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Partial fraction expansions for delay systems

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Abstract: Partial fraction decompositions for single input–single output delay systems are studied. The results have application in rational L_∞-approximations, which play an important role in robust compensator design for delay systems.

Keywords: Infinite-dimensional systems, Nuclearity, Delay systems, Approximation.

1. Introduction

Recently researchers have become interested in L_∞-approximations of infinite dimensional systems, because of their usefulness in the design of robust finite dimensional controllers (Glover, Curtain and Lam [4], Curtain and Glover [3], Bontsema, Curtain and Schumacher [2]). The controllers are designed to stabilize a low order L_∞-approximation of the infinite dimensional system and if the approximation error is sufficiently small, the controller also stabilizes the original infinite dimensional system; for a short account of this design technique see Curtain [7].

An essential step in the controller design is the L_∞-approximation step and a theory for this is developed by Glover et al. in [10]. If one wants to calculate a low order L_∞-approximation with a given error bound, then it is convenient to have the transfer function expressed as a partial fraction expansion, as is the case for flexible systems [7]. This is, however, not the case for delay systems and this motivated our problem.

In this paper we shall investigate the uniform approximation of a class of delay systems by means of partial fraction expansions. In particular we give easily verifiable conditions for large classes of delay systems to possess such expansions. The bounds we obtain here are used by us in [12] to obtain conditions for L_∞-approximation of delay system, as well as in a discussion of nuclear systems, which have been much studied by Glover, Curtain and Partington [10].

The transfer functions we consider necessarily, have infinitely many poles. Functions such as $e^{-Ts}f(s)$ with $f(s)$ rational are not considered her but are studied by Glover, Lam and Partington in [11] using rather different techniques.

We conclude with a simple example of a delay system admitting a partial fraction decomposition.

The mathematics is of necessity somewhat technical, but readers who are interested in the main result and its applications can read Theorem 2.5, the example in Section 3 and [10].

(Some of the results of this paper were presented by H. Zwart at the conference [5], but there the emphasis was on applications to spectral realisation.)

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2. Class of delay systems with a partial fraction decomposition

In Bellman and Cooke [1] large classes of delay systems are studied using both a state space and frequency domain description. The book contains a wealth of information about difference delay systems including interesting properties of the poles of the transfer functions, which we can use to advantage for the partial fraction expansion problem.

Since the delay transfer functions that we shall consider are fractions of polynomials in s and e^{-s} we shall begin by considering the following class of functions:

$$g(s) = \sum_{i=0}^{n} p_i s^{m_i}(1+\varepsilon_i(s))\, e^{\beta_i s}. \tag{2.1}$$

where $p_i \neq 0$, $m_i \geq 0$, $0=\beta_0<\beta_1<\cdots<\beta_n$ and $s\varepsilon_i(s)$ is bounded if $|s| \to \infty$. With the points $P_i := (\beta_i, m_i)$ we can define the distribution diagram of (2.1) (see Figure 1); this is the polygonal line L which is the upper boundary part of the convex hull of the points (0, 0), P_i, $(\beta_n, 0)$.

Let the successive segments of L be denoted by $L_1, L_2, \ldots, L_k$, numbered from left to right, and let the slope of L_j denote by μ_j. Let $c_1 \in (0, \infty)$, then we can construct in the complex plane a number of segments $V_1, \ldots, V_k$, $U_0, \ldots, U_k$, defined by the inequalities

$$U_0:\quad |s^{\mu_1} e^s| < e^{-c_1},$$
$$U_j:\quad |s^{\mu_j} e^s| > e^{c_1} \text{ and } |s^{\mu_{j+1}} e^s| < e^{-c_1},\quad j=1,\ldots,k-1,$$
$$U_k:\quad |s^{\mu_k} e^s| > e^{c_1},$$
$$V_j:\quad e^{-c_1} < |s^{\mu_j} e^s| < e^{c_1},\quad j=1,\ldots,k.$$

In Figure 2 we indicate pictorially the appearance of the complex plane corresponding to the diagram in Figure 1.

From Bellman and Cooke [1] we have that outside a sufficiently large circle, these V and U regions do not intersect each other, and the following theorem about the distribution of the zeros of $g(s)$ and its bounds holds.

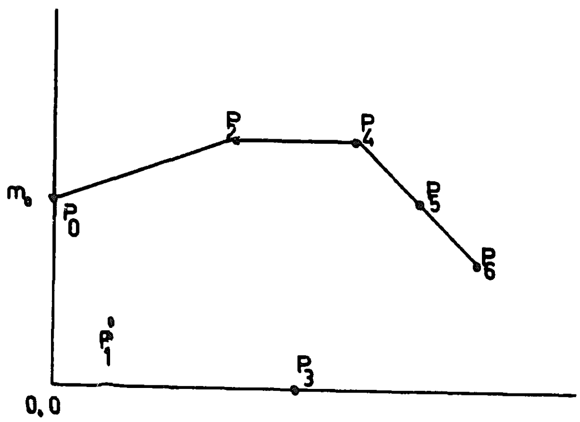

Fig. 1.

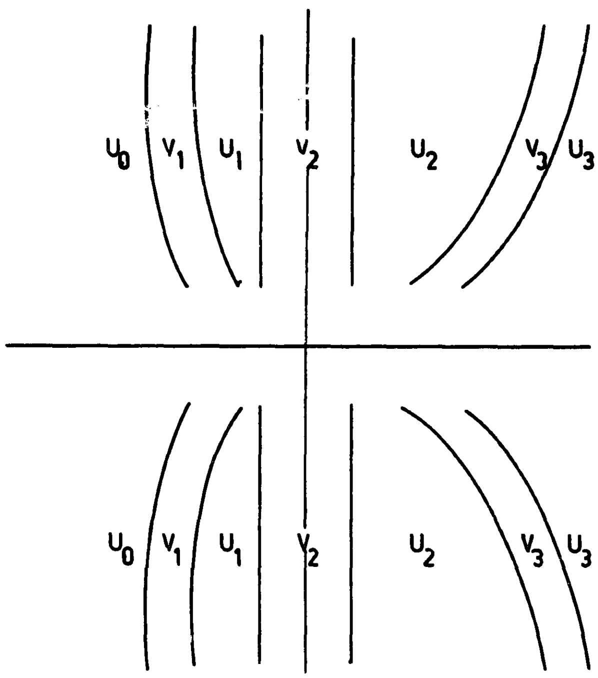

Fig. 2.

Theorem 2.1. *Let the complex plane be divided into regions $V_1, V_2, \ldots, V_k$ and $U_0, \ldots, U_k$ in the manner described above. Outside a certain circle $|s| = c_2$ the following holds:*

(a) *There are no zeros of $g(s)$ in U_j $(j = 0, \ldots, k)$. If $s \in U_j$, $|g(s)s^{-m_r} e^{-\beta_r s}|$ is uniformly bounded away from zero, where (β_r, m_r) is the point of the distribution diagram at the right end of the segment L_j, for $j > 0$. If $j = 0$ then $(\beta_r, m_r) = (\beta_0, m_0)$.*

(b) *All the zeros of $g(s)$ lie in V_j $(j = 1, \ldots, k)$, and in any subregion of V_j in which s is uniformly bounded away from all zeros, $|g(s) e^{-m_l} e^{-\beta_l s}|$ is uniformly bounded away from zero, where (β_l, m_l) is the left end point of L_j.*

Proof. See [1], Theorem 12.10.

We shall now consider the function

$$h(s) = \sum_{i=0}^{n} \frac{p_i(s) e^{-\gamma_i s}}{q(s) e^{-\alpha s}} \tag{2.2}$$

where $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n$, $p_i(s)$ is a polynomial of degree δ_i, $\alpha \in \mathbb{R}$ and $q(s)$ is a polynomial of degree δ. $h(s)$ can be understood as the reciprocal of the transfer function of a delay system. Let K be the distribution diagram of the points (γ_i, δ_i), $i = 0, \ldots, n$. We shall denote the successive segments of K by K_j, $j = 1, \ldots, k$, and the slope of K_j by θ_j, $j = 1, \ldots, k$. Furthermore θ_0 will be infinity and θ_{k+1} is minus infinity. With this notation we have the following growth bounds for $h(s)$.

Theorem 2.2. *If s is uniformly bounded away from all the zeros of $h(s)$ and $|s|$ is sufficiently large, then*

$$|h(s)| \geqslant c|s|^{\tau} \tag{2.3}$$

where τ satisfies $(\alpha, \delta + \tau) \in K$, with the convention that $\tau = -\infty$ if $\alpha \notin [0, \gamma_n]$.

Remark. Since K is the upper boundary part of the convex hull of the points $(0, 0)$, (γ_i, δ_i), $(\gamma_n, 0)$ we have that τ is uniquely determined (see Figure 3).

Proof. Before we prove this theorem we shall prove the following important property of τ.

For $j = 1, \ldots, k+1$ we have that

$$\inf_{\theta \in [\theta_j, \theta_{j-1}]} (\delta_l - \delta + \theta(\alpha - \gamma_l)) \geq \tau \tag{2.4}$$

where (γ_l, δ_l) is the left end point of K.

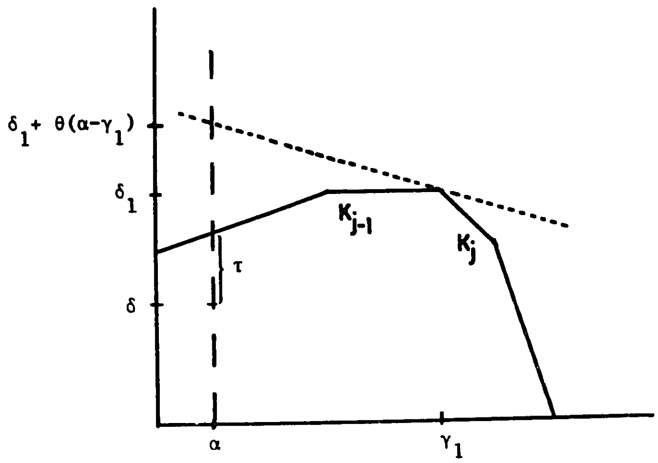

Fig. 3.

If $\alpha \notin [0, \gamma_n]$, then (2.4) is trivially satisfied. So assume that $\alpha \in [0, \gamma_n]$. The point $(\alpha, \delta_l + \theta(\alpha - \gamma_l))$ has a clear geometric meaning. It is the intersection of the line with slope θ passing through the point (γ_l, δ_l) and the line $x = \alpha$. Thus if $0 \le \alpha \le \gamma_n$, then it is clear geometrically that for $\theta \in [\theta_j, \theta_{j-1}]$ this point of intersection lies above K. Thus $(\delta_l - \delta + \theta(\alpha - \gamma_l)) \ge \tau$.

Now we shall prove the theorem

$$h(s) = \sum_{i=0}^{n} \frac{p_i(s)\, e^{-\gamma_i s}}{q(s)\, e^{-\alpha s}} = \sum_{i=0}^{n} \frac{p_i}{q} s^{\delta_i - \delta}(1 + \varepsilon_i(s))^{(\alpha - \gamma_i)s},$$

where $s\varepsilon_i(s)$ is bounded for $|s| \to \infty$. Define

$$g_1(s) = e^{(\gamma_n - \alpha)s} \cdot s^{\delta} \cdot h(s) = \sum_{i=0}^{n} \frac{p_i}{q} s^{\delta_i}(1 + \varepsilon_i(s))\, e^{(\gamma_n - \gamma_i)s} \tag{2.5}$$

and

$$m_{n-i} = \delta_i, \qquad \beta_{n-i} = \gamma_n - \gamma_i, \qquad a_{n-i} = p_i/q. \tag{2.6}$$

With these definitions we have that

$$g_1(s) = \sum_{i=0}^{n} a_i s^{m_i}(1 + \varepsilon_i(s))\, e^{\beta_i s}$$

where $m_i \ge 0$ and $0 = \beta_0 < \beta_1 < \cdots < \beta_n = \gamma_n$. If L is the distribution diagram of $g_1(s)$, then by the definitions of m_i and β_i it is just a reflection and translation of K, and furthermore, it is not hard to prove that:

(i) If μ_j is the slope of L_j then $-\mu_j = \theta_{k-j+1}$ (the slope of K_{k-j+1}).

(ii) If (β_r, m_r) is the right end point of L_j then $(\gamma_{n-r}, \delta_{n-r})$ is the left end point of K_{k-j+1}. With the notation of Theorem 2.1 we have that if $s \in U_j$, then $|g_1(s)| \ge c_3 |s^{m_r} e^{\beta_r s}|$, $|s| > c_2$. For $s \in U_j$, there exists a $\bar{\mu}_j \in (\mu_{j+1}, \mu_j)$ such that $|s^{\bar{\mu}_j} e^s| = 1$.

So if $s \in U_j$ and $|s| > c_2$, then

$$\begin{aligned} |h(s)| &= |e^{(\alpha - \gamma_n)s} \cdot s^{-\delta} \cdot g_1(s)| \ge c_3 |e^{(\alpha + \beta_r - \gamma_n)s} \cdot s^{m_r - \delta}| \\ &= c_3 |e^{(\alpha - \gamma_{n-r})s} \cdot s^{\delta_{n-r} - \delta}| = c_3 |s^{(\delta_{n-r} - \delta - \bar{\mu}_j(\alpha - \gamma_{n-r}))}|. \end{aligned} \tag{2.7}$$

Since (β_r, m_r) is the right end point of L_j we have that $(\gamma_{n-r}, \delta_{n-r})$ is the left end point of K_{k-j+1}. Furthermore since $-\bar{\mu}_j \in (-\mu_j, -\mu_{j+1}) = (\theta_{k-j+1}, \theta_{k-j})$ we have from (2.4) that $\delta_{n-r} - \delta - \bar{\mu}_j(\alpha - \gamma_{n-r}) \ge \tau$. So for $s \in U_j$ and $|s| > c_2$ we have that $|h(s)| \ge c_3 |s|^{\tau}$.

Let $s \in V_j$ with $|s| > c_2$ and s uniformly bounded away from all zeros; then $|g_1(s)| \ge c_4 |s^{m_l} e^{\beta_l s}|$. Since $s \in V_j$ we have that $|s^{\mu_j} e^s| = c_5$, $c_5 \in [e^{-c_1}, e^{c_1}]$. With the same reasoning as in (2.7) we have

$$|h(s)| \ge c_4\, e^{-c_1} |s^{\delta_{n-l} - \delta - \mu_j(\alpha - \gamma_{n-l})}|. \tag{2.8}$$

Since (β_l, m_l) is the left end point of L_j we have that $(\gamma_{n-l}, \delta_{n-l})$ is the right end point of K_{k-j+1}, or equivalently the left end point of K_{k-j+2}. And from (2.4) we see that $\delta_{n-l} - \delta - \mu_j(\alpha - \gamma_{n-l}) = \delta_{n-l} - \delta + \theta_{k-j+1}(\alpha - \gamma_{n-l}) \ge \tau$. Thus $|h(s)| \ge c_4\, e^{-c_1} |s|^{\tau}$.

For applications to common delay systems, the following special case is important:

$$\alpha = 0, \qquad 0 \le \delta < \delta_0,$$

$$|h(s)| \ge c|s|^{\delta_0 - \delta} \quad \text{for } |s| > c_2 \quad \text{and} \quad s \text{ uniformly bounded away from all zeros.}$$

We are now in a position to prove theorem 2.5 concerning sufficient conditions for a delay transfer function to have a partial fraction expansion, but to prove the absolute convergence we need two rather technical lemmas, which one can prove similar to Theorem 2.2.

Consider the following type of functions:

$$g_2(s) = e^{(\gamma_n - \alpha)s} \cdot s^{\delta} \cdot \sum_{i=0}^{n} \frac{p_i(s)\, e^{-\gamma_i s}}{q(s)\, e^{-\alpha s}} \tag{2.9}$$

where $\deg(p_i(s)) = \delta_i$, $\deg(q(s)) = \delta$ and $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n$, and

$$\tilde{g}_2(s) = e^{(\gamma_n - \alpha)s} \cdot s^{\delta} \cdot \left\{ \frac{p_0'(s)\, e^{-\gamma_0 s} + \sum_{i=1}^{n} (-\gamma_i p_i(s) + p_i'(s))\, e^{-\gamma_i s}}{q(s)\, e^{-\alpha s}} \right\}. \tag{2.10}$$

It can easily be seen that L, the distribution diagram of $g_2(s)$, is the same as $\tilde{L}$, the distribution diagram of $\tilde{g}_2(s)$, except for the last segment. Let $V_j(g_2)$ denote the regions in the complex plane defined by L; see Figure 2. Then we have the following result.

Lemma 2.3. *Assume that $0 \le \alpha \le \gamma_n$. If s is uniformly bounded away from all the zeros of $\tilde{g}_2(s)$ and $|s|$ is sufficiently large, then*

$$|s^{-\delta} e^{(-\gamma_n + \alpha)s} \cdot \tilde{g}_2(s)| \ge c|s|^{\tau}, \tag{2.11}$$

for $s \in \bigcup_{j=1}^{k} V_j(g_2)$, and τ is the same as in Theorem 2.2.

The next lemma shows that outside a sufficiently large circle the zeros of $\tilde{g}(s)$ are uniformly bounded away from the zeros of $g(s)$.

Lemma 2.4. *Let $h_1(s) = \sum_{i=0}^{n} p_i(s)\, e^{-\gamma_i s}$, where $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n$, and $\deg(p_i(s)) = \delta_i$. If on each segment K_j of K, the distribution diagram of $h_1(s)$, lie exactly two points (γ_i, δ_i), then outside a sufficiently large circle the zeros of $h_1'(s)$ are uniformly bounded away from the zeros of $h_1(s)$.*

Finally we give sufficient conditions for a general delay transfer function to have an uniformly convergent partial fraction expansion. The sufficient conditions are easily verifiable in terms of the distribution diagram of the denominator and the coefficients of the numerator.

Theorem 2.5. *Let $h_1(s) = \sum_{i=0}^{n_1} p_i(s)\, e^{-\gamma_i s}$ with $p_i(s)$ a polynomial of degree δ_i and $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{n_1}$ and let K be the distribution diagram of $h_1(s)$. Suppose that on each segment K_j of K lie exactly two points (γ_i, δ_i).*

Let $h_2(s) = \sum_{i=1}^{n_2} q_i(s)\, e^{-\alpha_i s}$ with $q_i(s)$ a polynomial of degree d_i, and assume that h_1 and h_2 are coprime. If all the points (α_i, d_i) lie below K and $0 \le \alpha_i \le \gamma_{n_1}$, $i = 1, \ldots, n_2$, then:

(a) *$f(s) := h_2(s)/h_1(s)$ has a partial fraction expansion, given by*

$$f(s) = \sum_{j=m+1}^{\infty} \frac{h_2(z_j)/h_1'(z_j)}{s - z_j} + \sum_{j=1}^{m} \sum_{i=1}^{n_j} \frac{c_{ij}}{(s - z_j)^i}, \tag{2.12}$$

where z_j are the zeros of $h_1(s)$ with multiplicity n_j; and

(b) *this sum is uniformly convergent in s in any compact subset of $\mathbb{C}/\{z_j\}$.*

Proof. We shall assume that there are no poles of f with multiplicitly larger than one. From Bellman and Cooke [1] and the conditions on K we have that there is only a finite number with multiplicitly larger than one. Those will account for the last term in (2.12).

(a) From Theorem 2.2 we have that if $|s| > c_2$ and s is uniformly bounded away from all zeros of $h_1(s)$, then

$$|f(s)| \leq \sum_{i=1}^{n_2} \left| \frac{q_i(s)\,e^{-\alpha_i s}}{\sum_{j=0}^{n_1} p_j(s)\,e^{-\gamma_j s}} \right| \leq \sum_{i=1}^{n_2} c_i |s|^{-\tau_i} \leq c|s|^{-\tau}, \quad \tau > 0, \tag{2.13}$$

where τ_i is such that $(\alpha_i, \tau_i, +d_i) \in K$ and $\tau = \min\{\tau_i \mid 1 \leq i \leq n_2\}$.

From Bellman and Cooke [1], Theorem 12.13, we have the existence of closed contours C_l such that distance $(0, C_l) \to \infty$, but length(C_l)/distance$(0, C_l)$ is bounded, the C_l have a least distance, greater than the zero from the set of all poles of $f(s)$ and there is exactly one pole between C_l and C_{l+1}. Consider the integral $\int_{C_l} f(s)/(z-s)\,ds$, for a fixed point z (not a pole of f). From Cauchy theorem we have

$$\frac{1}{2\pi i}\int_{C_l} \frac{f(s)}{z-s}\,ds = \text{sum of residues of } \frac{f(s)}{z-s} \tag{2.14}$$

inside the contour C_l. The function $f(s)/(z-s)$ has it poles at the poles of $f(s)$ and at z. The residues are, respectively Res$(f;\ z_p)/(z-z_p)$, where z_p is a pole of f, and $-f(z)$. So

$$\frac{1}{2\pi i}\int_{C_l} \frac{f(s)}{z-s}\,ds = \sum_{j=1}^{N_l} \frac{\text{Res}(f;\ z_j)}{z-z_j} - f(z), \tag{2.15}$$

where the sum is taken over all the poles z_j inside the contour C_l, and we have assumed that z lies inside C_l.

From the first part of this proof we have that on C_l, $l > l_0$,

$$|f(s)| \leq |s|^{-\tau} \leq (\text{distance}\ (0, C_l))^{-\tau} \to 0 \quad \text{if } l \to \infty. \tag{2.16}$$

Thus

$$\left| \int_{C_l} \frac{f(s)}{z-s}\,ds \right| \leq \int_{C_l} \left| \frac{f(s)}{z-s} \right| |ds| \leq c \cdot \text{length}\ (C_l) \cdot \frac{(\text{distance}\ (0, C_l))^{-\tau}}{\text{distance}\ (0, C_l)} \to 0, \quad \text{if } l \to \infty.$$

So $f(z) = \sum_{j=1}^{\infty} \text{Res}(f;\ z_j)/(z-z_j)$.

(b) From the form of f, and assuming that z_j is a simple pole we have that Res$(f;\ z_j) = h_2(z_j)/h_1'(z_j)$. So we have proved (2.12).

From Lemma 2.3, Lemma 2.4 and the fact that inside every circle $f(s)$ has only finitely many poles, we have that

$$|\text{Res}(f;\ z_j)| = |h_2(z_j)/h_1'(z_j)| = \left| \frac{\sum_{i=1}^{n_2} q_i(z_j)\,e^{-\alpha_i z_j}}{h_1'(z_j)} \right|$$

$$\leq \sum_{i=1}^{n_2} \left| \frac{q_i(z_j)\,e^{-\alpha_i z_j}}{p_0'(z_j)\,e^{-\gamma_0 z_j} + \sum_{l=0}^{n_1} \left(-\gamma_l p_l(z_j) + p_l'(z_j)\right) e^{-\gamma_l z_j}} \right|$$

$$\leq \sum_{i=1}^{n_2} c_i |z_j|^{-\tau_i} \leq c|z_j|^{-\tau}$$

where τ is the same as in (2.13).

From Bellman and Cooke [1], pp. 399–415, we have that the poles of $f(s)$, i.e. the zeros of $h_1(s)$, are asymptotically close to s_r in $V_j(h_1)$, where s_r is given by (2.17) if $\delta_j \neq \delta_{j-1}$ and by (2.18) if $\delta_j = \delta_{j-1}$.

$$\mathrm{Re}(s_r) = m(\log |w| - \log |2r\pi m + m \arg w \mp m\pi/2|), \tag{2.17a}$$

$$\mathrm{Im}(s_r) = m(2\pi r + \arg w \pm \tfrac{1}{2}\pi), \tag{2.17b}$$

$$m = \frac{\delta_{j-1} - \delta_j}{\gamma_j - \gamma_{j-1}}, \qquad w = \left[\frac{-p_j}{p_{j-1}}\right]^{1/(\delta_{j-1}-\delta_j)}, \qquad r \in \mathbb{Z}, \tag{2.17c}$$

$$s_r = \frac{1}{\gamma_j - \gamma_{j-1}}\left[\log\left|\frac{-p_{j-1}}{p_j}\right| + \mathrm{i}\left(2\pi r + \arg\left(\frac{-p_{j-1}}{p_j}\right)\right)\right]. \tag{2.18}$$

From these equations we have that $|z_j|$ is of the order j. So

$$\sum_{j=1}^{\infty}\left|\frac{\mathrm{Res}(f;\ z_j)}{z - z_j}\right| \leq c \sum_{j=1}^{\infty} \frac{|z_j|^{-\tau}}{|z - z_j|} \leq \tilde{c} \sum_{j=1}^{\infty} j^{-\tau-1}$$

for z in a compact subset of $\mathbb{C}/\{z_j\}$. Since $\tau > 0$ we have that $\sum_{j=1}^{\infty} |\mathrm{Res}(f; z_j)/z - z_j| < \infty$.

For some applications having a locally uniformly convergent partial fraction expansion is not quite enough; one also requires the following condition to hold:

$$\sum_{j=1}^{\infty}\left|\frac{\mathrm{Res}(f;\ z_j)}{\mathrm{Re}\ z_j}\right| < \infty. \tag{2.19}$$

The conditions for this are somewhat stronger.

Lemma 2.6. *Let $f(s)$, $h_1(s)$, $h_2(s)$ and K be the same as in Theorem 2.5 and let τ_i be the 'vertical distance' from (α_i, d_i) to K, τ_i satisfies $(\alpha_i, d_i + \tau_i) \in K$.*

If there is no horizontal segment in K and $\tau := \min_{i=1,\ldots,n_2}\{\tau_i\}$ is larger than one, then there is only a finite number, N, of poles of $f(s)$ on the imaginary axis and

$$\sum_{j=N+1}^{\infty}\left|\frac{\mathrm{Res}(f;\ z_j)}{\mathrm{Re}\ z_j}\right| < \infty. \tag{2.20}$$

So if, in addition, $f(s)$ has no poles on the imaginary axis, then (2.19) *holds.*

Proof. From the proof of Theorem 2.5 we have that $|\mathrm{Res}(f;\ z_j)| < \tilde{c}j^{-\tau}$. Furthermore Re z_j is asymptotically of order log j (see 2.17a). So there will only be a finite number, N, of zeros on the imaginary axis and

$$\sum_{j=N+1}^{\infty}\left|\frac{\mathrm{Res}(f;\ z_j)}{\mathrm{Re}\ z_j}\right| \leq c \sum_{j=N+1}^{\infty} \frac{j^{-\tau}}{\log j}. \tag{2.21}$$

This last sum converges iff $\tau > 1$.

Remark 1. If K_j is a horizontal segment in K, then Lemma 2.6 still holds, provided that $|p_{j-1}| \neq |p_j|$, where p_j is the coefficient of s^{δ_j} in $p_j(s)$; see (2.18).

Remark 2. Let τ and τ_i be the same as in Lemma 2.6; then we have that $\tau = \tau_{i_0}$ for some $i_0 \in \{1, \ldots, n_2\}$. For simplicity we shall assume that $\tau_{i_0} > \tau_i$ for $i \neq i_0$. From the definition of τ there exists a segment, K_m, of K such that τ is the vertical distance from $(\alpha_{i_0}, \mathrm{d}_{i_0})$ to K_m. Let z_{j_m} be the j-th zero of $h_1(s)$ in the

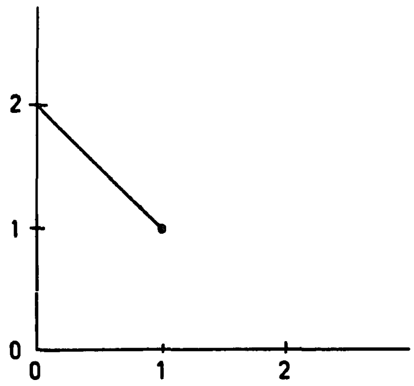

Fig. 4.

sector V_m (see Theorem 2.1); then by (2.16) and similar argument as that in Theorem 2.5 we have that $\text{Res}(f;\ z_{j_m})$ is of the order of $z_{j_m}^{-\tau}$ and the residues of the poles in the other sectors are of lower order.

3. Example

We conclude with one simple example of a delay system. Consider the delay system

$$\dot{x}_1 = -\tfrac{1}{2}\pi x_1(t-1) + x_2(t), \qquad \dot{x}_2 = u(t), \qquad y(t) = x_1(t). \tag{3.1}$$

This system has the transfer function

$$G_1(s) = \frac{1}{s\left(s + \frac{1}{2}\pi\, \mathrm{e}^{-s}\right)}. \tag{3.2}$$

The distribution diagram of the denominator is as shown in Figure 4. So $\tau = 2$ and by Theorem 2.5, $G_1(s)$ has a locally uniformly convergent partial fraction expansion. $G_1(s)$ is not stable, as it has three poles on the imaginary axis. Subtracting the residues of these three poles yields a transfer function which is stable and satisfies the conditions of Lemma 2.6 and hence (2.19) holds and in particular the partial fraction expansion converges uniformly in the right half plane. In [12] it has been shown that this transfer function is nuclear and hence can be approximated by a finite dimensional system using the methods of Glover, Curtain and Partington [10]; for a numerical example, see [6].

We remark that a factor e^{-sT} in the numerator of the delay transfer function (3.2) changes τ into $2 - T > 1$, if $0 \le T < 1$, and so all preceding remarks apply to these systems too.

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