)$ and $B_H = N_H(B)$.

Claim 18. A_H and B_H are independent sets.

Proof. It is sufficient if we can prove that for any two vertices v_i, v_j $(1 \leq i < j < k)$ such that $|v_i C v_j|$ is odd, either they share a common neighbor in H or their neighbors in H are nonadjacent. We use induction to prove that fact. First, suppose that $|v_i C v_j| = 3$. If v_i, v_j have different neighbors in H, say w_1, w_2, respectively, then using Claim 17, $w_1 w_2 \in E(H)$. But then $v_i w_1 w_2 v_j C v_i$ is a $(k+1)$-cycle, a contradiction. Hence the claim holds for the case $|v_i C v_j| = 3$. Now suppose that the claim holds for $|v_i C v_j| \leq 2m - 1$ $(m \geq 2)$. Then it is sufficient to deal with the case $|v_i C v_j| = 2m + 1$. Suppose that $v_i w_1 \in E(G), v_j w_2 \in E(G)$ and $w_1 w_2 \in E(H)$. If $v_i^{+2} w_1 \in E(G)$, then $|v_i^{+2} C v_j| = 2m - 1$ and $w_1 w_2 \in E(H)$, contradicting the assumption. If $v_i^{+2} w_2 \in E(G)$, then $|v_i C v_i^{+2}| = 3$ and $w_1 w_2 \in E(H)$, contradicting the assumption. Hence, v_i^{+2} has a neighbor different from w_1, w_2 in H, say w_3, and $w_1 w_3, w_2 w_3 \notin E(G)$. To avoid $\{w_3, v_i^+, w_1, w_2\}$ inducing $2K_1 \cup K_2$, we have $v_i^+ w_2 \in E(G)$. For the vertex v_j^-, by the same arguments we get that $v_j^- w_1 \in E(G)$. Then $|v_i^+ C v_j^-| = 2m - 1$ and w_2, w_1 are their neighbors, respectively. Since $w_1 w_2 \in E(H)$, that contradicts the induction hypothesis. Hence, the claim holds for the case $|v_i C v_j| = 2m + 1$ and the proof is complete. □

Claim 19. $A_H \cap B_H = \emptyset$ and $V(H) = A_H \cup B_H$.

Proof. Suppose that there is a vertex $w \in A_H \cap B_H$. Then w has two neighbors v_i, v_j on C such that i is odd and j is even. Clearly, $j \neq i + 1, j \neq i - 1$ and $v_i^+ v_j^+ \notin E(G)$; otherwise there is a $(k+1)$-cycle. By Claim 18, w has no neighbor in $A_H \cup B_H$. Since H is connected and nontrivial, there is a vertex

$w' \in V(H)\setminus(A_H \cup B_H)$ such that $ww' \in E(G)$. Since $N_H(C) = A_H \cup B_H$, $w'v_i^+ \notin E(G)$ and $w'v_j^+ \notin E(G)$. Then $\{v_i^+, v_j^+, w, w'\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $A_H \cap B_H = \emptyset$.

Suppose that $w \in V(H)\setminus(A_H \cup B_H)$. Since $N_H(C) = A_H \cup B_H$, $N(w) \cap V(C) = \emptyset$. Assume that $w_1 \in B_H$ and $w_1v_2 \in E(G)$. We have that $ww_1 \in E(G)$; otherwise, $\{w, w_1, v_1, v_k\} \cong 2K_1 \cup K_2$, a contradiction. Assume that $w_2v_{k-1} \in E(G)$. We have that $w_2 \neq w_1$; otherwise, $v_kv_3 \notin E(G)$ and $\{v_k, v_3, w_1, w\} \cong 2K_1 \cup K_2$, a contradiction. If $ww_2 \in E(G)$, then we get a $(k+1)$-cycle $v_2Cv_{k-1}w_2ww_1v_2$, a contradiction. Thus, $ww_2 \notin E(G)$. To avoid $\{w, w_2, v_1, v_k\}$ or $\{w_2, v_k, w, w_1\}$ inducing $2K_1 \cup K_2$, we have that $w_2v_1 \in E(G)$ and $w_2w_1 \in E(G)$. Thus, $w_2 \in A_H$ and $v_{k-1} \in A$. We have $v_1v_{k-1} \notin E(G)$; otherwise, $v_1w_2w_1v_2Cv_{k-1}v_1$ is a $(k+1)$-cycle, a contradiction. Since $A_H \cap B_H = \emptyset$, $w_1v_{k-1} \notin E(G)$. Then $\{v_1, v_{k-1}, w, w_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $V(H) = A_H \cup B_H$. □

Claim 20. $|B_H| \geq 2$.

Proof. Suppose, by contradiction, that $B_H = \{w\}$. Since $V(H) = A_H \cup B_H$ and $|V(H)| \geq 3$, $|A_H| \geq 2$. Suppose first that k is odd. Then $v_{k-1}w \in E(G)$. Suppose that $v_1w_1 \in E(G)$ and $w_1 \in A_H$. We have that $ww_1 \notin E(G)$; otherwise, $v_1Cv_{k-1}ww_1v_1$ is a $(k+1)$-cycle, a contradiction. By Claim 18 and Claim 19, w_1 has no neighbor in H, contradicting the fact that H is connected. Hence, k is even. Moreover, we have that $B \cup \{v_k\}$ is an independent set; otherwise, suppose that $v_i, v_j \in B \cup \{v_k\}$ and $v_iv_j \in E(G)$. Then $\{w_1, w_2, v_i, v_j\} \cong 2K_1 \cup K_2$, where $w_1, w_2 \in A_H$, a contradiction. Now, if we delete all the vertices of $A \cup \{w\}$, then we will get $|B \cup \{v_k\}| + |A_H|$ trivial components. Since $|B \cup \{v_k\}| = |A|$ and $|A_H| \geq 2$, $|B \cup \{v_k\}| + |A_H| > |A \cup \{w\}|$. This contradicts the fact that G is 1-tough. □

By Claims 18–20, there are two vertices $w_1, w_2 \in B_H$ such that $w_1w_2 \notin E(G)$ and $w_1, w_2 \notin N(v_1)$. Then $\{w_1, w_2, v_1, v_k\} \cong 2K_1 \cup K_2$, our final contradiction that completes the proof for Case 2.2.

Case 2.3. There is no nontrivial segment on C.

In this case, all the vertices of C are neighbors of a component H of $G-V(C)$. We first claim that H has at least three vertices; otherwise G belongs to the class of graphs that we have excluded.

Claim 21. $|V(H)| \geq 3$.

Proof. Suppose, by contradiction, that $V(H)=\{a,b\}$. Then all the vertices of C are adjacent to a or b. Without loss of generality, assume that $v_1a \in E(G)$. Then we have that k is even and $N_C(a) = \{v_i \mid 1 \leq i \leq k \text{ and } i \text{ is odd}\}$, $N_C(b) = \{v_j \mid 1 \leq j \leq k \text{ and } i \text{ is even}\}$. Denote $A = N_C(a)$, $B = N_C(b)$. Then $V(C) = A \cup B$.

Claim 21.1. A and B are independent sets.

Proof. Suppose that $v_i, v_j \in A$. Clearly, $v_i^+, v_j^+ \in B$. If $v_iv_j \in E(G)$, then we get a $(k+1)$-cycle $v_i^+ b v_j^+ C v_i v_j \overline{C} v_i^+$, a contradiction. Hence, A is independent. Similarly, B is also independent. □

Claim 21.2. $G-V(C)=H$ or $V(G)\setminus(V(C)\cup V(H))=\{a',b'\}$. Moreover, in the latter case $A \subseteq N(a')$ and $B \subseteq N(b')$.

Proof. Since G is $2K_1 \cup K_2$-free and all components of $G-V(C)$ are nontrivial, $G-V(C)$ has either one component or two complete components. If $G-V(C)$ has one component, then the claim holds. Suppose that there is another component $H' = G-(V(C)\cup V(H))$. Then there is a vertex $a' \in H'$ such that a' has a neighbor on C. Without loss of generality, assume $a'v_1 \in E(G)$. Then $A \subseteq N(a')$; otherwise, suppose $v_ia' \notin E(G)$ and $v_i \in A$. Then $\{b, v_i, a', v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $A \subseteq N(a')$ and $B \cap N(a') = \emptyset$. Suppose that $a'b' \in E(H')$. Then $B \subseteq N(b')$; otherwise, suppose $v_ib' \notin E(G)$ and $v_i \in B$. Then $\{a, v_i, a', b'\} \cong 2K_1 \cup K_2$, a contradiction. If $V(H') \neq \{a', b'\}$, assume $c' \in V(H')\setminus\{a',b'\}$. Then there is a $(k+1)$-cycle $v_1a'c'b'v_4Cv_1$, a contradiction. Hence, the claim holds. □

If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_k \in A$, then $\{v_i, v_j, v_k, a\} \cong 2K_1 \cup K_2$, a contradiction. Similarly, if there are two vertices $v_i, v_j \in A$ that are not adjacent to a vertex $v_k \in B$, then $\{v_i, v_j, v_k, b\} \cong 2K_1 \cup K_2$, a contradiction. Hence, every vertex of A has at least $|B|-1$ neighbors in B, and every vertex of B has at least $|A|-1$ neighbors in A. Thus, if $G-V(C)=H$, then $G=K_{s,s}-M$ for the two independent sets $A\cup\{b\}$ and $B\cup\{a\}$, and a matching M of $K_{s,s}$. Since $N(a)=A\cup\{b\}$ and $N(b)=B\cup\{a\}$, $0\leq|M|\leq s-2$. If $V(G)\setminus(V(C)\cup V(H))=\{a',b'\}$, then $G=K_{s,s}-M$ for the two independent sets $A\cup\{b,b'\}$ and $B\cup\{a,a'\}$, and a matching M of $K_{s,s}$. Since $ab'\notin E(G)$ and $ba'\notin E(G)$, $2\leq|M|\leq s$. This contradicts the assumption. □

Claim 22. H is not complete.

Proof. Suppose, by contradiction, that H is complete. If $|N_H(C)|=2$ and $N_H(C)=\{a,b\}$, then all the vertices of C are neighbors of a or b alternately. Denote $A=N_C(a)$ and $B=N_C(b)$. Then $|A|=|B|\geq 2$ and A,B are independent sets. Taking two vertices v_i,v_j from A and a vertex c from $V(H)\setminus\{a,b\}$, we get that $\{v_i,v_j,b,c\}\cong 2K_1\cup K_2$, a contradiction. Hence, $|N_H(C)|\geq 3$. Then there are three vertices $a,b,c\in V(H)$ and a vertex $v_i\in V(C)$ such that $av_i, bv_i^+, cv_i^{+2}\in E(G)$. Then we get a $(k+1)$-cycle $v_iacv_i^{+2}Cv_i$, a contradiction. Hence, H is not complete. □

Using Claim 22, we get that $G-V(C)=H$; otherwise, suppose H' is another component of $G-V(C)$. Then two nonadjacent vertices of H with an edge of H' will induce a $2K_1\cup K_2$, a contradiction. Denote $A=\{v_i\in V(C)\mid 1\leq i\leq k$ and i is odd$\}$, $B=\{v_i\in V(C)\mid 2\leq i\leq k$ and i is even$\}$, and denote $A_H=N_H(A)$ and $B_H=N_H(B)$.

Claim 23. A_H and B_H are independent sets.

Proof. The only difference between the conditions of Claim 18 and Claim 23 is that v_k is a neighbor of H in the latter one but not in the former one. In the induction proof of Claim 18, the absence of v_k does not affect the result. Hence, the proof of Claim 18 is also valid here. □

Claim 24. $V(H)=A_H\cup B_H$ and $A_H\cap B_H=\emptyset$.

Proof. Suppose, by contradiction, that $V(H)\neq A_H\cup B_H$. There is a vertex $w\in V(G)\setminus(A_H\cup B_H)$ such that $ww_1\in E(H)$, where $w_1\in A_H\cup B_H$. Since $N_H(C)=A_H\cup B_H$, w has no neighbor on C. Without loss of generality, assume that $w_1 \in A_H$ and $w_1v_1 \in E(G)$. If $v_kv_2 \notin E(G)$, then $\{v_k, v_2, w_1, w\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $v_kv_2 \in E(G)$. Suppose that $v_2w_2, v_3w_3 \in E(G)$ with $w_2, w_3 \in V(H)$. Obviously, $w_2 \neq w_1$, $w_2 \neq w_3$, and since $v_kv_2 \in E(G)$ we have $w_1 \neq w_3$ and $w_1v_3, w_3v_1 \notin E(G)$. By Claim 23, $w_1w_3 \notin E(G)$. Moreover, we have $w_2w_3 \notin E(G)$; otherwise, $v_2w_2w_3v_3Cv_kv_2$ is a $(k+1)$-cycle, a contradiction. If $w_1w_2 \notin E(G)$, then $\{w_2, w_3, w_1, v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $w_1w_2 \in E(G)$. If $k = 4$, then we get a $(k+1)$-cycle $v_1w_1w_2v_2v_kv_1$, a contradiction. Thus, $k \geq 5$ and $v_4 \neq v_k$. We have $v_1v_4 \notin E(G)$ and $w_1v_4 \notin E(G)$; otherwise, $v_kv_2w_2w_1v_1v_4Cv_k$ or $v_2w_2w_1v_4Cv_2$ is a $(k+1)$-cycle, respectively, a contradiction. Then $\{w_3, v_4, w_1, v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $V(H) = A_H \cup B_H$.

Suppose that $w \in A_H \cap B_H$. By Claim 23, w has no neighbor in $A_H \cup B_H$. Since $V(H) = A_H \cup B_H$, w is a independent vertex of H, contradicting the fact that H is connected. Hence, $A_H \cap B_H = \emptyset$. □

Claim 25. k is even.

Proof. For two vertices $v_i, v_j \in V(C)$, if i and j are both odd or both even, then we say that v_i, v_j are in the same group, and by Claim 23 we know that their neighbors on H are independent. Suppose that k is odd, and $v_kw \in E(G)$ with $w \in V(H)$. Clearly, $w \in A_H$. Since $v_k^{+2} = v_2$, $v_k^{+4} = v_4$, …, if we relabel the vertices on C by increasing the subscript of every vertex by one, then v_k becomes v_1, v_2 becomes v_3, …. Then the original v_k and the original $v_2, v_4, \ldots$ are in the same group. Then w is independent with every vertex of B_H. By Claims 23 and 24, w has no neighbor in H, contradicting that H is connected. Hence, k is even. □

Claim 26. A and B are independent sets.

Proof. Suppose that $v_i, v_j \in A$ and $v_iv_j \in E(G)$. If $|B_H| = 1$, assume $B_H = \{w\}$. Then $v_i^+w, v_j^+w \in E(G)$ and we get a $(k+1)$-cycle $v_i^+wv_j^+Cv_iv_j\overline{C}v_i^+$, a contradiction. If $|B_H| \geq 2$, assume $w_1, w_2 \in B_H$. Then $\{w_1, w_2, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction. Hence, A is an independent set. Similarly, B is also and independent set. □

By Claims 23–26, $A\cup B_H$ and $B\cup A_H$ are independent. Since G is 1-tough and $V(G)=A\cup B\cup A_H\cup B_H$, we have $|A\cup B_H|=|B\cup A_H|$.

Claim 27. For each vertex $x\in A\cup B_H$, x has at least $|B\cup A_H|-1$ neighbors in $B\cup A_H$, and for each vertex $y\in B\cup A_H$, y has at least $|A\cup B_H|-1$ neighbors in $A\cup B_H$.

Proof. Suppose that y_1,y_2 are two vertices of $B\cup A_H$ such that they are not adjacent to a vertex $x\in A\cup B_H$. If $x\in A$ and $y_1,y_2\in B$, then $\{y_1,y_2,x,w\}\cong 2K_1\cup K_2$, where w is a neighbor of x in A_H, a contradiction. If $x\in A$ and $y_1,y_2\in A_H$, then $\{y_1,y_2,x,x^+\}\cong 2K_1\cup K_2$, a contradiction. If $x\in A$ and $y_1\in B$, $y_2\in A_H$, then there is another vertex $y_3\in A_H\backslash\{y_2\}$ such that $xy_3\in E(G)$, and $\{y_1,y_2,x,y_3\}\cong 2K_1\cup K_2$, a contradiction.

If $x\in B_H$ and $y_1,y_2\in B$, then $\{y_1,y_2,x,z\}\cong 2K_1\cup K_2$, where z is a neighbor of x in A_H, a contradiction. If $x\in B_H$ and $y_1,y_2\in A_H$, then $\{y_1,y_2,x,z'\}\cong 2K_1\cup K_2$, where z' is a neighbor of x in B, a contradiction. If $x\in B_H$ and $y_1\in B$, $y_2\in A_H$, then there is another vertex $y_3'\in B\backslash\{y_1\}$ such that $xy_3'\in E(G)$, and $\{y_1,y_2,x,y_3'\}\cong 2K_1\cup K_2$, a contradiction. Hence, for each vertex $x\in A\cup B_H$, x has at least $|B\cup A_H|-1$ neighbors in $B\cup A_H$. By symmetry, for each vertex $y\in B\cup A_H$, y has at least $|A\cup B_H|-1$ neighbors in $A\cup B_H$. □

By Claim 27, we have that $G=K_{s,s}-M$ for the two independent sets $A\cup B_H$ and $B\cup A_H$, and a matching M of $K_{s,s}$ with $0\le|M|\le s$. This final contradiction to the assumption completes the proof for Case 2.3, and also completes the proof of the fact. □

This completes the proof of Theorem 6.7. ■

6.5 Proof of Theorem 6.8

Before we present our proof of Theorem 6.8, we state the following lemma to narrow down the category of graphs that we need to consider.

Lemma 6.11 (Hendry [47]). *If G is a graph of order n with $\delta(G)\ge(n+1)/2$, then G is fully cycle extendable.*

Here a graph G is called *fully cycle extendable* if every vertex of G lies on a triangle of G and furthermore every nonhamiltonian cycle C of G can be extended to another cycle C' such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)|+1$. Hence, if a graph is fully cycle extendable then it is surely pancyclic. By Lemma 6.11, if $\delta(G) \geq (n+1)/2$, then G is pancylic. So we only need to consider graphs whose minimum degree is less than $(n+1)/2$. Suppose that G is a graph of order n satisfying the conditions of Theorem 6.8 and with $\delta(G) < (n+1)/2$, i.e., $\delta(G) \leq n/2$. Let S be a minimal cut set of G. Since $\delta(G) \leq n/2$, $|S| \leq n/2$. The following claim was given by Li et al. in [54]. We add its proof for convenience.

Claim 1 (Li et al. [54]). Every vertex of S is adjacent to every vertex of $V(G) \setminus S$.

Proof. Clearly, the choice of S implies that for every vertex $x \in S$ and every component H of $G-S$, x is adjacent to at least one vertex of H; otherwise $S\backslash\{x\}$ is a vertex cut, contradicting the choice of S. Suppose that $xy \notin E(G)$ for some $y \in V(G) \setminus S$. Let H be the component of $G-S$ containing y, let P be a shortest path from y to x with all internal vertices in H, and let y' be a neighbor of x in a component of $G-S$ other than H. Then $yPxy'$ is an induced path on at least 4 vertices, contradicting that G is P_4-free. □

We now proceed with the proof of Theorem 6.8 and consider two cases.

Case 1. $|S| = n/2$.

Clearly, n is even and $|V(G)\setminus S)| = n/2$. Denote $S = \{x_1, x_2, \ldots, x_{n/2}\}$, $V(G)\setminus S = \{y_1, y_2, \ldots, y_{n/2}\}$. If S and $V(G) \setminus S$ are independent sets, then $G = K_{n/2,n/2}$, and we get the result. Without loss of generality, assume that S is not independent and $x_1x_2 \in E(S)$. By Claim 1, $K_{n/2,n/2}$ is a spanning subgraph of G. Then from $K_{n/2,n/2}$ we can get all cycles of even length from 4 up to $n-2$ containing x_1 but not x_2. By inserting x_2 after vertex x_1 in every even cycle, we get all cycles of odd length from 5 up to $n-1$. Since $x_1x_2y_1x_1$ is a 3-cycle and $K_{n/2,n/2}$ contains an n-cycle, G is pancyclic.

Case 2. $|S| < n/2$.

Let $s = |S|$, hence $|V(G) \setminus S| = n - s$, and clearly $n - s > n/2$. Since G is 1-tough, $G - S$ is not independent. Let H be the subgraph of G induced by S and s vertices of $G - S$ that contain at least one adjacent pair. Then $K_{s,s}$ is a spanning subgraph of H. By the same method as in Case 1, we can prove that H contains cycles of length 3 up to $2s$, and hence G contains cycles of length 3 up to $2s$.

We know from earlier results that every 1-tough P_4-free graph on at least three vertices is hamiltonian, so G contains a Hamilton cycle C. Let the vertices of $S = \{x_1, x_2, \ldots, x_s\}$ be arranged in this order around C according to a fixed orientation of C, and denote every segment of C from x_i to x_{i+1} by $S_i = x_i y_{i_1} y_{i_2} \ldots y_{i_{r_i}} x_{i+1}$ (possible with $r_i = 0$, i.e., no vertex y_i in between for some $1 \leq i \leq s$). By Claim 1, x_i is adjacent to every vertex of $S_i \setminus \{x_{i+1}\}$. Hence, if $r_i \neq 0$, then we can get cycles of length from n down to $n - r_i + 1$ using $x_i y_{i_k} C x_i$ $(1 \leq k \leq r_i)$. In this way, if $r_i \neq 0$ for each $1 \leq i \leq s$, then we can delete the vertices within every segment one by one, until we are left with only two end vertices and one inside. Thus, we get cycles with length from n down to $2s$. Hence, G is pancyclic. If $r_i = 0$ for some $1 \leq i \leq s$, we can use similar arguments for the segments with $r_i \neq 0$ to get cycles with all possible missing lengths. Hence, G is pancyclic. This completes the proof of Theorem 6.8. ■

Chapter 7

More on the hamiltonicity of 1-tough graphs

In the previous two chapters, we mentioned the open problem of [54] asking whether every $K_1 \cup P_4$-free 1-tough graph on at least three vertices is hamiltonian. In fact, this problem has already appeared as a conjecture in a paper of 2013 [66]. Another conjecture in [66] states that every $K_1 \cup K_{1,3}$-free graph G with $\tau(G) > 4/3$ is hamiltonian. In this chapter we give partial answers corresponding to these two conjectures by involving the triangle as an additional forbidden subgraph.

7.1 Introduction

Nikoghosyan [66] investigated the hamiltonicity of 1-tough graphs by considering disconnected single forbidden subgraphs, and he presented the following conjectures.

Conjecture 7.1 (Nikoghosyan [66]). Every 1-tough $K_1 \cup P_4$-free graph on at least three vertices is hamiltonian.

As we saw in the previous chapters, Li et al. [54] considered all possible subgraphs of $K_1 \cup P_4$, and they proved that there is no forbidden subgraph apart from the subgraphs of $K_1 \cup P_4$ that can ensure every 1-tough graph is hamiltonian, leaving $K_1 \cup P_4$ as the only open case. We recall the two results for convenience.

Theorem 7.1 (B. Li et al. [54]). *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

Theorem 7.2 (B. Li et al. [54]). *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

While forbidding any proper subgraph of $K_1 \cup P_4$ can guarantee 1-tough graphs to be hamiltonian, the case with the graph $K_1 \cup P_4$ itself is still open. Since Conjecture 7.1 seems to be very hard to resolve, we considered partial solutions. In particular, if we impose the additional condition that the graphs under consideration are triangle-free, we can prove the following partial result. Here we use $\triangle$ to denote a triangle, i.e., a complete graph on 3 vertices.

Theorem 7.3. *Every 1-tough $\{\triangle, K_1 \cup P_4\}$-free graph on at least three vertices is hamiltonian.*

We postpone the proof of Theorem 7.3 to Section 7.2.

Another conjecture in [66] deals with the hamiltonicity of $K_1 \cup K_{1,3}$-free graphs. Clearly, Theorem 7.2 implies that the toughness of these graphs must be strictly larger than one.

Conjecture 7.2 (Nikoghosyan [66]). Every $K_1 \cup K_{1,3}$-free graph G on at least three vertices with $\tau(G) > 4/3$ is hamiltonian.

In [66], the Petersen graph was used to show that the condition $\tau(G) > 4/3$ in Conjecture 7.2 cannot be relaxed to $\tau(G) = 4/3$. Similarly as in Theorem 7.3, we involved the triangle as another forbidden subgraph, in order to get the following partial result related to Conjecture 7.2.

Theorem 7.4. *If G is a 1-tough $\{\triangle, K_1 \cup K_{1,3}\}$-free graph on at least three vertices, then G is hamiltonian or the Petersen graph.*

The proof of Theorem 7.4 is postponed to Section 7.3. In [66], Nikoghosyan also raised the following conjecture.

Conjecture 7.3 (Nikoghosyan [66]). Every $K_2 \cup K_2$-free graph G on at least three vertices with $\tau(G) > 1$ is hamiltonian.

As far as we know, this conjecture is still open, but it is known from a recent paper due to Shan [70] that 3-tough $2K_2$-free graphs on at least three vertices are hamiltonian. This result considerably improves the result due to Broersma et al. [16] that 25-tough $2K_2$-free graphs on at least three vertices are hamiltonian. A partial result in [16] deals with triangle-free $2K_2$-free graphs, and supplements our results, as follows.

Theorem 7.5 (Broersma et.al [16]). *If G is a 1-tough $\{\triangle, K_2 \cup K_2\}$-free graph on at least three vertices, then G is hamiltonian.*

Another open conjecture from [66] states that every $K_1 \cup P_5$-free graph G on at least three vertices with $\tau(G) > 1$ is hamiltonian. By involving triangle-freeness, we propose the following conjecture for future work.

Conjecture 7.4. If G is a 1-tough $\{\triangle, K_1 \cup P_5\}$-free graph on at least three vertices, then G is hamiltonian.

7.2 Proof of Theorem 7.3

Suppose, to the contrary, that G is a 1-tough $\{\triangle, K_1 \cup P_4\}$-free graph on at least three vertices, but not hamiltonian. Then G contains a cycle. We choose a longest cycle C of G. Since G is not hamiltonian, we use H to denote a component of $G - V(C)$. We denote all neighbors of H on C as $N_C(H) = \{u_1, u_2, \ldots, u_t\}$ with $t \geq 2$, in this order around C according to a fixed orientation of C, and we denote the segment of C from u_i^+ to u_{i+1}^- as $S_i = u_i^+ C u_{i+1}^-$ for $i = 1, 2, \ldots, t$. The next result is obvious, but we give it and its proof for later reference.

Claim 1. $N_C(H)^+$ and $N_C(H)^-$ are independent, and there is no path outside C connecting any two vertices of any one of these two sets.

Proof. Without loss of generality, assume that u_i^+ and u_j^+ are connected by a path P outside C (possibly P is an edge). We find a cycle longer than C: $u_iHu_j\overline{C}u_i^+Pu_j^+Cu_i$, a contradiction. □

We continue with proving another set of claims that are more specific for this graph class. We say that two sets A and B are connected by a path outside C if there is a path between a vertex $x \in A$ and $y \in B$ with all internal vertices not on C.

Let S_i and S_j be two distinct segments.

Claim 2. If segments S_i and S_j are connected by a path outside C, then $\{u_j^{+2}, u_j^{+4}, \ldots, u_{j+1}^{-2}\} \subseteq N(u_i^+)$ and $\{u_i^{+2}, u_i^{+4}, \ldots, u_{i+1}^{-2}\} \subseteq N(u_j^+)$, and $|S_i|$ and $|S_j|$ are odd.

Proof. Suppose S_i and S_j are connected by a path outside C. Then we can find a shortest path from u_i^+ to u_j^+ along $u_i^+CxPy\overline{C}u_j^+$, where $x \in S_i$, $y \in S_j$ and P is a path such that $V(P) \cap (V(C) \cup V(H)) = \{x, y\}$. We use P_{ij} to denote this shortest path for segments S_i and S_j. Then P_{ij} is an induced path. If $|V(P_{ij})| \geq 4$, then any vertex of H together with a subpath of P_{ij} of length 3 induces a copy of $K_1 \cup P_4$, contradicting the hypothesis. Using Claim 1, $|V(P_{ij})| = 3$. Thus, P is a path with $x = u_i^+$ or $y = u_j^+$. Denote $P_{ij} = p_1p_2p_3$, with $p_1 = u_i^+$ and $p_3 = u_j^+$. Since P is a path outside C connecting S_i and S_j, and by Claim 1, we have that $p_2 \in S_i \cup S_j$, and $|S_i| > 1$ or $|S_j| > 1$. Without loss of generality, we assume $|S_i| > 1$. If $p_2 \neq u_i^{+2}$, then to avoid any vertex $w \in V(H)$ with the path $u_i^{+2}u_i^+p_2u_j^+$ inducing $K_1 \cup P_4$, we have $p_2u_i^{+2} \in E(G)$. Then $\{u_i^+, u_i^{+2}, p_2\} \cong \triangle$, a contradiction. Hence, $u_j^+u_i^{+2} \in E(G)$. If u_i^{+4} exists, then to avoid inducing triangles and to avoid any vertex $w \in V(H)$ with the path $u_j^+u_i^{+2}u_i^{+3}u_i^{+4}$ inducing $K_1 \cup P_4$, we have $u_j^+u_i^{+4} \in E(G)$. Similarly, we have $u_i^{+6}, u_i^{+8} \ldots \in N(u_j^+)$ if these vertices exist. For the last vertex of S_i, if $u_j^+u_{i+1}^- \in E(G)$, then $u_j^+ \neq u_{j+1}^-$ by Claim 1. To avoid any vertex $w \in V(H)$ with the path $u_i^+u_i^{+2}u_j^+u_j^{+2}$ inducing $K_1 \cup P_4$, we have $u_i^+u_j^{+2} \in E(G)$. Then there is a cycle longer than C: $u_iHu_{i+1}Cu_j^+u_{i+1}^-\overline{C}u_i^+u_j^{+2}Cu_i$, a contradiction. Hence, $u_j^+u_{i+1}^- \notin E(G)$ and $|S_i|$ is odd. If $u_j^{+2}, u_j^{+4}, u_j^{+6}, \ldots$ exist, then by symmetry, we have that $u_j^{+2}, u_j^{+4}, u_j^{+6}, \ldots \in N(u_i^+)$ and $|S_j|$ is odd. □

For any segment S_i that is connected to another segment by a path outside C, by Claim 2 we know that $|S_i|$ is odd. We divide S_i into two sets: $S_i^o = \{u_i^+, u_i^{+3}, \ldots, u_{i+1}^-\}$ and $S_i^e = \{u_i^{+2}, u_i^{+4}, \ldots, u_{i+1}^{-2}\}$. By Claim 2, if S_i is connected to S_j by a path outside C, then $S_i^e \subseteq N(u_j^+)$, and since G is $\triangle$-free, $S_i^o \cap N(u_j^+) = \emptyset$.

Claim 3. If S_i and S_j are connected by a path outside C, then $S_i^o \cup S_j^o$ is independent.

Proof. Suppose that $u_i^{+s}, u_i^{+t} \in S_i^o$ $(t > s)$ and $u_i^{+s}u_i^{+t} \in E(G)$. Since G is $\triangle$-free, $t > s+2$, and since $u_i^{+(s+1)}, u_i^{+(t-1)} \in N(u_j^+)$, $u_i^{+s+1}u_i^{t-1} \notin E(G)$. Then any vertex $w \in V(H)$ with the path $u_i^{s+1}u_i^{+s}u_i^{+t}u_i^{+(t-1)}$ induces a copy of $K_1 \cup P_4$, contradicting the hypothesis. Hence, S_i^o is independent. Similarly, S_j^o is also independent.

Suppose that $u_i^{+l} \in S_i^o, u_j^{+m} \in S_j^o$ and $u_i^{+l}u_j^{+m} \in E(G)$. By Claim 2, $u_i^{+l} \neq u_i^+$, $u_j^{+m} \neq u_j^+$ and $u_j^{+(m-1)}u_i^+ \in E(G)$. Also we know that $u_i^{+(l+1)} = u_{i+1}$ or $u_i^{+(l+1)}u_j^+ \in E(G)$. Then the cycle $u_iHu_j\overline{C}u_i^{+(l+1)}u_j^+Cu_j^{+(m-1)}u_i^+Cu_i^{+l}u_j^{+m}Cu_i$ is longer than C, a contradiction. □

Claim 4. If S_i is connected to S_j by a path outside C, and S_i has a neighbor $w' \in V(G) \setminus (V(C) \cup V(H))$, then $S_i^e \subseteq N(w')$ and $S_i^o \cap N(w') = \emptyset$.

Proof. First, $u_i^+ \notin N(w')$; otherwise, to avoid any vertex $w \in V(H)$ with a path $w'u_i^+u_j^{+2}u_j^+$ (if $|S_j| \neq 1$) or with a path $w'u_i^+u_i^{+2}u_j^+$ (if $|S_i| \neq 1$) inducing $K_1 \cup P_4$, we have $w'u_j^+ \in E(G)$. That contradicts Claim 1.

Suppose that $u_i^{+k} \in N(w')$ $(k \neq 1)$. To avoid any vertex $w \in V(H)$ with a path $w'u_i^{+k}u_i^{+(k+1)}u_i^{+(k+2)}$ or with a path $w'u_i^{+k}u_i^{+(k-1)}u_i^{+(k-2)}$ inducing $K_1 \cup P_4$, we have $w'u_i^{+(k+2)} \in E(G)$ and $w'u_i^{+(k-2)} \in E(G)$ if these vertices exist. By this argumentation, we know that every vertex on S_i having even distance to u_i^{+k} on C is a neighbor of w'. Since $u_i^+ \notin N(w')$ and G is $\triangle$-free, $S_i^e \subseteq N(w')$ and $S_i^o \cap N(w') = \emptyset$. □

We use S^o to denote the union of all S_i^o, and S^e to denote the union of all S_i^e, for all segments S_i $(1 \leq i \leq t)$ that are connected to some other segment by a path outside C. By Claim 3 and Claim 4, there is no path outside C connecting any two vertices of S^o. Hence, if we remove all the vertices of $N_C(H) \cup S^e$, we get at least $|N_C(H) \cup S^e| + 1$ components, contradicting the hypothesis that G is 1-tough. This completes the proof of Theorem 7.3. ■

7.3 Proof of Theorem 7.4

Let G be a 1-tough $\{\triangle, K_1 \cup K_{1,3}\}$-free graph. For any vertex $u \in V(G)$ of degree larger than 2, since G is $\triangle$-free, we have that u and any three of its neighbors together induce a $K_{1,3}$. Next we distinguish two cases according to the connectivity of G in order to complete the proof of Theorem 7.4.

Case 1. G is 3-connected.

Suppose that G is not hamiltonian. Using a number of claims, we are going to prove that G is the Petersen graph. Here we use the same notations as in the proof of Theorem 7.3. Let C be a longest cylce of G, let H be a component of $G - V(C)$, and let $N_C(H) = \{u_1, u_2, \ldots, u_t\}$ be all the neighbors of H on C, in this order according to a fixed orientation of C. Since G is 3-connected, $t \geq 3$. We denote the segment of C from u_i^+ to u_{i+1}^- as $S_i = u_i^+ C u_{i+1}^-$ for $i = 1, 2, \ldots, t$. Claim 1 in the proof of Theorem 7.3 clearly also holds here, but we recall it here without proof for later reference.

Claim 5. $N_C(H)^+$ and $N_C(H)^-$ are independent, and there is no path outside C connecting any two vertices of any one of these two sets.

We present several other claims, each followed by a proof.

Claim 6. If S_i and S_j are connected by a path P_{ij} outside C, then $P_{ij} = u_i^+ u_{j+1}^-$ or $P_{ij} = u_{i+1}^- u_j^+$.

Proof. Let $P_{ij} = p_1 p_2 \ldots p_s$ ($s \geq 2$) be such a path with $p_1 \in S_i$ and $p_s \in S_j$. If $p_1 \notin \{u_i^+, u_{i+1}^-\}$ or $p_s \notin \{u_j^+, u_{j+1}^-\}$, then $\{w, p_1, p_1^-, p_1^+, p_2\} \cong K_1 \cup K_{1,3}$ and $\{w, p_s, p_s^-, p_s^+, p_{s-1}\} \cong K_1 \cup K_{1,3}$ for any vertex $w \in V(H)$, a contradiction. By Claim 5, if $p_1 = u_i^+$, then $p_s = u_{j+1}^-$; if $p_1 = u_{i+1}^-$, then $p_s = u_j^+$ ($j \neq i + 1$). In both cases, $|S_i| \geq 2$ and $|S_j| \geq 2$.

To avoid any vertex of $V(H)$ with $\{u_i^+; u_i, u_i^{+2}, p_2\}$ or $\{u_j^+; u_j, u_j^{+2}, p_{s-1}\}$ inducing $K_1 \cup K_{1,3}$, we have $V(H) \subseteq N(u_i)$ or $V(H) \subseteq N(u_j)$. Since G is $\triangle$-free, $|V(H)| = 1$ and $N_C(H)$ is independent. Denote $H = \{w\}$.

Suppose $s \geq 3$ and $p_2 \in V(H')$, where H' is another component of $G - V(C)$ different from H. If P_{ij} is connecting u_i^+ to u_{j+1}^-, then $u_i \neq u_{j+1}$, and $p_2 u_i \notin E(G)$, $p_2 u_{j+1} \notin E(G)$; otherwise there clearly is a longer cycle. Then to avoid p_2 with $\{w; u_i, u_j, u_{j+1}\}$ inducing $K_1 \cup K_{1,3}$, we have $p_2 u_j \in E(G)$. Since $\{u_i^{+2}, u_j^+\} \in N_C(H')^+$, $u_i^{+2} u_j^+ \notin E(G)$ by Claim 5. To avoid u_j^+ with $\{u_i^+; u_i, u_i^{+2}, p_2\}$ inducing $K_1 \cup K_{1,3}$, we have $u_j^+ u_i \in E(G)$. Then $u_i^- u_j^+ \notin E(G)$; otherwise $\langle\{u_i, u_i^-, u_j^+\}\rangle \cong \triangle$. To avoid u_i^- with $\{u_j; u_j^-, u_j^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_i^- u_j \in E(G)$. Then we find a cycle longer than C: $u_i w u_{j+1} C u_i^- u_j \overline{C} u_i^+ P_{ij} u_{j+1}^- \overline{C} u_j^+ u_i$, a contradiction. If P_{ij} is connecting u_{i+1}^- to u_j^+, then $u_{i+1} \neq u_j$ and $p_2 u_{i+1} \notin E(G)$, $p_2 u_j \notin E(G)$. We have $p_2 u_i \in E(G)$; otherwise, p_2 with $\{w; u_i, u_{i+1}, u_j\}$ induces $K_1 \cup K_{1,3}$. By Claim 5, $u_i^+ u_{i+1} \notin E(G)$ since $\{u_i^+, u_{i+1}\} \in N_C(H')^+$. To avoid u_{i+1} with $\{u_i; u_i^+, u_i^-, p_2\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1} u_i^- \in E(G)$. Then $u_{i+1}^+ u_i^- \notin E(G)$; otherwise $\langle\{u_i^-, u_{i+1}, u_{i+1}^+\}\rangle \cong \triangle$. To avoid u_{i+1}^+ with $\{u_i; u_i^-, u_i^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1}^+ u_i \in E(G)$. Then the cycle $u_{i+1} w u_j \overline{C} u_{i+1}^+ u_i C u_{i+1}^- P_{ij} u_j^+ C u_i^- u_{i+1}$ is longer than C, a contradiction. Hence, $s = 2$ and $P_{ij} = u_i^+ u_{j+1}^-$ or $P_{ij} = u_{i+1}^- u_j^+$. □

Since G is 1-tough, there are two distinct segments S_i and S_j that are connected by a path outside C. By Claim 6, $u_i^+ u_{j+1}^- \in E(G)$ or $u_{i+1}^- u_j^+ \in E(G)$. To avoid any vertex of $V(H)$ with $\{u_i^+; u_i, u_i^{+2}, u_{j+1}^-\}$ or with $\{u_j^+; u_j, u_j^{+2}, u_{i+1}^-\}$ inducing $K_1 \cup K_{1,3}$, we have $V(H) \subseteq N(u_i)$ or $V(H) \subseteq N(u_j)$. Since G is $\triangle$-free, $|V(H)| = 1$ and $N_C(H)$ is independent. Denote $H = \{w\}$.

Claim 7. $|S_i| = 2$ for all $i \in \{1, 2, \ldots, t\}$.

Proof. Suppose that there is a segment S_i with $i \in \{1, 2, \ldots, t\}$ such that $|S_i| \geq 3$. Then $u_i^+ \neq u_{i+1}^{-2}$. By Claim 6, u_{i+1}^{-2} has no neighbor in $V(C) \setminus (S_i \cup N_C(H)$. To avoid u_{i+1}^{-2} with $\{u_{i+2}; u_{i+2}^-, u_{i+2}^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1}^{-2} u_{i+2} \in E(G)$. To avoid u_i^- with $\{u_{i+1}^{-2}; u_{i+1}^-, u_{i+1}^{-3}, u_{i+2}\}$ inducing $K_1 \cup K_{1,3}$, we have $u_i^- u_{i+2} \in E(G)$. To avoid u_{i+1}^- with $\{u_{i+2}; u_{i+2}^-, u_i^-, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1}^{-}u_{i+2} \in E(G)$. But now $\langle\{u_{i+1}^{-2}, u_{i+1}^{-}, u_{i+2}\}\rangle \cong \triangle$, a contradiction. Hence, $|S_i| \leq 2$ for all $i \in \{1,2,\ldots,t\}$.

Suppose that there is a segment S_i with $i \in \{1,2,\ldots,t\}$ such that $|S_i| = 1$. Then $u_i^+ = u_{i+1}^-$. By Claim 5 and Claim 6, u_i^+ has no neighbor in $V(C)\backslash N_C(H)$. To avoid u_i^+ with $\{u_j; u_j^-, u_j^+, w\}$ inducing $K_1 \cup K_{1,3}$ $(j \neq i+1, j \neq i-1)$, we have $u_i^+ u_j \in E(G)$ for all $j \in \{1,2,\ldots,t\}$. Since G is 1-tough, there are two segments S_j and S_k $(k > j)$ that are connected by an edge $u_j^+ u_{k+1}^-$ or $u_{j+1}^- u_k^+$. Since $|S_i| \leq 2$ for all $i \in \{1,2,\ldots,t\}$ and by Claim 5, $|S_j| = |S_k| = 2$ in both cases. If $u_j^+ u_{k+1}^- \in E(G)$, then to avoid u_j^+ with $\{u_k; u_k^+, u_i^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_j^+ u_k \in E(G)$. Since G is $\triangle$-free, $k \neq j+1$ and $u_{j+1}^- u_k \notin E(G)$. Then u_{j+1}^- with $\{u_k; u_k^-, u_i^+, w\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. If $u_{j+1}^- u_k^+ \in E(G)$, then $k > j+1$. To avoid u_{j+1}^- with $\{u_k; u_k^-, u_i^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{j+1}^- u_k \in E(G)$, but then $\langle\{u_{j+1}^-, u_k, u_k^+\}\rangle \cong \triangle$, a contradiction. Hence, $|S_i| = 2$ for all $i \in \{1,2,\ldots,t\}$. □

Claim 8. For any component $H' \neq H$ of $G - V(C)$, $N_C(H') = N_C(H)$.

Proof. Suppose that H' has a neighbor in $V(C) \setminus N_C(H)$. By Claim 7, this neighbor of H' is either u_i^+ or u_i^- for some $i \in \{1,2,\ldots,t\}$. Without loss of generality, we assume that $w'u_1^+ \in E(G)$, where $w' \in V(H')$. By Claim 5, $w'u_2^+ \notin E(G)$. To avoid u_2^+ with $\{u_1^+; u_1, u_1^{+2}, w'\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1 u_2^+ \in E(G)$. To avoid u_3^+ with $\{u_1; u_1^+, u_2^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1 u_3^+ \in E(G)$. To avoid u_2^{+2} with $\{u_1; u_1^+, u_3^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1^+ u_2^{+2} \in E(G)$. We also have $w'u_2^{+2} \notin E(G)$; otherwise, $\langle\{w', u_1^+, u_2^{+2}\}\rangle \cong \triangle$, a contradiction. But now w with $\{u_1^+; u_1^{+2}, u_2^{+2}, w'\}$ induces $K_1 \cup K_{1,3}$, a contradiction. Hence, $N_C(H') \subseteq N_C(H)$. Since we have chosen H arbitrarily, by symmetry we have $N_C(H) \subseteq N_C(H')$. Hence, $N_C(H') = N_C(H)$. □

Claim 9. $t = 3$.

Proof. Since G is 1-tough, there are at least two distinct segments that are connected by a path outside C. By Claim 6 and Claim 7, without loss of generality, we can assume $u_1^+ u_i^- \in E(G)$ $(i \geq 3)$. To avoid u_i^+ with $\{u_1^+; u_1, u_2^-, u_i^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1 u_i^+ \in E(G)$ or $u_2^- u_i^+ \in E(G)$. If $u_1 u_i^+ \in E(G)$, then $u_{i+1} \neq u_1$ and $u_{i+1}^- u_1 \notin E(G)$. To avoid u_{i+1}^- with $\{u_1^+; u_1, u_2^-, u_i^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1}^- u_1^+ \in E(G)$. But then w with $\{u_1^+; u_2^-, u_i^-, u_{i+1}^-\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Hence, $u_1 u_i^+ \notin E(G)$. Suppose that $u_2^- u_i^+ \in E(G)$. If $i \neq t$, then $u_{i+1}^- \neq u_1^-$. To avoid u_i^+ with $\{u_1; u_1^+, u_1^-, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_i^+ u_1^- \in E(G)$. But then w with $\{u_i^+; u_1^-, u_2^-, u_{i+1}^-\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Hence, $i = t$.

If $i \neq 3$, then $u_{i-1} \neq u_2$. To avoid u_2^+ with $\{u_1^+; u_1, u_2^-, u_i^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_2^+ u_i^- \in E(G)$ or $u_2^+ u_1 \in E(G)$. If $u_2^+ u_i^- \in E(G)$, then w with $\{u_i^-; u_1^+, u_2^+, u_{i-1}^+\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. If $u_2^+ u_1 \in E(G)$, then $u_3^- u_1 \notin E(G)$. To avoid u_3^- with $\{u_1^+; u_1, u_2^-, u_i^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_3^- u_1^+ \in E(G)$. Then w with $\{u_1^+; u_2^-, u_3^-, u_i^-\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Hence, $i = t = 3$. □

Since G is 1-tough and by Claims 5–9, without loss of generality, we can assume that $u_1^+ u_3^- \in E(G)$. To avoid u_3^+ with $\{u_1^+; u_1, u_2^-, u_3^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_2^- u_3^+ \in E(G)$. To avoid u_1^- with $\{u_3^-; u_3, u_2^+, u_1^+\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1^- u_2^+ \in E(G)$. By Claims 5–9, there is no other edge joining a pair of nonadjacent vertices on C. Suppose that $H' \neq H$ is another component of $G - V(C)$. By Claim 8, $N_C(H') = \{u_1, u_2, u_3\}$. Assume that $w' \in V(H')$ and $w' u_1 \in E(G)$. Then u_3^+ with $\{u_1; u_1^+, w, w'\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Hence, H is the only component of $G - V(C)$. Recalling that $|V(H)| = 1$, it is clear that G is the Petersen graph. This completes the proof for Case 1.

Case 2. $\kappa(G) = 2$.

Suppose that $\{u_1, u_2\}$ is a cut set of G. Since G is 1-tough, $G - \{u_1, u_2\}$ has exactly two components, say H_1 and H_2. Since G is $\{\triangle, K_1 \cup K_{1,3}\}$-free, each of H_1 and H_2 is an induced path or an induced cycle. We again prove a number of claims in order to reach a contradiction.

Claim 10. If H_i is an induced cycle C' for $i \in \{1, 2\}$, then there are two consecutive vertices on C' such that one vertex is adjacent to u_1 and the other one is adjacent to u_2.

Proof. Suppose, by contradiction, that for any neighbor of u_1 on C', its predecessor and successor on C' are not adjacent to u_2. Since G is $\triangle$-free, any two vertices of $N_{H_i}(u_1) \cup N_{H_i}(u_2)$ are not consecutive on C'. Since C' is induced, by removing all the vertices of $N_{H_i}(u_1) \cup N_{H_i}(u_2)$ from G we get $|N_{H_i}(u_1) \cup N_{H_i}(u_2)| + 1$ components, contradicting the hypothesis that G is 1-tough. □

Claim 11. If H_i is an induced path P for $i \in \{1, 2\}$, then one end vertex of P is adjacent to u_1 and the other one is adjacent to u_2.

Proof. Suppose that $H_1 = P = p_1 p_2 \dots p_t$. If $t \le 2$, then the claim clearly holds. Suppose that $t \ge 3$ and $\{p_1, p_t\} \in N(u_1) \setminus N(u_2)$. Let $p_i \in V(H_1)$ $(1 < i < t)$ be a vertex adjacent to u_2. If $u_1 u_2 \in E(G)$, then there is a vertex $x \in V(H_2)$ such that $x u_2 \notin E(G)$. Then x with $\{p_i; p_{i-1}, p_{i+1}, u_2\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Thus, $u_1 u_2 \notin E(G)$. If $|V(H_2)| \ge 2$, then there will also be a vertex in H_2 not adjacent to u_2, and similarly we can get an induced $K_1 \cup K_{1,3}$. Hence, $|V(H_2)| = 1$. Denote $V(H_2) = \{w\}$. If $t \ge 6$, then $p_3 u_1 \in E(G)$ and $p_4 u_1 \in E(G)$; otherwise, p_3 or p_4 with $\{u_1; p_1, p_t, w\}$ induces $K_1 \cup K_{1,3}$, a contradiction. Now $\langle \{u_1, p_3, p_4\} \rangle \cong \triangle$, a contradiction. If $t = 5$, then to avoid p_3 with $\{u_1; p_1, p_t, w\}$ inducing $K_1 \cup K_{1,3}$, we have $p_3 u_1 \in E(G)$. To avoid u_2 with $\{u_1; p_1, p_3, p_t\}$ inducing $K_1 \cup K_{1,3}$, we have $p_3 u_2 \in E(G)$. To avoid inducing a triangle, $\{p_2, p_4\} \cap (N(u_1) \cup N(u_2)) = \emptyset$. Obviously, now $\{u_1, p_3\}$ is a cut set that induces three components consisting of $p_1 p_2$, $p_4 p_5$ and $u_2 w$. This contradicts the hypothesis that G is 1-tough. If $t = 4$, then precisely one vertex of $\{p_2, p_3\}$ is adjacent to u_2. Without loss of generality, assume $p_2 u_2 \in E(G)$. To avoid inducing a triangle, $p_2 u_1 \notin E(G)$ and $p_3 u_1 \notin E(G)$. Then $\{u_1, p_2\}$ is a cut set that induces three components consisting of p_1, $p_3 p_4$ and $u_2 w$. This contradicts the hypothesis that G is 1-tough. If $t = 3$, then $\{u_1, p_2\}$ is a cut set that induces three components consisting of p_1, p_3 and $u_2 w$, contradicting the hypothesis that G is 1-tough. □

Using Claim 10 and Claim 11, it is clear that there is a cycle in G containing all the vertices of G. Hence, G is hamiltonian. This completes the proof of Theorem 7.4. ■

Chapter 8

Thomassen's Conjecture on two classes of graphs

In this short chapter, we present two results that deal with affirmative answers to the conjecture due to Thomassen [74] that every hamiltonian graph G of minimum degree at least 3 contains an edge e such that $G-e$ and G/e are both hamiltonian. Here $G-e$ denotes the graph obtained from G by deleting the edge e, while G/e denotes the graph obtained from G by contracting the edge $e=uv \in E(G)$. Hence, G/e is obtained from G by replacing u and v by a new vertex x and making x adjacent to the vertices of $N_G(u) \cup N_G(v)$ (so avoiding multiple edges). Bielak [8] proved that the above conjecture holds for claw-free graphs. In this chapter, we prove that the conjecture holds for $K_1 \cup K_{1,3}$-free graphs and $K_1 \cup P_5$-free graphs. The general conjecture remains open and seems to be very hard to resolve.

8.1 Introduction

As we have seen, the research area of hamiltonian graph theory has been attracting many researchers and has resulted in great research efforts since its appearance. Many related topics arose and earned a lot of attention as well. Undoubtedly, these topics have increased the richness and interest of graph theory, and added some color to graph theory. Carsten Thomassen is one of the many scholars who have made outstanding contributions to the field of graph theory. Not only is he known for his excellent work in solving difficult problems with elegant and brilliant methods, but also for his good questions and conjectures. In [1], the authors summarized 19 questions and conjectures due to Thomassen, most of which remain open to date. For more questions posed by Thomassen, we refer the reader to [10] and [65]. In this chapter, we study one of the conjectures that has been restated in [1].

Whereas the Hamilton problem deals with the question whether a graph contains a Hamilton cycle or not, the conjecture we will consider is related to the question under which conditions a hamiltonian graph has more than one Hamilton cycle. For cubic hamiltonian graphs, Smith [75] gave an affirmative answer by proving that every cubic hamiltonian graph has at least three Hamilton cycles. In fact, he proved the following result.

Theorem 8.1 (Smith [75]). *Every edge of a cubic graph lies in an even number of Hamilton cycles. Consequently, a cubic hamiltonian graph has at least three Hamilton cycles.*

This result has been extended to graphs with only odd degree vertices by using the so-called lollipop method due to Thomason [73]. Thomassen [74] applied the lollipop technique to bipartite graphs and proved that a bipartite uniquely hamiltonian graph must have a vertex of degree 2 in each class. He also stated the following conjecture to indicate how the existence of more than one Hamilton cycle may lead to a general reduction method for hamiltonian graphs.

Conjecture 8.1 (Thomassen [74]). Every hamiltonian graph G of minimum degree at least 3 contains an edge e such that $G-e$ and G/e are both hamiltonian.

Here, $G-e$ is the graph obtained from G by deleting the edge e, and G/e is the graph obtained from G by contracting the edge $e=uv$, i.e., replacing the vertices u and v by a new vertex w, and making w adjacent to all vertices of $N_G(u)\cup N_G(v)$.

It is easy to show that if a graph G has a second Hamilton cycle, then Conjecture 8.1 holds for graph G (see the proof in the next section). So, for trying to prove Conjecture 8.1 one can restrict oneself to uniquely hamiltonian graphs. For regular graphs, it is still not known whether uniquely hamiltonian k-regular graphs exist for $k \geq 4$, but from [75] it is known that if such a graph were to exist, it must be for even k. As a subcase, Sheehan's Conjecture [71] states that there are no 4-regular uniquely hamiltonian graphs. In the light of this conjecture, note the importance of restricting oneself to simple graphs, i.e., graphs in which each pair of vertices is joined by at most one edge. The conjecture does not hold for multigraphs, since Fleischner [42] constructed a 4-regular uniquely hamiltonian multigraph. For almost 3-regular graphs, Entringer and Swart [35] obtained the following related results.

Theorem 8.2 (Entringer and Swart [35]). *A hamiltonian graph G having one vertex of degree 2 or 4 and all others of degree 3 has at least two Hamilton cycles.*

Theorem 8.3 (Entringer and Swart [35]). *For each $n = 2k$, for $k > 11$, there exists a graph on n vertices having two vertices of degree 4, all others of degree 3, and having a unique Hamilton cycle.*

There is little known about Conjecture 8.1 for general graphs, but Bielak [8] proved the conjecture for claw-free graphs.

Theorem 8.4 (Bielak [8]). *Every hamiltonian claw-free graph G of minimum degree at least 3 contains an edge e such that $G-e$ and G/e are both hamiltonian.*

Furthermore, Bielak proved that any possible counterexample to Conjecture 8.1 must contain a claw two edges of which are on a Hamilton cycle.

Theorem 8.5 (Bielak [8]). *Let G be a hamiltonian graph of minimum degree at least 3 such that a Hamilton cycle of G does not contain two edges of a claw. Then G contains an edge e such that $G-e$ and G/e are both hamiltonian.*

In this chapter, we extend Theorem 8.5 as follows.

Theorem 8.6. *Let G be a hamiltonian graph of minimum degree at least 3, and suppose G is $K_1 \cup K^*_{1,3}$-free, where $K^*_{1,3}$ is a claw having two edges on a Hamilton cycle. Then G contains an edge e such that $G-e$ and G/e are both hamiltonian.*

Corollary 8.7. *Every hamiltonian $K_1 \cup K_{1,3}$-free graph G of minimum degree at least 3 contains an edge e such that $G - e$ and G/e are both hamiltonian.*

Next, we also prove the validity of the conjecture for $K_1 \cup P_5$-free graphs.

Theorem 8.8. *Every hamiltonian $K_1 \cup P_5$-free graph G of minimum degree at least 3 contains an edge e such that $G - e$ and G/e are both hamiltonian.*

The remainder of this chapter contains our proofs of the two main results, but before we start the proofs, in the next section we present some notational conventions and auxiliary results.

8.2 Preliminaries

A k-cycle is a cycle with k vertices. Throughout the rest of this chapter, we use $\{u; v, w, x, y\}$ to denote an induced subgraph isomorphic to $K_1 \cup K_{1,3}$ with the independent vertex u and the claw induced by $\{v, w, x, y\}$. Similarly, we use $\{u; v, w, x, y, z\}$ to denote an induced subgraph isomorphic to $K_1 \cup P_5$ with the independent vertex u and the induced $P_5 = vwxyz$. Our proofs of Theorem 8.6 and Theorem 8.8 are based on the following three lemmas. Most of these observations have been made before, but we include their easy and short proofs for convenience.

Lemma 8.9. *If a hamiltonian graph G contains a second Hamilton cycle, then G contains an edge e such that $G - e$ and G/e are both hamiltonian.*

Proof. Let C and C' be two different Hamilton cycles of G, i.e., with distinct edge sets. For any edge $e \in E(C) \backslash E(C')$, clearly C/e is a Hamilton cycle of G/e, and C' is a Hamilton cycle of $G - e$. □

Lemma 8.10. *If a hamiltonian graph G of order n with $\delta(G) \geq 3$ contains an $(n-1)$-cycle, then G contains an edge e such that $G - e$ and G/e are both hamiltonian.*

Proof. Let C and C' be a Hamilton cycle and an $(n-1)$-cycle of G, respectively. Suppose that $V(C) \setminus V(C') = \{v\}$. Since $\delta(G) \geq 3$, there is a chord e of C

incident with v. Now $G—e$ has the Hamilton cycle C and G/e has the Hamilton cycle C'. □

Lemma 8.11. *Let G be a hamiltonian graph with $\delta(G) \geq 3$, and let $C = v_1v_2v_3\ldots v_nv_1$ be a Hamilton cycle of G. If one of the following cases holds (with $1 \leq i \neq j \leq n$ and the usual convention that $v_{n+1} = v_1$, etc.), then G contains an edge e such that $G-e$ and G/e are both hamiltonian.*

(1) *There exists a chord v_iv_{i+2};*

(2) *There exist two chords $e_1 = v_iv_j$ and $e_2 = v_i^+v_j^+$ or $e_2 = v_i^+v_j^{+2}$;*

(3) *There exists a chord v_iv_j such that* $N_{v_j^+Cv_i^-}(v_i^+) \cap [N^-_{v_j^+Cv_i^-}(v_j^-) \cup N^+_{v_j^+Cv_i^-}(v_j^-)] \neq \emptyset$;

(4) *There exists a chord v_iv_j such that* $N_{v_j^+Cv_i^-}(v_i^+) \cap [N^{-2}_{v_j^+Cv_i^-}(v_j^-) \cup N^{+2}_{v_j^+Cv_i^-}(v_j^-)] \neq \emptyset$;

(5) *There exists a chord v_iv_j such that* $N_{v_j^+Cv_i^-}(v_i^{+2}) \cap [N^-_{v_j^+Cv_i^-}(v_j^-) \cup N^+_{v_j^+Cv_i^-}(v_j^-)] \neq \emptyset$ *or* $N_{v_j^+Cv_i^-}(v_j^{-2}) \cap [N^-_{v_j^+Cv_i^-}(v_i^+) \cup N^+_{v_j^+Cv_i^-}(v_i^+)] \neq \emptyset$.

Proof. We provide short proofs for each of the cases, in the same order, as follows.

(1) If there exists a chord v_iv_{i+2}, then G clearly contains an $(n-1)$-cycle. Then the result follows from Lemma 8.10.

(2) If there exist two chords $e_1 = v_iv_j$ and $e_2 = v_i^+v_j^+$, then G contains the following second Hamilton cycle $v_i^+Cv_jv_i\overline{C}v_j^+v_i^+$, and the result follows from Lemma 8.9. If there exist two chords $e_1 = v_iv_j$ and $e_2 = v_i^+v_j^{+2}$, then G contains the $(n-1)$-cycle $v_i^+Cv_jv_i\overline{C}v_j^{+2}v_i^+$, and the result follows from Lemma 8.10.

(3) If $v_k \in N_{v_j^+Cv_i^-}(v_i^+) \cap [N^-_{v_j^+Cv_i^-}(v_j^-) \cup N^+_{v_j^+Cv_i^-}(v_j^-)]$ for a chord v_iv_j, then G either contains the second Hamilton cycle $v_i^+Cv_j^-v_k^+Cv_iv_jCv_kv_i^+$ or the second Hamilton cycle $v_i^+Cv_j^-v_k^-\overline{C}v_jv_i\overline{C}v_kv_i^+$. The result follows from Lemma 8.9.

(4) If $v_k \in N_{v_j^+Cv_i^-}(v_i^+) \cap [N^{-2}_{v_j^+Cv_i^-}(v_j^-) \cup N^{+2}_{v_j^+Cv_i^-}(v_j^-)]$ for a chord v_iv_j, then there is an $(n-1)$-cycle $v_i^+Cv_j^-v_k^{+2}Cv_iv_jCv_kv_i^+$ or $v_i^+Cv_j^-v_k^{-2}\overline{C}v_jv_i\overline{C}v_kv_i^+$. The result follows from Lemma 8.10.

(5) If $v_k \in N_{v_j^+Cv_i^-}(v_i^{+2}) \cap [N^{-}_{v_j^+Cv_i^-}(v_j^-) \cup N^{+}_{v_j^+Cv_i^-}(v_j^-)]$ for a chord v_iv_j, then there is an $(n-1)$-cycle $v_i^{+2}Cv_j^-v_k^{+}Cv_iv_jCv_kv_i^{+2}$ or $v_i^{+2}Cv_j^-v_k^{-}\overline{C}v_jv_i\overline{C}v_kv_i^{+2}$. The result follows from Lemma 8.10. The other case is symmetric.

□

8.3 Proofs of Theorem 8.6 and Theorem 8.8

Suppose, to the contrary, that G is a hamiltonian graph satisfying the conditions of Theorem 8.6 or Theorem 8.8, and G does not contain an edge e such that $G-e$ and G/e are both hamiltonian. Then G does not satisfy the conditions of any of the three Lemmas 8.9-8.11. Let C be the unique Hamilton cycle of G. We deal with the two cases according to the forbidden subgraph involved in each of the two theorems.

Case 1. G is $K_1 \cup K_{1,3}$-free.
Using Theorem 8.5, we assume that C contains two edges of a claw. Let this claw $\{u; u^-, u^+, v\}$ be chosen in such a way that $\min\{|u^+Cv^-|, |v^+Cu^-|\}$ is as small as possible. Without loss of generality, we assume that $|v^+Cu^-| \leq |u^+Cv^-|$. Since G is $K_1 \cup K_{1,3}$-free, $\{u; u^-, u^+, v\}$ is a dominating vertex set of G. By Lemma 8.11(1) and (2), $v^{+2}v \notin E(G)$ and $v^{+2}u^+ \notin E(G)$.

Suppose that $v^{+2}u \in E(G)$. By the choice of v, $\{u; u^-, u^+, v^{+2}\}$ does not induce $K_{1,3}$, so $v^{+2}u^- \in E(G)$. By the choice of u, $\{v^{+2}; v^+, v^{+3}, u\}$ also does not induce $K_{1,3}$. By Lemma 8.11(2), $v^{+3}u \notin E(G)$, so $v^+u \in E(G)$. Since v^{+3} is dominated by $\{u; u^-, u^+, v\}$, and using Lemma 8.11, we have $v^{+3}u^- \in E(G)$. Clearly, $\{u^-; u^{-2}, u, v^{+3}\}$ does not induce $K_{1,3}$. Using Lemma 8.11, we have $v^{+3}u^{-2} \in E(G)$. Then, again using Lemma 8.11, v^{+4} is not adjacent to any of the vertices of $\{u; u^-, u^+, v\}$, a contradiction. Hence, $v^{+2}u \notin E(G)$, and $v^{+2}u^- \in E(G)$.

Suppose that $v^+u^- \in E(G)$. Then, to avoid $\{u^-; u, u^{-2}, v^{+2}\}$ inducing a claw which contradicts the choice of u, and using Lemma 8.11, we have $u^{-2}v^{+2} \in E(G)$. By Lemma 8.11, v^{+3} is not adjacent to any of the vertices of $\{u; u^-, u^+, v\}$, a contradiction. Hence, $v^+u^- \notin E(G)$. Then to avoid $\{v^{+2}; v, v^{+3}, u^-\}$ inducing a claw which contradicts the choice of u, we have $u^-v^{+3} \in E(G)$. Then $\{u^-; u, u^{-2}, v^{+2}\}$ induces a $K_{1,3}$, contradicting the choice of u. This completes our proof for the case that G is $K_1 \cup K_{1,3}$-free.

Case 2. G is $K_1 \cup P_5$-free.
Let uv be a chord of C such that the distance of u and v on C is minimum among all chords of C, i.e., $\min\{|u^+Cv^-|, |v^+Cu^-|\}$ is as small as possible. Without loss of generality, we assume $|u^+Cv^-| \leq |v^+Cu^-|$. Then both uCv^- and u^+Cv are induced subgraphs. By the choice of the chord uv and using Lemma 8.11, $[N(u^-) \cup N(v^+)] \cap u^+Cv^- = \emptyset$. We are going to use the following claim.

Claim 1. $|u^+Cv^-| \leq 3$.

Proof. If $|u^+Cv^-| \geq 5$, then $\{u^-; u^+Cu^{+5}\}$ induces a $K_1 \cup P_5$, a contradiction. If $|u^+Cv^-| = 4$, then to avoid $\{u^-; u^+Cv\}$ inducing a $K_1 \cup P_5$, we have $u^-v \in E(G)$. Similarly, to avoid $\{v^+; uCv^-\}$ inducing a $K_1 \cup P_5$, we have $v^+u \in E(G)$. Now we are in case (2) of Lemma 8.11 with $e_1 = uv^+$ and $e_2 = u^-v$, a contradiction. □

We distinguish two subcases according to $|u^+Cv^-|$.

Case A. $|u^+Cv^-| = 3$.
We first prove the following fact.

Claim 2. $u^-v^+ \in E(G)$.

Proof. Suppose, by contradiction, that $u^-v^+ \notin E(G)$. To avoid $\{v^+; u^-Cv^-\}$ and $\{u^-; u^+Cv^+\}$ inducing a $K_1 \cup P_5$, we have $v^+u \in E(G)$ and $u^-v \in E(G)$. Now we are in case (2) of Lemma 8.11 with $e_1 = uv^+$ and $e_2 = u^-v$, a contradiction. □

Now consider the subgraph induced by $\{v^{+2}; u^-Cv^-\}$. To avoid an induced $K_1 \cup P_5$, we have that $v^{+2}u^+ \in E(G)$ or $v^{+2}u^{+2} \in E(G)$ or $v^{+2}u^- \in E(G)$.

If $v^{+2}u^{+} \in E(G)$, then we are in case (2) of Lemma 8.11, a contradiction. If $v^{+2}u^{+2} \in E(G)$, then we are in case (5) of Lemma 8.11, a contradiction. Hence, $v^{+2}u^{-} \in E(G)$. By symmetry, we have $u^{-2}v^{+} \in E(G)$. Then we are in case (2) of Lemma 8.11 with $e_1 = v^{+2}u^{-}$ and $e_2 = u^{-2}v^{+}$, a contradiction. This completes Case A.

Case B. $|u^{+}Cv^{-}| = 2$.
We first prove the following statement.

Claim 3. $u^{+}u^{-2} \notin E(G)$ or $v^{-}v^{+2} \notin E(G)$.

Proof. Suppose, to the contrary, that $u^{+}u^{-2} \in E(G)$ and $v^{-}v^{+2} \in E(G)$. If $u^{-}u^{-4} \notin E(G)$ or $v^{+}v^{+4} \notin E(G)$, then we take $u^{+}u^{-2}$ or $v^{-}v^{+2}$ to replace uv, respectively, and the claim holds. If $u^{-}u^{-4} \in E(G)$ and $v^{+}v^{+4} \in E(G)$, then we take $u^{-}u^{-4}$ or $v^{+}v^{+4}$ to replace uv. By repeating the same arguments that we used for uv, either we end up at a chord for which the claim holds, or we conclude that G contains a hamiltonian 3-regular graph. In the latter case, by Theorem 8.1 G has at least three Hamilton cycles, contradicting the assumption. □

Using Claim 3, without loss of generality, we assume $u^{+}u^{-2} \notin E(G)$. Let x be the last neighbor of u^{+} on $v^{+}Cu^{-}$ according to the orientation of C. We need three more claims for completing our proof.

Claim 4. $uv^{+} \notin E(G)$.

Proof. Suppose, by contradiction, that $uv^{+} \in E(G)$. By Lemma 8.11(2), we have that $\{u^{-}, u^{-2}\} \cap [N(v^{-}) \cup N(v)] = \emptyset$. Then $\{v^{-}, v, u, u^{-}, u^{-2}\}$ induces a P_5. By Lemma 8.11(3), we have that $v^{-}x^{-} \notin E(G)$ and $vx^{-} \notin E(G)$, by regarding uv and uv^{+}, respectively. By Lemma 8.11(2), we have that $ux^{-} \notin E(G)$ and $u^{-}x^{-} \notin E(G)$. If $u^{-2}x^{-} \in E(G)$, then G contains an $(n-1)$-cycle $xCu^{-2}x^{-}\overline{C}v^{+}uvv^{-}u^{+}x$, a contradiction by Lemma 8.10. Now $\{x^{-}; v^{-}, v, u, u^{-}, u^{-2}\}$ induces a $K_1 \cup P_5$, a contradiction. □

Claim 5. $vu^{-} \notin E(G)$.

Proof. If $v^-v^{+2} \notin E(G)$, then by symmetry with Claim 4, $vu^- \notin E(G)$. If $v^-v^{+2} \in E(G)$, then to avoid the $(n-1)$-cycle $v^{+2}Cu^-vuu^+v^-v^{+2}$ of G, we have $vu^- \notin E(G)$. □

Claim 6. $u^-v^+ \notin E(G)$.

Proof. Suppose, to the contrary, that $u^-v^+ \in E(G)$. By Lemma 8.11(2), $u^+v^+ \notin E(G)$ and $u^+v^{+2} \notin E(G)$. By Lemma 8.11(3), $v^-v^{+2} \notin E(G)$. Then u^+Cv^{+2} is an induced P_5. By the choice of x, $u^+x^+ \notin E(G)$. By Lemma 8.11(2), $x^+v^- \notin E(G)$ and $x^+v \notin E(G)$. By Lemma 8.11(3) and (4), $x^+v^+ \notin E(G)$ and $x^+v^{+2} \notin E(G)$ by considering u^+x. Now $\{x^+; u^+Cv^{+2}\}$ induces a $K_1 \cup P_5$, a contradiction. □

Now consider the subgraph induced by $\{v^+; u^{-2}Cv^-\}$. Using Lemma 8.11 and Claims 4 and 6, we have that $u^-Cv^- \cap N(v^+) = \emptyset$. To avoid an induced $K_1 \cup P_5$, we have $v^+u^{-2} \in E(G)$. Then, using Lemma 8.11 and Claims 4 to 6, we have that $\{u^-; u^+Cv^{+2}\}$ induces a $K_1 \cup P_5$, a contradiction. This completes the proofs of Theorem 8.6 and Theorem 8.8. ■

Summary

The Hamilton problem, which is the problem of determining whether an arbitrarily given graph admits a Hamilton cycle, is one of the most well-known NP-complete decision problems within graph theory and computational complexity. Most of the research work related to this Hamilton problem is centred around finding sufficient conditions for a graph to be hamiltonian, i.e., to admit a Hamilton cycle (a cycle containing all the vertices of the graph). Exploring sufficient conditions for hamiltonicity is also the main topic of this thesis, although it also includes the study of sufficient conditions for other hamiltonian properties.

Due to the NP-completeness of the Hamilton problem for general graphs, a lot of research work has been focused on certain parts of the universe of graphs, or on some classes of graphs. Forbidding some induced subgraphs in a graph is a way to categorize graphs into different classes. In this thesis, we use forbidden subgraphs to restrict our research work to some classes of graphs rather than to arbitrary graphs.

In the first part of our work, which includes Chapters 2-4, we use the implicit degree of vertices as a parameter to obtain sufficient conditions for hamiltonicity. Since the introduction of the notion of the implicit degree, many existing results based on degree conditions have been extended by relaxing the degree condition to an implicit degree condition, including Dirac-type, Ore-type, Fan-type and Bondy-type sufficient degree conditions. Inspired by this idea, we tried to improve other existing results about hamiltonicity that are based on degree conditions in this way. In Chapter 2 and

Chapter 3, we impose implicit degree conditions on specific induced subgraphs, and extend heavy subgraph conditions to implicit heavy subgraph conditions. In Chapter 2, we study various implicit degree conditions, including but not limited to Ore-type and Fan-type conditions. We prove that imposing these conditions on specific pairs of induced subgraphs of a 2-connected graph ensures its hamiltonicity. In particular, we complete the characterization of pairs of o-heavy and f-heavy subgraphs for hamiltonicity of 2-connected graphs. In Chapter 3, we give two sufficient conditions for hamiltonicity of almost distance-hereditary graphs, by imposing implicit degree conditions on every induced claw of a graph. In Chapter 4, we consider minimum implicit degree conditions on claw-free graphs to approach the best-possible implicit degree conditions that guarantee the hamiltonicity of claw-free graphs.

The second part of our work, reported in Chapters 5-7, deals with three different hamiltonian properties studied for some restricted classes of graphs by involving toughness conditions. The presented results are inspired by a recent paper due to Li et al. [54], who tried to characterize all possible graphs H such that every 1-tough H-free graph is hamiltonian. The almost complete answer was given there by the conclusion that every proper induced subgraph H of $K_1 \cup P_4$ can act as a forbidden subgraph to ensure that every 1-tough H-free graph is hamiltonian, and that there is no other forbidden subgraph with this property, except possibly for the graph $K_1 \cup P_4$ itself.

Instead of researching this open case, in Chapter 5 we consider the stronger property of being hamiltonian-connected under the same additional forbidden subgraph conditions, but assuming the toughness to be strictly larger than one. We find that the results are completely analogous to the hamiltonian case: every graph H such that any 1-tough H-free graph is hamiltonian also ensures that every H-free graph with toughness larger than one is hamiltonian-connected. And similarly, there is no other forbidden subgraph having this property, except possibly for the graph $K_1 \cup P_4$ itself. In Chapter 6, we examine whether the conditions on toughness and forbidden subgraphs for hamiltonicity in [54] in fact imply the stronger property of pancyclicity. We conclude that this is indeed the case, except for a few specific classes of exceptional graphs. In Chapter 7, we explore the open problem in [54] asking whether every $K_1 \cup P_4$-free 1-tough graph on at least three vertices is hamiltonian, and the related conjecture in [66] stating that every $K_1 \cup K_{1,3}$-free graph G with $\tau(G) > 4/3$ is hamiltonian. By involving the triangle as an additional forbidden subgraph we obtain results that partially resolve these two problems.

The last part of our work is motivated by an interesting conjecture due to Thomassen which states that every hamiltonian graph G of minimum degree at least 3 contains an edge e such that $G - e$ and G/e are both hamiltonian. In Chapter 8, we prove that the conjecture holds for $K_1 \cup K_{1,3}$-free graphs and $K_1 \cup P_5$-free graphs.

Samenvatting

Het Hamilton-probleem (H-probleem), dat is het probleem om te bepalen of een willekeurige gegeven graaf een Hamiltoncykel bevat, is een van de meest bekende en bestudeerde NP-volledige beslissingsproblemen binnen de grafentheorie en de complexiteitstheorie. Het overgrote deel van het onderzoek dat gerelateerd is aan dit H-probleem is gecentreerd rond het vinden van voldoende voorwaarden opdat een graaf een Hamiltoncykel bevat, dat is een cykel die alle punten van de graaf bevat. Het onderzoeken van zulke voldoende voorwaarden is ook het hoofdthema van dit proefschrift, hoewel het ook onderzoeksresultaten bevat voor andere hamiltonse eigenschappen.

Omdat het H-probleem NP-volledig is voor algemene grafen, is veel onderzoek gericht geweest op kleinere deelklassen van de verzameling van alle grafen. Een van de benaderingen hierbij is gebaseerd op het verbieden van bepaalde grafen of kleine verzamelingen van grafen als geïnduceerde deelgrafen. Hierdoor wordt een groot aantal grafen uitgesloten, en bovendien hebben we daardoor meer grip op de structuur van de te onderzoeken grafen. Alle resultaten in dit proefschrift zijn in meer of mindere mate gebaseerd op deze benadering.

In het eerste deel van dit proefschrift, bestaande uit Hoofdstuk 2–4, gebruiken we impliciete graadvoorwaarden om voldoende voorwaarden voor het bestaan van Hamiltoncykels af te leiden. Deze resultaten zijn geïnspireerd door een groot aantal eerdere verbeteringen van bestaande graadvoorwaarden in de literatuur tot impliciete graadvoorwaarden, waaronder voorwaarden van Dirac-type, Ore-type, Fan-type en Bondy-type. In Hoofdstuk 2 en

Hoofdstuk 3 leggen we impliciete graadvoorwaarden op aan bepaalde geïnduceerde deelgrafen, en veralgemeniseren we daarmee zogenoemde heavy-deelgraaf-voorwaarden tot impliciete heavy-deelgraaf-voorwaarden. In het bijzonder geven we in Hoofdstuk 2 een volledige karakterisering van paren o-heavy en f-heavy deelgrafen voor 2-samenhangende hamiltonse grafen. In Hoofdstuk 3 worden impliciete graadvoorwaarden opgelegd aan klauwen om daarmee het bestaan van Hamiltoncykels in de klasse van zogenoemde almost distance-hereditary grafen af te dwingen. In Hoofdstuk 4 geven we impliciete graadvoorwaarden die garanderen dat klauw-vrije grafen hamiltons zijn.

Het tweede deel van dit proefschrift, bestaande uit Hoofdstuk 5–7, richt zich op drie verschillende varianten van hamiltonse eigenschappen die bestudeerd worden voor deelklassen van grafen waarbij tevens een taaiheidsvoorwaarde wordt opgelegd. De gepresenteerde resultaten zijn geïnspireerd door een recent artikel van Li et al. [54], waarin de auteurs een poging doen om alle grafen H te karakteriseren zodanig dat elke 1-taaie H-vrije graaf een Hamiltoncykel bevat. Ze geven in dat artikel een bijna volledige oplossing met de conclusie dat elke echte deelgraaf H van de graaf $K_1 \cup P_4$ die eigenschap heeft, en dat er geen andere grafen voor H in aanmerking komen, behalve wellicht de graaf $K_1 \cup P_4$ zelf. Dit open geval ziet er zeer moeilijk uit.

In plaats van het verder onderzoeken van dit open geval, richten we ons in Hoofdstuk 5 op de sterkere eigenschap dat een graaf hamilton-samenhangend is, en we concluderen daar dat er een sterke analogie met de resultaten van [54] bestaat: elke graaf H waarvoor geldt dat elke 1-taaie H-vrije graaf hamiltons is, garandeert ook dat elke H-vrije graaf G met taaiheid $\tau(G) > 1$ hamilton-samenhangend is. Net als in [54] is er geen andere graaf die dit laatste garandeert, behalve wellicht de graaf $K_1 \cup P_4$ zelf. In Hoofdstuk 6 onderzoeken we of de voorwaarden uit [54] de sterkere eigenschap afdwingen dat de onderhavige grafen cykels van elke lengte bevatten, van een driehoek tot een Hamiltoncykel. Dit blijkt inderdaad het geval te zijn, op een paar duidelijk te karakteriseren uitzonderingen na. In Hoofdstuk 7 gaan we verder in op het open probleem uit [54] en een gerelateerd vermoeden uit [66] dat elke $K_1 \cup K_{1,3}$-vrije graaf G met taaiheid $\tau(G) > 4/3$ hamiltons is. We lossen deze problemen gedeeltelijk op door te bewijzen dat de uitspraken geldig zijn als we naast deze verboden deelgrafen ook de driehoek verbieden.

Het laatste hoofdstuk is gemotiveerd door een interessant vermoeden van Thomassen dat elke hamiltonse graaf G met minimale graad minstens 3 een lijn e bevat zodanig dat zowel $G - e$ als G/e hamiltons is. In Hoofdstuk 8 bewijzen we dat dit vermoeden waar is voor $K_1 \cup K_{1,3}$-vrije grafen en voor $K_1 \cup P_5$-vrije grafen.

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Wei Zheng was born on December 1, 1990 in Qihe County, Dezhou City, P. R. of China. From 1998 until 2006, she attended primary and secondary school in her hometown. From 2006 to 2009, she studied at No.1 Middle School Of Shandong Qihe.

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