Fourier Transform and Ludolph van Ceulen

M.Vajta, *Member, IEEE*
Department of Mathematical Sciences
University of Twente
P.O.Box 217, 7500 AE Enschede
The Netherlands
e-mail: m.vajta@math.utwente.nl

Abstract -- **The author would like to dedicate this paper to the memory of Ludolph van Ceulen (1540-1610), the famous Dutch mathematician, who spent almost his whole life to calculate the first 35 decimals of π. Now, 400 years later, the known number of digits of π exceeds 200 billion and increasing. Highly efficient and very fast converging methods have been developed in the last decades. Without the number crunching power of digital computers it would never have been possible to calculate the billions of digits of π. Amazingly, the key element in calculating the billions of digits is an efficient multiplication method. And here we can find an unexpected application from signal processing and control theory: the Fast Fourier transform (FFT). The paper presents the most efficient algorithms and describes how the Fourier transform makes fast multiplication possible.**

Index Terms **-- Fourier transform, approximation theory, elliptic functions.**

I. INTRODUCTION:

On july 5, 2000 a very special ceremony[1] took place in the *St.Pieterskerk* (St.Peter's Church) at Leiden, the Netherlands [16,18]. A replica of the original tombstone of Ludolph van Ceulen was placed into the Church since the original disappeared [17]. Ludolph van Ceulen (born 28 January 1540 in Hildesheim and died 31 December 1610 in Leiden) was a mathematician and fencing teacher. In 1600 Prince Maurits appointed him as one of the first *hoogleraar wiskunde* (professor of mathematics) at the University of Leiden. He dedicated almost all of his life to calculate more and more decimals of π. He published the 20 decimals in his book: *Van de Circel* in 1596. But he went on to calculate the first 35 decimals[2]. As legend has it, the 35 decimals were engraved on his tombstone which later disappeared [11,16]. It was therefore a tribute to the memory of Ludolph van Ceulen, when on Wednesday 5 july, 2000 prince Willem-Alexander (heir to the throne), unveiled the memorial tombstone in the St.Peter's Church, in Leiden.

Figure 1 shows the *gedenksteen* (memorial stone) in the *St.Pieterskerk* at Leiden [16]. Although the text is in old Dutch and difficult to decipher from the photo[3], it can be observed that Ludolph van Ceulen gave two bounds for the value of π. He stated: *if the diameter of a circle is 1, then the circumference is greater then*

$$3\frac{14159265358979323846264338327950288}{100000000000000000000000000000000000}$$

and less then

$$3\frac{14159265358979323846264338327950289}{100000000000000000000000000000000000}$$

Consequently, the value of π lies between these two values.

A one day mini-Symposium accompanied the ceremony on the St.Peter's Church and several papers on Ludolph van Ceulen also appeared in the *Nieuw Archief voor Wiskunde* Journal [10,11,16,17].

Calculating more and more decimals of π (first by hand and then by digital computers) not only fascinated mathematicians from ancient times but kept them busy as well. They invented hundreds of methods but the known number of decimals remained only a couple of hundred as of the late 19th century. All that changed with the advent of the digital computers. And although digital computers made possible to calculate thousands of decimals, the underlying methods hardly changed and their convergence remained slow (*linear*). Until the 1970's. Then, in 1976, an innovative *quadratic* convergent formula (based on the method of algebraic-geometric mean) for the calculation of π was published independently by Brent [12] and Salamin [19]. Soon after Brent and Salamin the Borwein brothers developed *cubically* and *quartically* convergent algorithms [8,9].

In spite of the incredible fast convergence of these algorithms, it was the application of the Fast Fourier transform (for multiplication) which enhanced their efficiency and reduced computer time [2,12,15].

[1] The one day program "Pi in the St.Peter's Church" was part of the *4th Algorithmic Number Theory Symposium* (ANTS IV) in Leiden, The Netherlands, 2-7 july, 2000.
See also the home-page: http //www.wiskgenoot.nl

[2] It should be noted at this point that 39 decimals of π suffice for computing the circumference of a circle enclosing the known universe with an error no greater than the radius of a hydrogen atom.

[3] The exact text can be found on the home-page of the Wiskundig Genootschap: http //www.wiskgenoot.nl and in [16].

II. TILL THE MIDDLE AGE

The first theoretical calculation seems to have been carried out by *Archimedes* of Syracuse (287-212 BC). He used inscribed and circumscribed polygons. Applying a polygon with 96 side he obtained the approximation $3^{10}/71 < \pi < 3^{1}/7$ (3,1408<π<3,1429). Archimedes did not have the advantage of an algebraic and trigonometric notation so he had to rely on pure geometrical means. Almost all of the methods from this period are based on the approximations of inscribed and circumscribed polygons. The highlight of this period was achieved by *Ludolph van Ceulen* (1540-1610) who determined the first 20 digits about 1596 based on a polygon with 60 x 2^{29} (=32212254720) sides [7, pp.34]. Some years later he succeeded to establish the first 35 digits [6, pp.102,13]. The Germans were so much impressed by van Ceulen's achievement that they began to call π as the *Ludolph's number*[4].

III. CALCULUS BASED METHODS

The introduction of *calculus* in the 16th century made a large number of new algorithms in the form of infinite series, products or continued fractions possible. François Viète, a lawyer and amateur mathematician, was the first who expressed π by an infinite product in 1579:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\left(\frac{1}{2}+\sqrt{\frac{1}{2}}\right)}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}}\cdots \qquad (1)$$

John Wallis also developed an infinite product but in much simpler form. Newton used his own formulae to calculate the first 15 decimals of π. When Gregory discovered the *arctangent* series in 1671 it led to a number of new algorithms [6:pp.92]. In 1706 Machin used Gregory's series and his formula (named after him) to calculate 100 decimals of π [2]:

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239}) \qquad (2)$$

Leonard Euler (circa 1748) invented dozens of new formulae for π [6:pp.112]. One of the most famous is the sum of the inverse square of integers:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \qquad (3)$$

As for the nature of π, Lambert proved it to be irrational in 1761. It took more then 100 years till Lindemann finally proved in 1882 that π is in fact transcendental putting the problem of squaring the circle to rest.

[4] see for example: Weierstrass,K.von: Zu Lindemann's Abhandlung: "Über die LUDOLPH'sche Zahl", see in [6, pp.207].

IV. 20TH CENTURY

In the beginning of the 20th century, there appeared some new and very interesting theoretical result by Ramanujan (based on elliptic and modular functions) [9]:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{(1103+26390n)}{396^{4n}} \qquad (4)$$

Besides the theoretical development it was the introduction of digital computers, which made possible to calculate an unbelievable large number of decimals of π. In the first 4000 years mathematicians could only calculate (by hand) the first 707 digits. As a test of the first digital computer (ENIAC) John von Neumann proposed to calculate the value of π. The first calculation by ENIAC in 1947 provided 2000 digits. The team of scientists needed 70 hours computer time including programming [6].

With the computer a new era began. As the speed and memory of computers grew, so increased the known number of digits. In 1961 D.Shanks and J.W.Wrench,Jr. break the 100000 decimals barrier on an IBM 7090 computer in less then 9 hours. But the formula used was still a variant of the *arctan* formula due to Störmer [6, pp.576]:

$$\pi = 24\arctan(\frac{1}{8}) + 8\arctan(\frac{1}{57}) + 4\arctan(\frac{1}{239}) \qquad (5)$$

Interestingly, almost all calculations from the beginning of the 18th century until the early 1970's have relied on one or another form of the Machin's (*arctan*) formula. Since the 60' the increase in number of decimals has been quite remarkable. Although Shanks and Wrench predicted in 1961 that to calculate the first 1 million digit should last for month (in computer time), nowadays it takes only minutes to calculate a couple of millions decimals on a fast Pentium PC. In 1996 the Chudnovsky brothers crossed the 1 billion limit. How could they do that? With faster and faster new algorithms and with an efficient multiplication method by the Fast Fourier transform.

V. NEW FAST CONVERGING ALGORITHMS

A. The Brent-Salamin Algorithm

One of the most crucial drawbacks of the known algorithms is their slow, **linear** convergence. It was therefore a milestone in the history of π research when Eugene Salamin and Richard Brent developed (independently from each other) a dramatically new algorithm in the 70's [12,19]. Their algorithm is based on the **arithmetic-geometric mean** (AGM) known already to Gauss [8]. The arithmetic-geometric mean was the basis of Gauss' method for the calculation of elliptic integrals. With the help of the elliptic integral relation of Legendre, π can be expressed in terms of the arithmetic-geometric mean and the resulting algorithm is quadratically fast [15]. The Brent-Salamin algorithm is the following: set $a_0=1$; $b_0=1/\sqrt{2}$ and

s_0=0.5 and calculate the iteration for k=1,2,...:

$$\begin{aligned} a_k &= (a_{k-1} + b_{k-1}) / 2; \\ b_k &= \sqrt{a_{k-1}\, b_{k-1}} \\ c_k &= a_k^2 - b_k^2 \\ s_k &= s_{k-1} - 2^k c_k \\ p_k &= 2 a_k^2 / s_k \end{aligned} \tag{6}$$

Then p_k converges to π **quadratically**. The first 5 terms of the Brent-Salamin algorithm give 1,3,9,20 and 42 decimals:

$\pi_{k=2} \approx$ 3,1 ...
$\pi_{k=2} \approx$ 3,141 ...
$\pi_{k=3} \approx$ 3,141592653 ...
$\pi_{k=4} \approx$ 3,1415926535 8979323846 ...
$\pi_{k=5} \approx$ 3,1415926535 8979323846 2643383279 5028841971 69 ...

Quadratic convergence means that with each new term the number of digits doubles! For a while it seemed there was no way to develop faster algorithms. But not for long. The results of Brent and Salamin gave a new impetus to the π research. Built on the same body of mathematics Jonathan and Peter Borwein introduced an even faster algorithm.

B. Borwein's Cubically Convergent Algorithm

Jonathan and Peter Borwein developed an even faster algorithms to approximate π in the 1980's. They also applied the *arithmetic-geometric mean* (AGM) and transformation theory of elliptic integral and modular equations [8,9]. Their cubically convergent algorithm is as follows: set a_0 = 1/3 and s_0=(√3-1)/2 and iterate:

$$\begin{aligned} r_{k+1} &= \frac{3}{1 + 2(1 - s_k^3)^{1/3}}; \\ s_{k+1} &= \frac{1}{2}\left(r_{k+1} - 1\right) \\ a_{k+1} &= r_{k+1}^2\, a_k - 3^k (r_{k+1}^2 - 1); \end{aligned} \tag{7}$$

then the series of $1/a_k$ converges **cubically** to π. Thus provides the first three terms 5,21 and 57 digits:

$\pi_{k=1} \approx$ 3.14159 ...
$\pi_{k=2} \approx$ 3.1415926535 8979323846 2 ...
$\pi_{k=3} \approx$ 3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974 ...

C. Borwein's Quartically Convergent Algorithm

It was a great achievement of the Borwein's to push further and establish a quartically convergent algorithm. Set a_0 = 6-4√2 and y_0 = √2-1 and iterate [9]:

$$\begin{aligned} y_{k+1} &= \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}; \\ a_{k+1} &= a_k (1 + y_{k+1})^4 - 2^{2k+3}\, y_{k+1} (1 + y_{k+1} + y_{k+1}^2) \end{aligned} \tag{8}$$

and $1/a_k$ converges to π **quartically**. The first terms provides:

$\pi_{k=1} \approx$ 3.1415926 ...
$\pi_{k=2} \approx$ 3.1415926535 8979323846 2643383279 5028841971
$\pi_{k=3} \approx$ 3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164 0628620899 8628034825 3421170679 8214808651 3282306647 0938446095 ...

The first 2 terms already gives 40 decimals, the first three terms gives 170 decimals! Only if Ludolph van Ceulen had known this method! In fact, this series converge so fast that taking only the first 15 terms provides 2 billions digits of π! Bailey applied this algorithm in his record-breaking calculation in 1987 [2].

It has been a long way to come from Machin's formulae to the quartically convergent Borwein-Borwein algorithm. But to increase the efficiency of the calculations one need better (i.e. faster) numerical methods as well. It seems simple to program algorithm (9) but how can we multiply two large numbers efficiently?

VI. The Key Ingredient: Fast Fourier Transform

Algorithm (6), (7) or (8) does not seem difficult to be implemented on a digital computer. We must note, however, that all operations must be correct up to the required number of digits plus *m* (30 as the guard digits for millions of decimal digit calculations) [15]. So we can see, that an efficient multiplication method is a key element in all algorithms. We learned how to multiply two numbers already in the elementary school and we know that to multiply two *n*-digit numbers we need $nxn=n^2$ operations. With scientific terms: the bit complexity of multiplication is $O(n^2)$. It seems all pretty simple but when we have to multiply two numbers with millions of digits, then it is another story. So we may ask: isn't there any better way to multiply large (i.e. long) numbers? The unexpected answer came in 1971. Then, Schönhage and Strassen showed that it is possible to multiply two *n*-digit integers with bit complexity O(*n*log*n*/loglog*n*) [9,20].

Their method based on the application of the Fourier transform. Fourier transform? Fourier transform has long been known to mathematicians and engineers alike [13,14]. Its application ranges from heat equations to sound engineering to speech recognition. But in number theory? How could that be?

To see how the Fourier transform may be used for fast multiplication, let $x:=(x_0, x_1, x_2, ..., x_{n-1})$ and $y:=(y_0, y_1, y_2, ..., y_{n-1})$ be the representations of two high-precision numbers in radix b. The radix b is usually selected to be some power of 2 or 10 whose square is less then the largest integer exactly representable as an ordinary floating-point number on the computer being used. Then, except for releasing each "carry", the product $z:=(z_0, z_1, z_2, ..., z_{2n-1})$ of x and y may be written as:

$$
\begin{aligned}
z_0 &= x_0 y_0 \\
z_1 &= x_0 y_1 + x_1 y_0 \\
z_2 &= x_0 y_2 + x_1 y_1 + x_2 y_0 \\
&\vdots \\
z_{n-1} &= x_0 y_{n-1} + x_1 y_{n-2} + \cdots + x_{n-1} y_0 \\
&\vdots \\
z_{2n-3} &= x_{n-1} y_{n-2} + x_{n-2} y_{n-1} \\
z_{2n-2} &= x_{n-1} y_{n-1} \\
z_{2n-1} &= 0
\end{aligned}
\tag{9}
$$

We apply now the well known discrete Fourier Transform [14]. First, extend x and y to length $N=2n$ by appending zeros at the end of each. Then a key observation may be made: the product sequence $z:=(z_0, z_1, z_2, ..., z_{2n-1})$ of x and y is precisely the discrete convolution $C(x,y)$:

$$z_k = C_k(x,y) = \sum_{j=1}^{N-1} x_j \, y_{k-j} \tag{10}$$

where the subscript k-j is to be interpreted as k-j+N if k-j is negative. Now a well-known result from Fourier analysis may be applied. Let $F(x)$ denote the **discrete Fourier transform** of the sequence x, and let $F^{-1}(x)$ denote the inverse discrete Fourier transform of x [2,9,14,15]:

$$
\begin{aligned}
F_k(x) &= \sum_{j=0}^{N-1} x_j \exp(-i 2\pi\, jk / N) \\
F_k^{-1}(x) &= \frac{1}{N} \sum_{j=0}^{N-1} x_j \exp(i 2\pi\, jk / N)
\end{aligned}
\qquad k=0,1,...,N-1 \tag{11}
$$

where $i = \sqrt{(-1)}$. Then the convolution theorem states, that the Fourier transform of a convolution product is the ordinary product of the Fourier transforms [14]:

$$F\big[C(x,y)\big] = F(x)F(y) \tag{12}$$

or, expressed in an other way:

$$C(x,y) = F^{-1}\big[F(x)F(y)\big] \tag{13}$$

Thus the entire multiplication pyramid z can be obtained by performing two forward discrete Fourier transforms, one vector complex multiplication and one inverse transform, each of length $N=2n$.

One must realize that it is the discrete Fast Fourier transform (FFT) which makes this scheme work. In particular, if $N=2^m$, then the discrete FFT can be evaluated in only $5m2^m$ arithmetic operations.

There are of course several "tricks" in implementing FFT based multiplication. One usual trick is to utilize the fact that the input data vectors x and y and the result vector z are purely real.

One other variation relies on the fact that the FFT can be applied in any number field in which there exists a primitive N^{th} root of unity. This requirements holds for the field of integers modulo p, where p is a prime of the form $p=kN+1$ [9]. The advantage to use a prime modulus field (instead of the field of complex numbers) is that there are no round-off errors (since all computations are exact). Some further details concerning the implementations can be found in [2,9,15].

Calculation of Reciprocals and Square Roots

Do we need something else besides a very fast multiplication method? Not really. For we can determine the inverse of a number 1/y or its square root √y or 1/√y by the Newton's quadratically convergent method:

$$
\begin{aligned}
1/y &\quad\rightarrow\quad x_{k+1} = 2x_k - x_k^2 y \\
\sqrt{y} &\quad\rightarrow\quad x_{k+1} = (x_k + y / x_k)/2 \\
1/\sqrt{y} &\quad\rightarrow\quad x_{k+1} = x_k(3 - x_k^2 y)/2
\end{aligned}
\tag{14}
$$

So we can see that even the inverse square root can be expressed by repeated multiplication.

By applying the FFT method to multiply large numbers, it is possible to accelerate the computations dramatically. In fact, in almost all record-breaking high-performance multi-precision computer programs recently, some variation of the FFT method has been applied [2,9,15]. The current record of number of digits is more than 200 billion. So we can conclude, that the successful combination of number theory and computer science (via the FFT) made these records possible.

VII. Conclusions

Ludolph van Ceulen needed almost his whole life to calculate the first 35 digits of π. With new methods developed in the last decades it became possible to calculate billions of decimals of π. But to program the fastest algorithms of the Borweins' one must have efficient multiplication methods as well. It is the **discrete Fast Fourier transform**, which made fast multiplication of very long numbers possible. Almost all the current records apply one or another version of FFT multiplication. In recent years, the computation of the expansion of π has assumed the role as a standard test of computer integrity. Finally, it is interesting to note that the π research enjoys considerable wide attention: popular books appeared on the subject [5,7] there is a video called π[5], and even a new cologne named π is marketed by Givenchy. Not mentioning the hundreds of home-pages over π.

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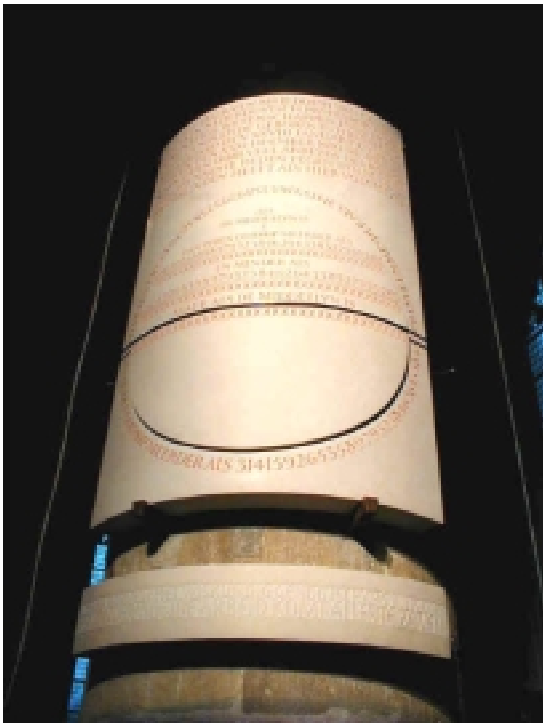

Figure 1. The memorial stone of Ludolph van Ceulen in the St.Peter's Church at Leiden.