SOME REMARKS ON PADÉ-APPROXIMATIONS

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ABSTRACT

Padé approximations are widely used to approximate a dead-time in continuous control systems. It provides a finite-dimensional rational approximation of a dead-time. However, the standard Padé approximation (recommended in many textbooks) with equal numerator- and denominator degree, exhibits a jump at time t=0. This is highly undesirable in simulating dead-times. To avoid this phenomena we shall reconsider the Padé approximation with different numerator degrees.

Keywords: Padé approximation, rational functions.

1. INTRODUCTION

There are many physical processes with dead-time. For example virtually all chemical processes involves some time delay and all transport processes also exhibit dead-time [3,12]. Control systems with dead-time are difficult to analyze and simulate. One of the reasons is that a closed-loop control system with dead-time is in fact an infinite dimensional system, i.e. the closed-loop has infinite number of poles [3,6]. It is also difficult to determine all the system poles. One of the most widely recommended remedies to overcome this difficulty is to approximate the dead-time by some method and analyze the resulting system [6,8]. The step-response of a dead-time is a delayed step-signal $h(t) = 1(t\text{-}T)$ where T denotes the dead-time. The Laplace transform of $h(t) = 1(t\text{-}T)$ is:

$$H(s) = e^{-sT} \tag{1}$$

Among the many methods Padé approximations are the most frequently used methods to approximate a dead-time by a rational function. Almost every textbook about classical control system theory provides the basic relation, but usually only for an approximation with equal numerator and denominator degree (try for example the subroutine *pade.m* in MATLAB). The most widely recommended Padé approximation is of 2nd order with equal numerator- and denominator degree [6,8]:

$$e^{-sT} \approx R_{2,2}(s) = \frac{12 - 6(sT) + (sT)^2}{12 + 6(sT) + (sT)^2} \tag{2}$$

It is a bit puzzling to realize, that the step-response of this approximation (say, transfer function) exhibits a jump at $t=0$ due to the equal numerator and denominator degree. That is, instead of delaying the input signal there appear an output signal at $t=0$. This seems to be quite bad. On the other hand, this approximation has nice properties in the frequency domain. So one may ask: is it possible to modify the approximation avoiding the jump at $t=0$ but keeping the frequency domain properties?

2. APPROXIMATIONS WITH CONSTANT NUMERATOR

There are many ways of approximating e^{-sT} by a rational function. Consider for example its Maclaurin series [1,11]. By taking only the first n-terms we can define the following approximation:

$$e^{-sT} \approx R_{0,n}(s) = \frac{1}{\sum_{k=0}^{n} \frac{(sT)^k}{k!}} = \frac{1}{1+(sT)+(sT)^2/2!+(sT)^3/3!+\ldots+(sT)^n/n!} \tag{3}$$

This formula is recommended in Kuo [pp.183] and in Palm [pp.509]. Although the expression seems natural to apply, an unexpected difficulty arises as one increases the degree of approximation. The rational function $R_{0,n}(s)$ exhibits right-half-plane poles as n increases, namely as $n>4$! Although the approximation's accuracy increases as n increases in the s-domain, but as a transfer function $R_{0,n}(s)$ becomes unstable. This is a rarely known phenomenon and makes the seemingly simple approximation useless for $n>4$. Consider for example the first 5 terms (5th order approximation):

$$R_{0,5}(s) = \frac{120}{120+120s+60s^2+20s^3+5s^4+s^5} \tag{4}$$

The poles of this rational function are: $p_{1,2}$ = 0,23981±i 3,12834; $p_{3,4}$ = -1,44180±i 2,43452 and p_5 = -2,1806. Since there are two conjugate complex poles on the right-half plane, this "*approximation*" is unstable!

Another method recommended in some textbooks [8, pp.521; 13, pp.216] is based on the infinite product formula of the exponential function [11]. Taking only the first n terms in the product leads to the following approximation:

$$e^{-sT} \approx R_{0,n}(s) = \frac{P_n}{Q_n(s)} = \frac{1}{(1+sT/n)^n} = \frac{n^n}{(n+sT)^n} \tag{5}$$

This approximation has multiple poles (with multiplicity n) at $p_n = -n/T$. In fact, equation (5) gives a rather poor approximation for low value of n [13,14]. Without going into more details, we can conclude, these simple approximations (without numerator dynamics) give poor approximations of a dead-time. One may expect to improve the accuracy by choosing an appropriate numerator.

3. PADÉ APPROXIMATION OF e^{-x}

The approximations given in the previous paragraph are rational functions but with zero numerator dynamics (numerator is constant). We shall now consider another kind of approximations, namely, approximations derived by expanding a function as a ration of two power series (thus with numerator and denominator dynamics). These approximations are usually called **Padé approximants**. They are usually superior to Taylor expansions when functions contain poles, because the use of rational functions allows them to be well-represented. Let us now consider the general equations of the Padé approximation. Let A(x) denote a function having a Maclaurin series expansion [1,2]:

$$A(x) = \sum_{k=0}^{\infty} a_k x^k \tag{6}$$

which converges in some neighborhood of the origin[1]. The **Padé approximation of order (m,n)** to A(x) is defined to be a rational function $R_{m,n}(x)$ expressed in a fractional form:

[1] If A(x) is a transcendental function then the a_k coefficients are given by the Taylor series about x_0:

$$a_k = \frac{1}{k!} A^{(k)}(x_0)$$

$$R_{m,n}(x) = \frac{P_m(x)}{Q_n(x)} \tag{7}$$

where $P_m(x)$ and $Q_n(x)$ are two polynomials[2]:

$$\begin{aligned} P_m(x) &= p_0 + p_1 x + p_2 x^2 + \ldots + p_m x^m \\ Q_n(x) &= q_0 + q_1 x + q_2 x^2 + \ldots + q_n x^n \end{aligned} \tag{8}$$

The unknown coefficients p_0 ... p_m and q_0 ... q_n of $R_{m,n}(x)$ can be determined from the condition that the first $(m+n+1)$ terms vanish in the Maclaurin series[3]:

$$A(x) - \frac{P_m(x)}{Q_n(x)} = 0; \qquad \text{or} \qquad A(x)Q_n(x) - P_m(x) = 0; \tag{9}$$

Substituting the two polynomials into this expression and equating the coefficients leads to a system of $m+n+1$ linear homogeneous equation [2] which can be expressed in matrix form (assuming $q_0=1$):

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & -a_0 & 0 & \cdots & 0 \\ & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 1 & -a_m & -a_{m-1} & \cdots & -a_{m-n+1} \\ 0 & 0 & \cdots & 0 & -a_{m+1} & -a_m & \cdots & -a_{m-n+2} \\ 0 & 0 & \cdots & 0 & -a_{m+2} & -a_{m+1} & \cdots & -a_{m-n+3} \\ & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & -a_{m+n-1} & -a_{m+n-2} & \cdots & -a_m \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_m \\ q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \\ a_{m+1} \\ a_{m+2} \\ \vdots \\ a_{m+n} \end{bmatrix} \tag{10}$$

Now we would like to apply this to the exponential function with the Maclaurin series:

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - + \ldots \tag{11}$$

We conclude that the coefficient $a_k = (-1)^k/k!$. In this case the polynomials $P_m(x)$ and $Q_n(x)$ of the Padé approximation $R_{m,n}(x)$ can be expressed by the following recursive relations [4]:

$$P_m(x) = \sum_{k=0}^{m} \frac{(m+n-k)!\, m!}{(m+n)!\, k!\, (m-k)!} (-x)^k \tag{12}$$

and

$$Q_n(x) = \sum_{k=0}^{n} \frac{(m+n-k)!\, n!}{(m+n)!\, k!\, (n-k)!} (-x)^k \tag{13}$$

Note, that the numerator coefficients have always alternating sign and $P_n(x) = Q_n(-x)$ for $m=n$. As a consequence, the zeros and poles of $R_{n,n}(x)$ are symmetrical to the imaginary axes!

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2 Note that there is no constraint on the degree's of the polynomials. That is to say, the numerator may have higher degree than that of the denominator.

3 $Q_n(x)$ can be multiplied by an arbitrary constant which will rescale the other coefficients, so an additional constraint can be applied. This is usually $Q_n(0)=1$.

4. PADÉ APPROXIMATIONS OF e^{-sT}

To determine the transfer functions of the Padé approximations with different numerator degree, one simply substitutes $x=sT$ into (12) and (13). For example, the 4th order approximation with 3rd order numerator can be expressed as [14]:

$$R_{3,4}(s) = \frac{840 - 360sT + 60(sT)^2 - (sT)^3}{840 + 480sT + 120(sT)^2 + 16(sT)^3 + (sT)^4} \tag{14}$$

Note, that the nth order Padé approximation has different denominator polynomials depending on the numerator's degree. It is interesting to determine the pole-zero configuration of the approximation. Figure 1 shows the pole-zero configuration of the 4th order Padé approximation with different numerator degree. Note, that all poles are on the left-half-plane and all zeros are on the right-half-plane. Notice, that the poles and zeros of the Padé approximation $R_{4,4}(s)$ are symmetrical to the imaginary axis and are close to a circle.

Due to the symmetrical pole-zero configuration, the phase of $R_{n,n}(s)$ goes to $-2n\pi/2$ and its amplitude remains constant at all frequencies. On the other hand, the step-response of $R_{n,n}(s)$ exhibits a jump at $t=0$ which is not very desirable. To avoid the jump in the step-response we recommend to use $R_{n-1,n}(s)$ instead of $R_{n,n}(s)$. In Table 1 we give the transfer functions of both up to 5th order. However, there is a price to be paid: due to the lower numerator's degree, the phase of $R_{n-1,n}(s)$ goes to $-(m+n)\pi/2$ only and its amplitude goes to zero at very high frequencies. But all in all, $R_{n-1,n}(s)$ seems to be a good compromise.

Figure 2 shows the step-responses of $R_{n-1,n}(s)$ and $R_{n,n}(s)$. One can easily see that $R_{n-1,n}(s)$ gives a better approximation in the time-domain [14], specially in the interval [0,T]. As a measure of the error we give in Table 2 the mean-square-errors defined in the time-domain by:

$$I_{m,n} = \int_0^\infty \{1(t-T) - y_{m,n}(t)\}^2 \, dt \tag{15}$$

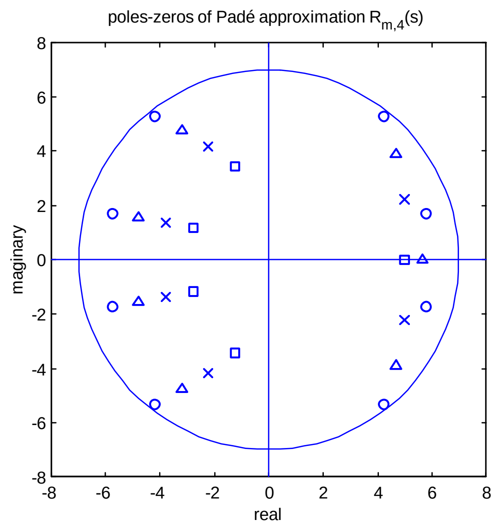

Figure 1. Pole-zero configuration of the 4th order Padé approximation with different numerator degree. All poles are on the left-half-plane and all zeros are on the right-half-plane. (? = $R_{1,4}(s)$; x = $R_{2,4}(s)$; ? = $R_{3,4}(s)$; o = $R_{4,4}(s)$)

n	$\mathbf{R_{n-1,n}(s)}$	$\mathbf{R_{n,n}(s)}$
1	$\dfrac{1}{1+sT}$	$\dfrac{2-sT}{2+sT}$
2	$\dfrac{6-2sT}{6+4sT+(sT)^2}$	$\dfrac{12-6sT+(sT)^2}{12+6sT+(sT)^2}$
3	$\dfrac{60-24sT+3(sT)^2}{60+36sT+9(sT)^2+(sT)^3}$	$\dfrac{120-60sT+12(sT)^2-(sT)^3}{120+60sT+12(sT)^2+(sT)^3}$
4	$\dfrac{840-360sT+60(sT)^2-(sT)^3}{840+480sT+120(sT)^2+16(sT)^3+(sT)^4}$	$\dfrac{1680-840sT+180(sT)^2-20(sT)^3+(sT)^4}{1680+840sT+180(sT)^2+20(sT)^3+(sT)^4}$
5	$\dfrac{15120-6720sT+1260(sT)^2-120(sT)^3+(sT)^4}{15120+8400sT+2100(sT)^2+300(sT)^3+25(sT)^4+(sT)^5}$	$\dfrac{30240-15120sT+3360(sT)^2-420(sT)^3+30(sT)^4-(sT)^5}{30240+15120sT+3360(sT)^2+420(sT)^3+30(sT)^4+(sT)^5}$

Table 1. Transfer functions $R_{n-1,n}(s)$ and $R_{n,n}(s)$ of the Padé approximations.

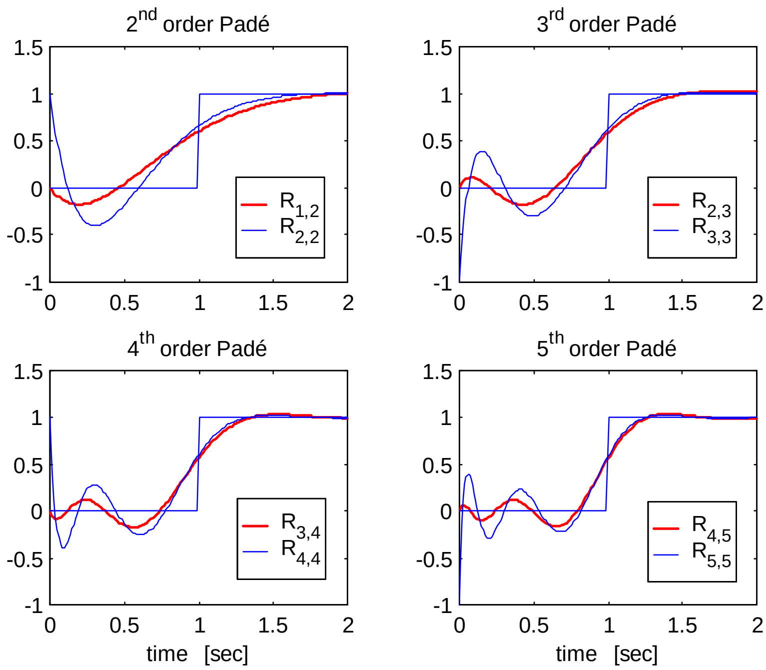

Figure 2. Step-responses of Padé approximationsf $R_{n-1,n}(s)$ and $R_{n,n}(s)$.

n	$\mathbf{I_{n-1,n}}$	$\mathbf{I_{n,n}}$
1	0,235759	0,27067
2	0,106261	0,15424
3	0,069044	0,10701
4	0,051133	0,08162
5	0,040512	0,06583

Table 2. Mean-square-errors of step-responses of $R_{n,n}(s)$ and $R_{n-1,n}(s)$.

CONCLUSIONS

We have considered the general Padé approximation of a dead-time with transfer function e^{-sT}. The polynomials of the rational approximations are given in analytic form. The "standard" Padé approximation $R_{n,n}(s)$ exhibits a jump at $t=0$ in its step-response. To avoid this phenomenon we recommend the Padé approximation $R_{n-1,n}(s)$ where the numerator's degree is one less than that of the denominator. This gives a better approximation of the step-response. Applied in closed-loop, they differ due to their different frequency characteristics. There seems to be a clear compromise between the use of $R_{n,n}(s)$ or $R_{n-1,n}(s)$ depending on the frequency range. One has to realize that by approximating a dead-time in control systems, we introduce modeling errors, which consequently limits the achievable bandwidth. Some consequences are discussed in [7, pp.115]. Padé approximations can also be used for model reduction [10].

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