Linear-potential values and the 'Shapley family'

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Abstract

We generalize the potentials of Hart & Mas-Colell [1989] by introducing a class of linear potentials for TU-games based on an idea of 'taxing and redistributing'. To each potential an efficient and additive value is associated, which attributes to each player his linearly modified contribution to the potential of the grand coalition of a so-called taxed game, plus an equal share in the 'tax revenues'.

The class of linear-potential values includes egalitarian and discounted Shapley values, but also weighted Shapley values. Using our potential we show that the class of consistent values in the sense of Hart & Mas-Colell [1989] can be extended. Furthermore, the 'Shapley family' is enlarged by the classes of semi-egalitarian discounted weighted Shapley values and equal-coalitional-improvement Shapley values.

We investigate connections between restrictions on linear-potential values and axioms, some of which lose independence, e.g., variants of standardness imply symmetry. We characterize several classes within the 'Shapley family' by single axioms, such as symmetry and parameter dependent forms of egalitarianism, consistency and standardness, as well as individual members by forms of consistency and standardness.
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1 Introduction

We introduce a generalization of the potential of Hart & Mas-Colell [1989]. To each transferable utility game[1] (N, v) the (a, b, α)-potential $P^{a,b,\alpha}$ attributes a real number such that

$$0 = P^{a,b,\alpha}(\emptyset, v),$$

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[1]A transferable utility game is an ordered pair (N, v) where N is the set of players, and v is a map attributing to each coalition $S \subseteq N$ a real number such that $v(\emptyset) = 0$.

$$v^\alpha(N) = \sum_{i\in N} a_i P^{a,b,\alpha}(N,v) - b_i P^{a,b,\alpha}(N\backslash\{i\},v).$$

Here, α is a real number, $a, b \in \mathbb{R}^{|Z|}$ are vectors[2] of 'weights,' each component connected to a unique player in the set of possible players $Z \supseteq N$. The right-hand side of the second equation adds up to $v^\alpha(N) \equiv (1-\alpha)v(N)$, i.e., the worth of (grand) coalition N for the α-taxed game (N, v^α). Such a game arises by raising a proportional tax of α on every coalition in (N, v). To guarantee that the potential is well-defined and unique we require that $\sum_{i\in S} a_i \neq 0$ for each nonempty $S \subseteq Z$.

We connect the linear-potential value $\psi^{a,b,\alpha}$ to the (a, b, α)-potential as follows. For all transferable utility games (N, v), and all $i \in N$:

$$\psi_i^{a,b,\alpha}(N,v) = a_i P^{a,b,\alpha}(N,v) - b_i P^{a,b,\alpha}(N\backslash\{i\},v) + \alpha\tfrac{v(N)}{|N|}.$$

So, each player gets his (linearly modified) marginal contribution to the potential of the α-taxed game and in addition to that an equal share in the total tax revenues.

Joosten *et al.* [1994], Joosten [1996] showed that for fixed α, the axioms of efficiency, symmetry, additivity and α-egalitarianism uniquely determine the α-egalitarian Shapley value[3], given by

$$Sh^\alpha(N,v) = Sh(N,v^\alpha) + \eta(N,v^{1-\alpha}). \tag{1}$$

The last axiom requires that a value attributes to each null-player in a game, a fraction α of the per-capita income, cf., Joosten [1996]. Thus, α can be interpreted as to reflect a social norm, i.e., the level of egalitarianism or solidarity in the society. Social norms on equality and solidarity exist in real life, and one may wish to devise solutions incorporating them.[4]

Hart & Mas-Colell [1988,1989] characterize the Shapley value by 'consistency' and 'standardness'. A reduced game is a game on a subset of the player set, that remains after paying off all other players in the original game in an appropriate way (which may vary with the solution in question). Then, consistency requires that when a solution is applied to the reduced game each player in the reduced game receives the same utility as in the original one. Standardness implies that when the solution is applied to an arbitrary two-person game, each player receives half of the surplus of that game on top of the amount that he receives in the one-person coalition. Hart & Mas-Colell [1989] extend this result to the entire family of weighted Shapley values: Each weighted Shapley value is uniquely determined by consistency and the amounts attributed in two-person games.

[2] Their interpretation is similar to the vector w for the weighted Shapley value. In fact, $P^{w,w,0}$ is the potential of the w-weighted Shapley value of Hart & Mas-Colell [1989].

[3] The class of linear-potential values contains all α-egalitarian Shapley values.

[4] E.g., Dutta & Ray [1989, 1991], Dutta [1990] and more explicitly Nowak & Radzik [1994], Ju *et al.* [2004], Van den Brink & Funaki [2009], Malawski [2013].

Another axiomatization of Sh^α in Joosten *et al.* [1994] follows Hart & Mas-Colell [1989] using the axioms of α-consistency and α-standardness, modifications of the consistency and standardness. Van den Brink *et al.* [2007,2013] show that consistency in the spirit of Sobolev [1973] may replace α-consistency to characterize each egalitarian Shapley value.

We investigate relations between restrictions on (a, b, α) and axioms used to characterize Sh^α for the class of linear-potential values, but also *connections among axioms*. For instance, symmetry for $\alpha \neq 1$ restricts the class of linear-potential values to those with a and b being vectors of constants, implying α-consistency in turn. The axiom of α-egalitarianism for $\alpha \neq 1$ restricts the parameters to $a = b$. We show that $\alpha \in \{0, 1\}$ yields a class of HM-consistent linear potential values putting no restrictions on a or b at all. Moreover, for every linear-potential value satisfying $\alpha \neq 0, 1$, α-standardness implies α-consistency. For this introduction this list will suffice, but several other implications regarding links with and between other axioms and parameters implied are to be presented in the remainder.

We distinguish certain subclasses of linear-potential values characterized by single axioms: α-egalitarian weighted Shapley values are characterized by α-egalitarianism, semi-egalitarian discounted Shapley values by symmetry, and equal-coalitional-improvement Shapley values by α-consistency. We extend results of Joosten *et al.* [1994] and Van den Brink *et al.* [2007] by deriving a characterization of special semi-egalitarian discounted Shapley values by λ-standardness and either α-consistency or Sobolov-consistency.

In Section 2 we introduce the model and present characterizations of the (weighted) Shapley value(s) for the sake of comparison and easy reference. In Section 3, we define and examine the families of (a, b, α)-potentials and values $\psi^{a,b,\alpha}$. Section 4 treats α-egalitarian Shapley values and characterizations. Section 5 sheds light on the connections between parameters (a, b, α) and axioms on one hand and the connections among axioms for linear-potential values. Section 6 presents other axiomatizations of Shapley family values. Section 7 concludes. The appendix contains all proofs.

2 Preliminaries

Let $\mathbb{R}$ denote the set of real numbers. Let Z be a nonempty set of natural numbers, representing the **set of potential players**. (Strict) inclusions are denoted by $(\subset)$ $\subseteq$. A **coalition** is a finite subset of Z. A **transferable utility game** is a pair (N, v) where $N \subseteq Z$ is a coalition and $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$. The function v is the **characteristic function**. We denote the set of all games with player set N by G^N, the set of all games is G.

Let $(N, v) \in G$, then:

- for $M \subseteq N$, the characteristic function of the game (M, v) is the map v restricted to 2^M;

- the **marginal contribution** of $i \in S \subseteq N$ is given by $\Delta_i^v(S) = v(S) - v(S \backslash \{i\})$;

- player $i \in N$ is a **null-player** (**dummy-player**) in (N, v) if $\Delta_i^v(S) = 0$ ($\Delta_i^v(S) = v(\{i\})$) for all $S \subseteq N, S \ni i$; the set $\mathcal{N}(N, v)$ ($\mathcal{D}(N, v)$) is the **set of null-players** (**dummy-players**) in (N, v);

- player $i \in N$ is a **nullifying-player** (**δ-reducing**) in (N, v) if $v(S) = 0$ ($v(S) = \delta v(S \backslash \{i\})$) for all $S \subseteq N, S \ni i$;

- players $i, j \in N$ are **symmetric** in (N, v), if $\Delta_i^v(S) = \Delta_j^v(S)$ for all $S \subseteq N, S \supseteq \{i, j\}$.

Let $v, w : 2^N \to \mathbb{R}$, and $\alpha, \lambda \in \mathbb{R}$, then:

- $(\lambda(v + w))(S) = \lambda v(S) + \lambda w(S)$, for all $S \subseteq N$,
- $(v^\alpha)(S) = (1 - \alpha)v(S)$ for all $S \subseteq N$.

With the first operation G^N is a linear space; the game (N, v^α) is the **$\boldsymbol{\alpha}$-taxed game** of (N, v), i.e., the game remaining after a proportional tax of α is levied on the worth of each coalition in (N, v). So, v^1 is equivalent to the **zero-game** v_0, i.e., $\mathcal{N}(N, v^1) = N$. For nonempty $T \subseteq N \subseteq Z$, the **T-unanimity game** (N, u_T) is the game with $u_T(S) = 1$ if $T \subseteq S \subseteq N$, and $u_T(S) = 0$ otherwise.

Shapley [1953] demonstrated that the collection of all T-unanimity games ($\emptyset \neq T \subseteq N$) constitutes a basis of the linear space G^N. Hence, for every game (N, v) with nonempty player-set N, there exists a unique set of numbers $\{c_T \in \mathbb{R} \mid \emptyset \neq T \subseteq N\}$, satisfying $v = \sum_{\emptyset \neq T \subseteq N} c_T u_T$ where for given $T \subseteq N$, $c_T = \sum_{\emptyset \neq S \subseteq T} (-1)^{|T| - |S|} v(S)$.

A **value** is a map ψ assigning to each game (N, v), a vector in $\mathbb{R}^N$. The interpretation is that if the value is applied to a game (N, v), the i-th component of the vector represents the utility attributed to player $i \in N$ in the game (N, v). Let ψ be a value, then:

- ψ is **efficient** if $\sum_{i \in N} \psi_i(N, v) = v(N)$ for every $(N, v) \in G$;

- ψ is **symmetric** if $\psi_i(N, v) = \psi_j(N, v)$ whenever $i, j \in N$ are symmetric players in $(N, v) \in G$;

- ψ is **linear** if $\psi(N, \lambda v + \mu w) = \lambda \psi(N, v) + \mu \psi(N, w)$ for all $\lambda, \mu \in \mathbb{R}$, and all $(N, v), (N, w) \in G$; ψ is **additive** if $\psi(N, v + w) = \psi(N, v) + \psi(N, w)$ for all $(N, v), (N, w) \in G$;

- ψ satisfies the **null-player property** if $\psi_i(N, v) = 0$ whenever $i \in \mathcal{N}(N, v)$; ψ satisfies the **dummy-player property** if $\psi_i(N, v) = v(\{i\})$ whenever $i \in \mathcal{D}(N, v)$;

- ψ satisfies the **nullifying-player property** if $\psi_i(N,v) = 0$ whenever i is a nullifying player (Van den Brink [2007]); ψ satisfies the **δ-reducing-player property** if $\psi_i(N,v) = 0$ whenever i is a δ-reducing player (Van den Brink & Funaki [2010]);
- ψ is **trivial** if $\psi_i(N,v_0) = 0$ for all $N \subseteq Z$, $i \in N$ (Chun[1989]);
- ψ is **strongly monotonic** if for any pair of games $(N,v),(N,w) \in G$, and any $i \in N$, it holds that if $\Delta_i^v(S) \geq \Delta_i^w(S)$ for all $S \subseteq N$, then $\psi_i(N,v) \geq \psi_i(N,w)$ (Young [1985]);
- ψ satisfies **marginality** if for any pair of games (N,v), $(N,w) \in G$, and any $i \in N$, it holds that if $\Delta_i^v(S) = \Delta_i^w(S)$ for all $S \subseteq N$, then $\psi_i(N,v) = \psi_i(N,w)$ (Young [1985]);
- ψ is **λ-standard** if $\psi_i(\{i,j\},v) = \frac{v(\{i,j\})-(1-\lambda)v(\{j\})+(1-\lambda)v(\{i\})}{2}$ for all 2-person games $(\{i,j\},v) \in G$ (Joosten *et al.* [1994], Yanovskaya & Driessen [2001]).

An example of a value is the **egalitarian value** denoted by η in the sequel. For arbitrary $(N,v) \in G$, $\eta_i(N,v) = \frac{v(N)}{|N|}$ for all $i \in N$. So, η distributes the worth of the grand coalition equally among the players in all games. Hence, η is efficient, symmetric, additive, linear and trivial, 1-standard and satisfies the nullifying property.

Another one is the **Shapley value** (cf., Shapley [1953], Roth [1988]), Sh in the sequel. For every $(N,v) \in G$ and every $i \in N$, Sh is given by

$$Sh_i(N,v) = \textstyle\sum_{S \subseteq N: i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \Delta_i^v(S).$$

The Shapley value assigns to each player his average marginal contribution in any game. The Shapley value is efficient, symmetric, additive, linear, trivial, and (0-)standard, moreover it satisfies strong monotonicity and marginality.

For linear values analysis may be simplified considerably, as $\psi(N,v) = \sum_{\emptyset \neq T \subseteq N} \psi(N, c_T u_T)$ for all linear values ψ and $(N,v) \in G$.

An equivalent expression of the Shapley value is the following. For every $(N,v) \in G$, and every $i \in N$,

$$Sh_i(N,v) = \textstyle\sum_{T \subseteq N: T \ni i} \frac{c_T}{|T|} \text{ for all } i \in N,$$

where the numbers c_T are given above. This may be seen immediately by noting that in every game $c_T u_T$ ($\emptyset \neq T \subseteq N$) all players in T receive $\frac{c_T}{|T|}$, all players outside T get zero. The Shapley value satisfies additivity, hence the remark above justifies this alternative expression.

Let $w \in \mathbb{R}^Z$ be a vector of exogenously given weights satisfying $w_i > 0$ for all $i \in Z$. Then, the **weighted Shapley value** (Shapley [1953]), denoted by Sh^w in the sequel, is for every $(N,v) \in G$ given by

$$Sh_i^w(N,v) = \textstyle\sum_{T \subseteq N: T \ni i} c_T \left(\frac{w_i}{\sum_{j \in T} w_j} \right) \text{ for all } i \in N.$$

For $w \in \mathbb{R}^Z$ satisfying $w_i = w_j > 0$ for all $i, j \in Z$, we have $Sh^w = Sh$. Each weighted Shapley value satisfies the axioms of efficiency, additivity and the null-player property. It is characterized (up to a scalar multiple of w) by these three axioms and the amounts which in all T-unanimity games, $\emptyset \neq T \subseteq N$, are attributed to the members of T.

Symmetric players *are not* treated equally if $w_i \neq w_j$ for some $i, j \in Z$. The following interpretation for asymmetric weights of the weighted Shapley values is similar to the one given by Kalai & Samet [1987]. Suppose players are to join in a project, and they can generate positive profits if they all cooperate, and can generate zero-profits otherwise. Then, the Shapley value gives each player an equal share of the profits. This seems reasonable when players have to provide similar inputs. In case, however, that there is an asymmetry in the efforts put forward by the players necessary to complete the project, a symmetric division of the profits may not be reasonable. Dividing the profits proportional to the efforts put forward by the players, seems a good alternative. For this purpose, the weighted Shapley value may be used with the weights equal to the effort-levels.

2.1 Hart & Mas-Colell potentials and consistency

Hart & Mas-Colell [1989] introduce the following family of 'potentials'.

Definition 1 *(Hart & Mas-Colell [1989]) Let $w \in \mathbb{R}^Z$ satisfy $w_i > 0$ for all $i \in Z$. Then the* w-potential *is the map $P^w : G \to \mathbb{R}$ satisfying:*

i. $P^w(\emptyset, v) = 0$,

ii. $\sum_{i \in N} w_i [P^w(N, v) - P^w(N \backslash \{i\}, v)] = v(N)$ *for every* $(N, v) \in G$.

So, a w-potential is a map attributing to each game a unique real number. The condition of strict positivity on the vector of weights w guarantees that the w-potential is well-defined and unique. The weighted Shapley value Sh^w (Shapley [1953], Kalai & Samet [1987]) is connected to this w-potential as follows (cf., Hart & Mas-Colell [1989]):

$$Sh_i^w(N, v) = w_i \left[P^w(N, v) - P^w(N \backslash \{i\}, v)\right] \text{ for all } (N, v) \in G,\ i \in N.$$

The w-potential provides an algorithm to compute the corresponding weighted Shapley value Sh^w recursively by using:

$$P^w(S, v) = \frac{v(S) + \sum_{k \in S} w_k P^w(S \backslash \{k\}, v)}{\sum_{k \in S} w_k} \quad \text{for all nonempty } S \subseteq N.$$

For $w = (1, ..., 1)$, we obtain the potential P and the Shapley value Sh.

Besides providing rather efficient algorithms to compute weighted Shapley values, w-potentials are useful in proving so called HM-consistency of a value as Hart & Mas-Colell [1989] have shown. Consistency is a reduced-game property, which may be described as follows. Let ψ be a value. For any group of players in a game, one defines a reduced game among them by giving the rest of the players the payoffs according to ψ. Then ψ is called consistent if, when applied to any reduced game, it yields the same payoffs as in the original game. The following formalizes such a property.

Definition 2 *(Hart & Mas-Colell [1989]) Let ψ be a value, $(N, v) \in G$, and $\emptyset \neq U \subset N$. Then the (U, ψ)-reduced game of v is the game $v^{U,\psi}$ satisfying:*

i. $v^{U,\psi}(S) = v(S \cup U) - \sum_{k \in U} \psi_k(S \cup U, v)$ *for all* $\emptyset \neq S \subseteq N \backslash U$;

ii. $v^{U,\psi}(\emptyset) = 0$.

Definition 3 *(Hart & Mas-Colell [1989]) Let ψ be a value. Then ψ is HM-consistent if for all games $(N, v) \in G$ and all $\emptyset \neq U \subset N$:*

$$\psi_i(N \backslash U, v^{U,\psi}) = \psi_i(N, v) \text{ for all } i \in N \backslash U.$$

One can directly show with these definitions that the egalitarian value η is HM-consistent. Other notions of consistency exist, each depending on its own type of reduced game, cf., e.g., Driessen [1991], Yanovskaya [2003].

The following pertains to the utilities attributed by a value to the players in two-person games.

Definition 4 *(Hart & Mas-Colell [1989]) Let ψ be a value, let $w \in \mathbb{R}^Z$ satisfy $w_i > 0$ for all $i \in Z$. Then ψ is w-proportional if for all 2-person games $(\{i, j\}, v)$, it holds that $\psi_i(\{i, j\}, v) = \frac{w_i v(\{i,j\}) - w_i v(\{j\}) + w_j v(\{i\})}{w_i + w_j}$.*

If $w_i = c > 0$ for all $i \in Z$, w-proportionality is equivalent to (0-)standardness. So, the weighted Shapley value with weights w, is w-proportional. Hart & Mas-Colell [1989] prove the following axiomatic characterization.

Proposition 1 *(Hart & Mas-Colell [1989]) Let ψ be a value, $w \in \mathbb{R}^Z$ satisfying $w_i > 0$ for all $i \in Z$. Then, the following statements are equivalent:*

i. ψ is HM-consistent and w-proportional;

ii. $\psi = Sh^w$.

3 Linear potentials and associated values

We now come to the central purpose of this paper: introducing a family of potentials generalizing those of Hart & Mas-Colell [1989], and associate with each potential a unique efficient and linear value. The families of potentials and values to be introduced depend on a tuple of parameters (a, b, α). The vectors $a, b \in \mathbb{R}^Z$ are exogenously given weights similar to the weights of the weighted Shapley values. As before, $\alpha \in \mathbb{R}$ is the level of taxation reflecting the norms on egalitarianism in the society.

Definition 5 *Let $a, b \in \mathbb{R}^Z$, $\alpha \in \mathbb{R}$ satisfy $\sum_{i \in S \subseteq Z, S \neq \emptyset} a_i \neq 0$. Then the (a, b, α)-potential is the unique map $P^{a,b,\alpha} : G \to \mathbb{R}$ given by*

i. *$P^{a,b,\alpha}(\emptyset, v) = 0$,*

ii. *$\sum_{i \in N} \left[a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N \backslash \{i\}, v) \right] = v^{\alpha}(N)$, for all $(N, v) \in G$, $N \neq \emptyset$.*

The linear-potential value $\psi^{a,b,\alpha}$ is for all $(N, v) \in G$, $i \in N$ given by

$$\psi_i^{a,b,\alpha}(N, v) = a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N \backslash \{i\}, v) + \alpha \frac{v(N)}{|N|}.$$

An interpretation of the value $\psi^{a,b,\alpha}$ is that for an arbitrary game (N, v) it gives to player $i \in N$ the sum of the proportion α of the per-capita income of the grand coalition, and his linearly modified marginal contribution to the potential of the taxed game v^{α}, i.e., $a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N \backslash \{i\}, v)$. As $\sum_{i \in S} a_i \neq 0$ for all $S \subseteq Z$, the preceding definition implies

$$P^{a,b,\alpha}(N, v) = \tfrac{(1-\alpha) v(N) + \sum_{k \in N} b_k P^{a,b,\alpha}(N \backslash \{k\}, v)}{\sum_{k \in N} a_k},$$

which can be used to determine both the (a, b, α)-potential and the connected value recursively. Two instances of α have a rather special influence. For $\alpha = 0$ 'taxing and redistributing' becomes void, whereas for $\alpha = 1$ the potential of each coalition is zero, which follows easily by recursion.

The following result may be proven straightforwardly and deals with properties which hold universally for linear-potential values.

Lemma 2 *For all admissible (a, b, α), the linear-potential value $\psi^{a,b,\alpha}$ is efficient, additive, linear, trivial, and homogeneous of degree* 0 *in (a, b).*

Def. 5 allows unified representations of the (weighted) Shapley value, the α-egalitarian Shapley values (cf., Eq. 1) and the egalitarian value:

- If $a = b = w$, then $P^{a,b,0} = P^w$ and $\psi^{a,b,0} = Sh^w$.
- If $a_i = b_i = 1$ for all $i \in Z$, then $P^{a,b,0} = P$ and $\psi^{a,b,0} = Sh$.
- If $a_i = b_i = 1$ for all $i \in Z$, then $P^{a,b,\alpha} = (1 - \alpha) P$ and $\psi^{a,b,\alpha} = Sh^{\alpha}$.
- If $a_i = 1, b_i = 0$ for all $i \in Z$, then $\psi^{a,b,\alpha}(N, v) = \eta(N, v)$.
- For arbitrary admissible a, b, we have $\psi^{a,b,1}(N, v) = \eta(N, v)$.

4 α-Egalitarian Shapley values

All weighted Shapley values satisfy the null-player property. So do several others, e.g., the nucleolus (Schmeidler [1969]) and the τ-value (Tijs [1981]). Instead, we introduce the following axiom where $\overline{\psi(N,v)} = \frac{\sum_{j \in N} \psi_j(N,v)}{|N|}$ is the **per-capita income** under the value ψ in (N, v).

Definition 6 *Let* $\alpha \in \mathbb{R}$. *The value* ψ *is* α*-egalitarian if for every* (N, v) *and* $i \in \mathcal{N}(N, v)$, $\psi_i(N, v) = \alpha\overline{\psi(N, v)}$.

The axiom stipulates that utility received by a null-player in any game is a fixed scalar multiple ('fraction') α of the per-capita income. Clearly, 0-egalitarianism is equivalent to the null-player property. We define the following family of values which satisfy this property.

Definition 7 *Let* $\alpha \in \mathbb{R}, (N, v) \in G$, then the **$\alpha$-egalitarian Shapley value $\mathbf{Sh}^\alpha$** is given by $Sh^\alpha(N, v) = Sh(N, v^\alpha) + \eta(N, v^{1-\alpha})$.

Under Sh^α each player receives in a particular game the sum of the Shapley value of the corresponding α-taxed game and an equal share in the 'tax revenues'. Van den Brink *et al.* [2007,2013] use the term egalitarian Shapley value only for *convex* combinations of Sh and η. Casajus & Huettner [2013] provide interesting characterizations, one pertaining to the subclass for which $\alpha \leq 1$, two others to the subclass for which $\alpha \in [0, 1]$.

Joosten *et al.* [1994] introduce social acceptability implying that null-players in a unanimity game share in the worth of the grand coalition but not to the extent that agents having marginal contributions which are at least as high as the null-players' marginal contribution get less than the latter, see also Driessen & Radzik [2009]. This axiom restricts the range of α to the unit interval.

Chameni Nembua & Demsou [2013] introduce 'ordinal equivalence': a pair of values is ordinally equivalent, if the ordinal ranking of the utilities under the values is the same for all possible games and players. The class of ordinally equivalent α-egalitarian Shapley value is characterized by $\alpha < 1$.

Malawski [2013] introduces 'procedural' values. For the Shapley value the following story is well known. Suppose all players enter a room one by one, each receiving his marginal contribution to the coalition arising by his entering. Then, after the last player has entered, the vector of the players' utilities is an efficient division of the worth of the grand coalition, $v(N)$. To obtain the Shapley value, it is necessary to perform this procedure for each and every sequence of the players' entering the room and then to determine for each player the average of his utilities taken over all possible sequences. For a 'procedural' value only one aspect changes, namely the amount each player may keep to himself upon entering. These amounts are predetermined and depend on each player's marginal contribution and *on the order in which the players enter.* The egalitarian Shapley values and the solidarity value of Nowak & Radzik [1994] are special instances of procedural values.

To continue our overview, we present two new axioms.

Definition 8 *The value ψ satisfies equal coalitional improvement if for all games $(N, v), (N, w)$ and any nonempty coalition $T \subseteq N$ with the property*

i. $w(S) = v(S) + c$ *for some* $c \in \mathbb{R}$, *and all* $S \supseteq T$,

ii. $w(S) = v(S)$ *otherwise,*

there exists some $\widetilde{c} \in \mathbb{R}$ satisfying $\psi_i(N, w) - \psi_i(N, v) = \widetilde{c}$ for all $i \in T$.

Suppose that a coalition $\emptyset \neq T \subseteq N$ has a gain such that the worths of all coalitions containing T increase by the amount c. Then, the property of equal coalitional improvement requires that all members of T improve by an amount $\widetilde{c}$ under the value. The Shapley value and the egalitarian value satisfy equal coalitional improvement. So, an α-egalitarian Shapley value must also satisfy this property.

Definition 9 *The value ψ satisfies α-marginality if for all $(N, v), (N, w)$ in G and $i \in N$, it holds that $\Delta_i^v(S) = \Delta_i^w(S)$ for all $S \subseteq N$ implies:*

$$\psi_i(N, v) - \alpha\overline{\psi(N, v)} = \psi_i(N, w) - \alpha\overline{\psi(N, w)}.$$

This property is an α-dependent variant of the axiom of marginality introduced by Young [1985]. The axiom of α-marginality compares the utilities attributed by a value in different games, requiring that if the vector of marginal contributions of a player is the same in two games, then the amounts which he receives *on top* of the fixed fraction α of the per-capita income in the two games, are identical.

The following characterization uses the axioms of α-egalitarianism, equal coalitional improvement and α-marginality. Part i. is the next of kin to a characterization due to Shapley [1953], Part iii. relates to Young [1985].

Proposition 3 *(Joosten [1996]) Let $\alpha \in \mathbb{R}$ and let ψ be a value. Then, the following statements are equivalent:*

i. ψ satisfies efficiency, additivity, symmetry, and α-egalitarianism;

ii. ψ satisfies efficiency, triviality, equal coalitional improvement, and α-marginality;

iii. ψ satisfies efficiency, symmetry, and α-marginality;

iv. $\psi = Sh^{\alpha}$.

4.1 Characterizations of Sh^α by forms of consistency

Here, results rely on generalizations of approaches by Hart & Mas-Colell [1989] and Sobolov [1973]. First, we introduce the following reduced game.

Definition 10 *Let* $(N, v) \in G$, $\alpha \in \mathbb{R}$, *and let* ψ *be a value. For nonempty* $U \subset N$, *the* (U, ψ, α)*-reduced game* $(N\backslash U, v^{\mathbf{U},\boldsymbol{\psi},\boldsymbol{\alpha}})$ *of* v *is given by*

i. $v^{U,\psi,\alpha}(S) = v(S \cup U) - \sum_{k\in U} \psi_k(S \cup U, v)$, *if* $\alpha = 1$ *and* $S \neq \emptyset$, *or if* $S = N\backslash U$;

ii. $v^{U,\psi,\alpha}(S) = v(S \cup U) - \sum_{k\in U} \psi_k(S \cup U, v) + \frac{\alpha}{1-\alpha}\left[\frac{|U|}{|S|+|U|} v(S \cup U) - \sum_{k\in U} \psi_k(S \cup U, v)\right]$, *if* $\alpha \neq 1$, $\emptyset \neq S \subset N\backslash U$,

iii. $v^{U,\psi,\alpha}(\emptyset) = 0$.

The interpretation is as follows. For any group of players in a game $S \subseteq N\backslash U$, one defines a reduced game among them by giving the rest of the players, i.e., U, the payoffs according to ψ in the game $(S \cup U, v)$. Then, the worth of S is compensated for the group 'leaving with the amount $\sum_{k\in U} \psi_k(S \cup U, v)$' by returning an amount $\frac{\alpha}{1-\alpha}\left[\frac{|U|}{|S|+|U|} v(S \cup U) - \sum_{k\in U} \psi_k(S \cup U, v)\right]$.

We now generalize 'HM-consistency' to an α-dependent variant.

Definition 11 (Joosten *et al.* [1994]) *Let* $\alpha \in \mathbb{R}$. *The value* ψ *is* α*-consistent if* $\psi_i(N\backslash U, v^{U,\psi,\alpha}) = \psi_i(N, v)$ *for all* $(N, v) \in G$, *all nonempty* $U \subset N$, *and all* $i \in N\backslash U$.

Van den Brink *et al.* [2007] use Sobolev-consistency in a characterization of egalitarian Shapley values. We introduce this type of consistency next.

Definition 12 (Sobolev [1973]) *Given* $(N, v) \in G$, *player* $j \in N$, *and efficient payoff vector* $x \in \mathbb{R}^n$, *i.e.,* $\sum_{i\in N} x_i = v(N)$, *the reduced game with respect to* j *and* x *is the game* $(N\backslash\{j\}, v^x)$ *given by*

$$v^x(S) = \frac{|S|}{|N|-1}\left(v\left(S \cup \{j\}\right) - x_j\right) + \frac{|N|-1-|S|}{|N|-1} v(S) \text{ for all } S \subseteq N\backslash\{j\}.$$

Considerations on the likelihood that coalition $S\cup\{j\}$ forms versus the event that j remains alone, motivate numbers $\frac{|S|}{|N|-1}$ and $\frac{|N|-1-|S|}{|N|-1}$ (cf., Van den Brink *et al.* [2007]). With this reduced game, we define consistency.

Definition 13 *The value* ψ *satisfies Sobolev-consistency if and only if for every* $(N, v) \in G$ *with* $|N| \geq 2$, $j \in N$, $\psi_i\left(N\backslash\{j\}, v^\psi\right) = \psi_i(N, v)$ for all $i \in N\backslash\{j\}$.

It is easily verified that for fixed $\alpha \in \mathbb{R}$, the value Sh^α satisfies α-standardness. We are now ready to present the following result, combining findings of Joosten *et al.* [1994] and Van den Brink *et al.* [2007].

Proposition 4 *Let $\alpha \in \mathbb{R}$ and let ψ be a value. Then, the following two statements are equivalent:*

i. ψ is α-standard and α-consistent;

ii. ψ satisfies α-standardness and Sobolev-consistency;

iii. $\psi = Sh^\alpha$.

5 Axioms and linear-potential values

Our goal here is to find relationships between widely-used axioms and the parameters (a, b, α). An additional aim is to reveal connections among the axioms implied by restrictions on the parameters. We arrange our material around four themes, each defines one subclass of linear-potential values.

5.1 α-Egalitarianism

First, we present *asymmetric* generalizations of egalitarian Shapley values.

Definition 14 *Let $w \in \mathbb{R}^Z$ satisfy $\sum_{i \in S} w_i \neq 0$ for every $S \subseteq Z$, and let $\alpha \in \mathbb{R}$. The α-egalitarian w-weighted Shapley value $Sh^{w,\alpha}$ is for every $(N, v) \in G$ given by $Sh^{w,\alpha}(N, v) = Sh^w(N, v^\alpha) + \eta(N, v^{1-\alpha})$.*

We call the values obtained by taking all admissible w and α *egalitarian weighted Shapley values*. Clearly, they form α-dependent linear combinations of the weighted Shapley value and the egalitarian value. Next, we show connections between axioms and restrictions on the parameters.

Proposition 5 *$\psi^{a,b,1}$ satisfies 1-egalitarianism and 1-marginality for arbitrary (admissible) (a, b). For $\alpha \neq 1$, the following statements are equivalent:*

i. $\psi^{a,b,\alpha}$ satisfies α-egalitarianism.

ii. $\psi^{a,b,\alpha}$ satisfies α-marginality.

iii. $\psi^{a,b,\alpha} = Sh^{w,\alpha}$ with $w = a = b$.

Moreover, $\psi^{a,b,\alpha}$ satisfies λ-egalitarianism, implies $\lambda = \alpha$.

Note that α appears both in the formulation of the linear-potential value and the axioms. The final statement stipulates that there can be at most one instance for which a constant proportion to the null players for all games with the same worth of the grand coalition is attributed by a linear-potential value, namely exactly linked to the α in the tuple (a, b, α).

5.2 Symmetry

First, we recall a subclass of linear-potential values from Joosten *et al.* [1994] to be seen as a *symmetric* generalization of the α-egalitarian Shapley values.

Definition 15 *For $\alpha \in \mathbb{R}$, $\delta \in \mathbb{R}\backslash\{0\}$, the semi α-egalitarian δ-discounted Shapley value Sh_{δ}^{α} is for all $(N,v) \in G$, $i \in N$ given by* $(Sh_{\delta}^{\alpha})_i (N,v) = \alpha\frac{v(N)}{|N|} + (1-\alpha)\sum_{S\subseteq N\backslash\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}\delta^{|N|-|S|-1}\left[v(S\cup\{i\}) - \delta v(S)\right].$

We refer to the class of values obtained by taking all admissible δ and α, as *semi-egalitarian discounted Shapley values*. It can be confirmed that $\alpha = 0$, $\delta = 1$ yields the Shapley value, and for arbitrary α and $\delta = 1$, we obtain the α-egalitarian Shapley value. The term *semi* is used because Sh_{δ}^{α} is not α-egalitarian for $\alpha \neq 0$. However, any null player in (N, v_δ) receives $\alpha\frac{v(N)}{|N|}$, so this establishes which players get $\alpha\frac{v(N)}{|N|}$. Van den Brink & Funaki [2010] call these players δ-reducing.

To give an elegant interpretation of these values, we present another notion which is useful in the remainder.

Definition 16 Let $\delta \in \mathbb{R}\backslash\{0\}$, then for $(N,v) \in G$, the **δ-discounted game** (N, v_δ) is given by $v_\delta(S) = \delta^{|N|-|S|}v(S)$ for all $S \subseteq N$.

The names chosen are inspired by Driessen & Radzik [2002] who coined the term of 'δ-discounted Shapley values' for the special instance that $\alpha = 0$. Also the following interpretation related to discounting for $\delta \in (0,1)$ is given: '*...the worth of a coalition in an n-person game is weakly discounted whenever the size s of the coalition is relatively large (or strongly discounted, if the size of the coalition is relatively small) in comparison with the size n of the player set'*.

Any δ-discounted Shapley value of the game $(N,v) \in G$ is equivalent to the Shapley value of the δ-discounted game $(N, v_\delta) \in G$. The reader may confirm this by applying the Shapley value to the game (N, v_δ) followed by substituting $v_\delta(S) = \delta^{|N|-|S|}v(S)$ for all $S \subseteq N$. Similarly, the α-egalitarian δ-discounted Shapley value of the game (N,v) is the α-egalitarian Shapley value applied to the δ-discounted game (N, v_δ). The following result links the linear-potential value of a game to the one of a discounted game.

Lemma 6 *For all $(N,v) \in G$, $\delta \in \mathbb{R}\backslash\{0\}$, $\psi^{a,\delta\cdot b,\alpha}(N,v) = \psi^{a,b,\alpha}(N,v_\delta)$.*

In what follows, we use a notational convenience by writing ψ_δ defined by $\psi_\delta(N,v) = \psi(N,v_\delta)$ for every (N,v) and value ψ with the customary restriction $\delta \neq 0$. Next, we show connections between restrictions on the parameters (a,b,α) and the axiom of symmetry. It uses the preceding lemma.

Proposition 7 *The linear-potential value $\psi^{a,b,1}$ is symmetric, for linear-potential values $\psi^{a,b,\alpha}$ with $\alpha \neq 1$, the following statements are equivalent:*

i. $\psi^{a,b,\alpha}$ *is symmetric;*

ii. $\psi^{a,b,\alpha} = Sh^{\alpha}_{\frac{\tau_2}{\tau_1}}$ *with* $a_i = \tau_1 \neq 0$, $b_i = \tau_2$ *for all* $i \in Z$.

5.3 HM-consistency and standardness

Recall that α-consistency and HM-consistency coincide for $\alpha \in \{0, 1\}$. This does not necessarily imply that for instance every linear-potential value $\psi^{a,b,0}$ satisfies HM-consistency. The following establishes this however.

Proposition 8 *For $\alpha = 0$ or $\alpha = 1$, the linear-potential value $\psi^{a,b,\alpha}$ is HM-consistent.*

So, for all $a, b \in \mathbb{R}^Z$ we have HM-consistency whenever $\alpha = 0$ or $\alpha = 1$. This means that even the rather large class of weighted Shapley values form merely a subclass of the HM-consistent linear-potential values.

The axiom of λ-standardness stipulates the utilities players in each 2-person game receive depending on the real number λ and the worths of the two-person grand coalition and the two 'stand alone' coalitions. The axiom has the following implications.

Proposition 9 $\psi^{a,b,1}$ *satisfies 1-standardness. For $\alpha \neq 1$, the following statements are equivalent:*

i. $\psi^{a,b,\alpha}$ *satisfies λ-standardness;*

ii. $\psi^{a,b,\alpha} = Sh^{\alpha}_{\frac{1-\lambda}{1-\alpha}}$.

5.4 Equal coalitional improvement

We now introduce linear-potential values forming in some sense a class of hybrids between egalitarian Shapley values and egalitarian weighted Shapley values. The former are symmetric, the latter are not.

Definition 17 *Let $w \in \mathbb{R}^Z$ satisfy $\sum_{i \in S} w_i \neq 0$ for every $S \subseteq Z$, and let $\alpha \in \mathbb{R}$, then the ECI(w, α)-Shapley value $Sh^{\mathbf{w},\boldsymbol{\alpha}}_{\mathbf{ECI}}$ is given by*

$$\left(Sh^{w,\alpha}_{ECI}\right)_i (N, v) = P^{w,\alpha}_{ECI}(N, v) - w_i P^{w,\alpha}_{ECI}(N \backslash \{i\}, v) + \alpha \tfrac{v(N)}{|N|},$$

where $P^{w,\alpha}_{ECI} = P^{a,\tau \cdot w,\alpha}(N, v)$ and $a_i = \tau$ for all $i \in Z$.

ECI is mnemonic for equal coalitional improvement, and we will refer to this class as ECI-Shapley values. Observe that equal coalitional improvement is implied by symmetry, but not vice versa. Therefore, the class just presented contains all semi-egalitarian discounted Shapley values. The following result establishes the relation between this axiom and the parameters (a, b, α).

Proposition 10 *$\psi^{a,b,1}$ satisfies equal coalitional improvement. For $\alpha \neq 0, 1$, the following statements are equivalent:*

i. $\psi^{a,b,\alpha}$ *satisfies equal coalitional improvement;*

ii. $\psi^{a,b,\alpha} = Sh^{w,\alpha}_{ECI}$.

iii. $\psi^{a,b,\alpha}$ *is α-consistent.*

Note that each ECI-Shapley value satisfies this axiom. The converse statement is obviously not true.

Figure 1 visualizes results of this section. Generalizations of standardness are very restrictive and in fact imply α-consistency for any value $\psi^{a,b,\alpha}$ with $\alpha \neq 0, 1$. In other words, the egalitarian Shapley values are characterized by α-standardness and the fact that they admit[5] an (a, b, α)-potential.

Corollary 11 *A linear-potential value $\psi^{a,b,\alpha}$, $\alpha \neq 0, 1$, satisfies*

- *α-egalitarianism if and only if it is an α-egalitarian weighted Shapley value.*
- *symmetry if and only if it is an egalitarian discounted Shapley value.*
- *α-consistency if and only if it is an ECI-Shapley value.*
- *α-standardness if and only if it is an α-egalitarian Shapley value*

6 Further characterizations of the 'Shapley family'

Here, we characterize certain semi-egalitarian discounted Shapley values.

Proposition 12 *The following statements are equivalent for $\alpha, \lambda \neq 1$:*

i. $\psi^{a,b,\alpha}$ *is α-consistent and λ-standard;*

ii. $\psi^{a,b,\alpha}$ *is Sobolev-consistent and λ-standard;*

iii. $\psi^{a,b,\alpha} = Sh^{\alpha}_{\frac{1-\lambda}{1-\alpha}}$.

[5] This expression was coined by Calvo & Santos [1997], cf., e.g., Driessen & Calvo [2001].

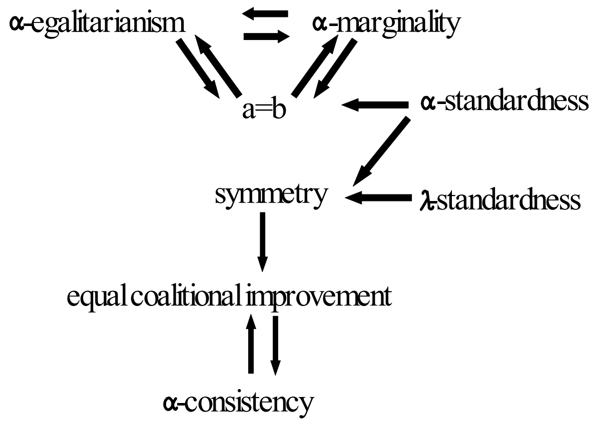


Figure 1: Arrows imply implications. These connections apply to all linear-potential values $\psi^{a,b,\alpha}$ satisfying $\alpha \neq 0, 1$. Note furthermore that $\lambda \neq \alpha$.

Now, we use another implication of Lemma 6. Observe that applying the α-egalitarian weighted Shapley value to the game (N, v_δ) yields another value for (N, v). The specifics are given by the following.

Definition 18 *Given $w \in \mathbb{R}^Z$ satisfying $\sum_{i \in S} w_i \neq 0$ for every $S \subseteq Z$, $\alpha \in \mathbb{R}$, and $\delta \neq 0$, the semi α-egalitarian δ-discounted w-weighted Shapley value for all (N, v) and all $i \in N$ is given by*

$$\left(Sh_\delta^{w,\alpha}\right)_i (N, v) = w_i \left[P_\delta^{w,\alpha}(N, v) - P_\delta^{w,\alpha}(N \backslash \{i\}, v)\right] + \alpha \tfrac{v(N)}{|N|},$$

where $P_\delta^{w,\alpha} = P^{w,\delta \cdot w,\alpha}$.

Again, we use the phrase semi, because $Sh_\delta^{w,\alpha}$ gives $\alpha \frac{v(N)}{|N|}$ to δ-reducing players. Recall that $a = b = w$ induces $Sh^{w,\alpha}$, here $b = \delta a = \delta w$ induces its δ-discounted next of kin, $Sh_\delta^{w,\alpha}$.

Definition 19 *Let $w \in \mathbb{R}^z$ satisfy $w_i > 0$ for all $i \in Z$, and let $\lambda \in \mathbb{R}$. Then, the value ψ is (w, λ)-proportional if for all 2-person games $(\{i, j\}, v)$, it holds that* $\psi_i(\{i, j\}, v) = \frac{w_i v(\{i,j\}) - (1-\lambda) \cdot w_i v(\{j\}) + (1-\lambda) \cdot w_j v(\{i\})}{w_i + w_j}$.

This notion extends w-proportionality straightforwardly as the latter is a special case of the first. The next result shows that HM-consistency and (w, λ)-proportionality characterize another member of the 'Shapley family'.

Proposition 13 *The following statements are equivalent:*

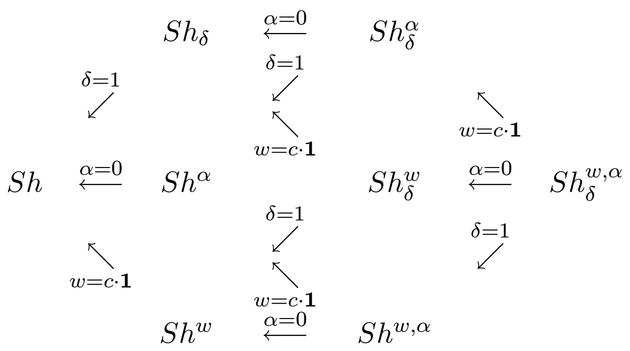
i. The value ψ *is HM-consistent and* (w, λ)*-proportional;*

ii. $\psi = Sh^{w,0}_{1-\lambda}$.

Corollary 14 *Let* $\alpha \neq 1$*, a value* ψ *is*

- *HM-consistent and* α*-standard if and only if* $\psi = Sh_{1-\alpha}$*;*
- α*-consistent and standard if and only if* $\psi = Sh^{\alpha}_{\frac{1}{1-\alpha}}$*;*
- *HM-consistent and* (w, α)*-proportional if and only if* $\psi = Sh^{w}_{1-\alpha}$*.*

The following visualization shows connections within the Shapley family. Each arrow denotes a restriction on the parameters as indicated.



7 Conclusions and discussion

We introduced a class of potentials and associated to each potential a value. This class of linear-potential values contains the Shapley value and the egalitarian value as special examples. Furthermore, the *classes* of the weighted Shapley values (Shapley [1953], Kalai & Samet [1987]), the (α-)egalitarian Shapley values (cf., Joosten [1996],Van den Brink *et al.* [2007]), and the discounted Shapley values (cf., Joosten *et al.* [1994], Joosten [1996], Yanovskaya & Driessen [2001]) are contained by it.

Other approaches use similar ideas and we discuss some differences now. Naumova [2005] presents a class of consistent (possibly asymmetric) values each originating from a potential as well. Her class of values intersects clearly with ours, as weighted Shapley values are contained by both, but it is easy to find values belonging to Naumova's class and not ours (e.g., the proportional value of Ortmann [2000]), or vice versa (e.g., discounted Shapley values, α-egalitarian Shapley values).

Our approach should not be confused with the one of Driessen & Radzik [2002] and related work (e.g., Feng [2013]) who define, connected to sequences of reals $\alpha_{\mathbb{N}}, \beta_{\mathbb{N}}, \gamma_{\mathbb{N}}$ and a potential $P^{\alpha_{\mathbb{N}},\beta_{\mathbb{N}},\gamma_{\mathbb{N}}}$, a value as a weighted pseudo-gradient, i.e., a vector of marginal contributions to $P^{\alpha_{\mathbb{N}},\beta_{\mathbb{N}},\gamma_{\mathbb{N}}}$. The sequences are **not** connected to the players, but to the cardinality of the player set for a particular game. Hence, asymmetric weighted Shapley values can not be written as a weighted pseudo-potential of the HM-potential. The class of values implied by Driessen & Radzik [2002] contains discounted Shapley values and the solidarity value. The former are a subclass of our linear-potential values, the latter is not.

Van den Brink & Van der Laan [2007] define a μ-potential and a function giving each player a share in the worth of the grand coalition.[6] Admittedly, taking $\mu(N,v) = v^{\alpha}(N)$ for all $(N,v) \in G$, yields $P^{v^{\alpha}}$ i.e., the potential of the α-egalitarian Shapley value. However, the vector

$$\sigma(N,v) = \left(P^{v^{\alpha}}(N,v) - P^{v^{\alpha}}(N\backslash\{1\},v), ..., P^{v^{\alpha}}(N,v) - P^{v^{\alpha}}(N\backslash\{n\},v)\right),$$

is projected unto the efficient n-dimensional hyperplane differently. We project $\sigma(N,v)$ orthogonally, whereas Van den Brink & Van der Laan [2007] do so along the ray connecting $(0,...,0)$ and $\sigma(N,v)$. So, the latter yields the Shapley value for all $\alpha \neq 1$ and the egalitarian value for $\alpha = 1$, and ours yields a unique value $Sh^{\alpha}(v)$ for each $\alpha \neq 1$.

We investigated connections between several axioms used for characterization of values and restrictions on the parameter set (a,b,α) determining both the potential $P^{a,b,\alpha}$ and its associated linear-potential value $\psi^{a,b,\alpha}$. It should be noted that by construction linear-potential values satisfy efficiency, additivity, linearity and triviality.

We focused on axioms used to characterize the (α-egalitarian) Shapley value(s), cf., Joosten *et al.* [1994]. For linear-potential values, we found the axiom of α-egalitarianism to be equivalent to α-marginality, a variant of marginality (cf., Young [1985]); we found any symmetric linear-potential value to satisfy α-consistency, the latter being equivalent to equal coalitional improvement for this entire class of values. For given (a,b,α), we showed that the axiom of λ-standardness implies symmetry for $\lambda \neq \alpha$, and for $\lambda = \alpha$ it also implies α-egalitarianism; therefore any linear-potential value satisfying α-standardness is an α-egalitarian Shapley value.

We distinguished within the class of linear-potential values certain subclasses characterized by single axioms: α-egalitarian weighted Shapley values are characterized by α-egalitarianism, semi-egalitarian discounted Shapley values by symmetry, and equal-coalitional-improvement Shapley values by α-consistency. We derive a characterization of each value in a subclass of the semi-egalitarian discounted Shapley values by λ-standardness and either α-consistency or Sobolov-consistency, extending results of Joosten *et al.* [1994] and Van den Brink *et al.* [2007].

[6] μ is a real-valued function depending on the number of players and the characteristic function of an arbitrary game. The Shapley value and the egalitarian value can be formulated in terms of such a potential and share function, by simply taking $\mu(N,v) = v(N)$ respectively $\mu(N,v) = 0$ for all $(N,v) \in G$

8 Appendix

Proof of Proposition 5: We only prove the part for $\alpha \neq 1$. **(iii) implies (i):** We first prove $P^{a,a,\alpha}(N,v) = P^{a,a,\alpha}(N\backslash\{i\},v)$ whenever $i \in \mathcal{N}(N,v)$, by induction on $|N|$.

If $N=\{i\}$, the statement holds trivially. Now, let $|N| \geq 2$, and assume for all games (S,v) with $|S|<|N|$ that $P^{a,a,\alpha}(S,v) = P^{a,a,\alpha}(S\backslash\{i\},v)$ whenever $i \in \mathcal{N}(S,v)$. Then,

$$\begin{aligned}
\textstyle\sum_{k\in N} a_k P^{a,a,\alpha}(N,v) &= (1-\alpha)v(N) + \textstyle\sum_{k\in N} a_k P^{a,a,\alpha}(N\backslash\{k\},v) \\
&= (1-\alpha)v(N\backslash\{i\}) + \textstyle\sum_{k\in N\backslash\{i\}} a_k P^{a,a,\alpha}(N\backslash\{i,k\},v) \\
&\quad + a_i P^{a,a,\alpha}(N\backslash\{i\},v) \\
&= \textstyle\sum_{k\in N\backslash\{i\}} a_k P^{a,a,\alpha}(N\backslash\{i\},v) + a_i P^{a,a,\alpha}(N\backslash\{i\},v) \\
&= \textstyle\sum_{k\in N} a_k P^{a,a,\alpha}(N\backslash\{i\},v).
\end{aligned}$$

Hence, $P^{a,a,\alpha}(N,v) = P^{a,a,\alpha}(N\backslash\{i\},v)$. The second equality follows from the induction assumption. This in turn implies: $\psi_i^{a,a,\alpha}(N,v) = a_i P^{a,a,\alpha}(N,v) - a_i P^{a,a,\alpha}(N\backslash\{i\},v) + \alpha\frac{v(N)}{|N|} = \alpha\frac{v(N)}{|N|}$ whenever $i \in \mathcal{N}(N,v)$.

(i) implies (iii): Let $i,j \in Z$, α-egalitarianism of $\psi^{a,b,\alpha}$ implies

$$\begin{aligned}
0 &= a_i P^{a,b,\alpha}(\{i,j\},u_{\{j\}}) - b_i P^{a,b,\alpha}(\{j\},u_{\{j\}}) \\
&= (1-\alpha)\left[a_i\frac{1+b_i\frac{1}{a_j}}{a_i+a_j} - b_i\frac{1}{a_j}\right] = (1-\alpha)\left[\frac{a_i+b_i\frac{a_i}{a_j}-b_i\frac{a_i}{a_j}-b_i\frac{a_j}{a_j}}{a_i+a_j}\right] \\
&= (1-\alpha)\left[\frac{a_i-b_i}{a_i+a_j}\right].
\end{aligned}$$

Hence, $a_i=b_i$. This in turn implies $a=b$.

(iii) implies (ii): Take $i \in Z$. Take (N,w) and $(N,\widetilde{w})$ with $i \in N$ and $\Delta_i^w(S) = \Delta_i^{\widetilde{w}}(S)$. Observe that $i \in \mathcal{N}(N,\widetilde{w}-w)$, and the part '(iii) implies (i)' imply $\psi^{a,a,\alpha}(N,\widetilde{w}-w) = \frac{\alpha(\widetilde{w}-w)(N)}{|N|} = \alpha\overline{\Phi^{a,a,\alpha}(N,\widetilde{w})} - \alpha\overline{\Phi^{a,a,\alpha}(N,w)}$.
By linearity of $\psi^{a,a,\alpha}$ we have for

$$\begin{aligned}
\psi_i^{a,a,\alpha}(N,\widetilde{w}) &= \psi_i^{a,a,\alpha}(N,\widetilde{w}-w+w) = \psi_i^{a,a,\alpha}(N,\widetilde{w}-w) + \psi_i^{a,a,\alpha}(N,w) \\
&= \psi_i^{a,a,\alpha}(N,w) - \alpha\overline{\psi^{a,a,\alpha}(N,w)} + \alpha\overline{\psi^{a,a,\alpha}(N,\widetilde{w})}.
\end{aligned}$$

This proves α-marginality of $\psi^{a,a,\alpha}$.

(ii) implies (iii): Take $i \in Z$, $N \ni i$, (N,v_0), and $(N,u_{N\backslash\{i\}})$. Then, by α-marginality

$$\psi_i^{a,b,\alpha}(N,v_0) - \alpha\overline{\psi^{a,b,\alpha}(N,v_0)} = \psi_i^{a,b,\alpha}(N,u_{N\backslash\{i\}}) - \alpha\overline{\psi^{a,b,\alpha}(N,u_{N\backslash\{i\}})},$$

and by efficiency and triviality we get $\psi_i^{a,b,\alpha}(N,u_{N\backslash\{i\}}) = \frac{\alpha}{|N|}$. This implies in turn that $a_i P^{a,b,\alpha}(N,u_{N\backslash\{i\}}) - b_i P^{a,b,\alpha}(N\backslash\{i\},u_{N\backslash\{i\}}) = 0$, so:

$$\begin{aligned}
0 &= a_i P^{a,b,\alpha}(N,u_{N\backslash\{i\}}) - b_i P^{a,b,\alpha}(N\backslash\{i\},u_{N\backslash\{i\}}) \\
&= (1-\alpha)\left[a_i\frac{1+b_i\frac{1}{\sum_{k\in N\backslash\{i\}}a_k}}{\sum_{k\in N}a_k} - b_i\frac{1}{\sum_{k\in N\backslash\{i\}}a_k}\right] \\
&= (1-\alpha)\frac{a_i\sum_{k\in N\backslash\{i\}}a_k + a_ib_i - b_i\sum_{k\in N}a_k}{\sum_{k\in N}a_k\left(\sum_{k\in N\backslash\{i\}}a_k\right)} = (1-\alpha)\frac{a_i-b_i}{\sum_{k\in N}a_k}.
\end{aligned}$$

Hence, $a_i = b_i$. This proves $a = b$. So, these three statements are equivalent. To prove the 'moreover' part, suppose that $\psi^{a,b,\alpha}$ is λ-egalitarian and $\lambda \neq \alpha$ and $\alpha \neq 1$. Then, it is a matter of calculation to establish that

$$\psi_j^{a,b,\alpha}\left(\{i,j\},u_{\{i\}}\right) = \tfrac{1-\alpha}{a_i+a_j}\left[a_j - b_j\right] + \tfrac{\alpha}{2} = \tfrac{\lambda}{2} \text{ for all } i,j \in Z.$$

However, we must have $\psi_j^{a,b,\alpha}\left(\{i,k\},u_{\{i\}}\right) = \frac{1-\alpha}{a_i+a_k}\left[a_j - b_j\right] + \frac{\alpha}{2} = \frac{\lambda}{2}$ as well, hence $a_j = a_k$ for all $j,k \in Z$. But then, substituting a for a_i and a_j, we have $\psi_j^{a,b,\alpha}\left(\{i,j\},u_{\{i\}}\right) = \frac{1-\alpha}{2a}\left[a - b_j\right] + \frac{\alpha}{2} = \frac{\lambda}{2}$. This implies $b_j = \frac{1-\lambda}{1-\alpha}a$, which in turn must hold for arbitrary $j \in Z$. Next, consider $\left(\{i,j,k\},u_{\{i\}}\right)$ with $i,j,k \in Z$ arbitrarily chosen, but $i \neq j \neq k \neq i$. Then,

$$\psi_j^{a,b,\alpha}\left(\{i,j,k\},u_{\{i\}}\right) = \tfrac{1}{3} - \tfrac{1-\lambda}{6(1-\alpha)}\left[2 - \lambda - \alpha\right] = \tfrac{\lambda}{3}.$$

Hence, $\frac{1-\lambda}{6(1-\alpha)}\left[2-\lambda-\alpha\right] = \frac{1-\lambda}{3}$ which is equivalent to $2-\lambda-\alpha = 2\left(1-\alpha\right).$ This yields a contradiction. So, $\psi^{a,b,\alpha}$ is λ-egalitarian implies that $\lambda = \alpha$.

Proof of Lemma 6: For $S \subseteq N$, we show $P^{a,\delta\cdot b,\alpha}(S,v) = \frac{1}{\delta^{|N|-|S|}}P^{a,b,\alpha}(S,v_\delta)$ by induction on the cardinality of the player sets $S' \subseteq S$. Observe that

$$\begin{aligned} P^{a,\delta\cdot b,\alpha}(\{i\},v) &= \tfrac{(1-\alpha)v(\{i\})}{a_i} = \tfrac{1}{\delta^{|N|-1}}\tfrac{(1-\alpha)\cdot\delta^{|N|-1|}v(\{i\})}{a_i} \\ &= \tfrac{1}{\delta^{|N|-1}}\tfrac{(1-\alpha)\cdot v_\delta(\{i\})}{a_i} = \tfrac{1}{\delta^{|N|-1}}P^{a,b,\alpha}(\{i\},v_\delta). \end{aligned}$$

Assume: $P^{a,\delta\cdot b,\alpha}(S',v) = \frac{1}{\delta^{|N|-|S'|}}P^{a,b,\alpha}(N,v_\delta)$ for all $S' \subseteq S$ with $|S'| < |S|$. Then,

$$\begin{aligned} P^{a,\delta\cdot b,\alpha}(S,v) &= \tfrac{(1-\alpha)v(S)+\sum_{k\in S}\delta\cdot b_k P^{a,\delta\cdot b,\alpha}(S\setminus\{k\},v)}{\sum_{k\in S}a_k} \\ &= \tfrac{(1-\alpha)\frac{1}{\delta^{|N|-|S|}}v_\delta(S)+\sum_{k\in S}\delta\cdot\frac{1}{\delta^{|N|-|S|+1}}b_k P^{a,b,\alpha}(S\setminus\{k\},v_\delta)}{\sum_{k\in S}a_k} \\ &= \tfrac{1}{\delta^{|N|-|S|}}\tfrac{(1-\alpha)v_\delta(S)+\sum_{k\in S}b_k P^{a,b,\alpha}(S\setminus\{k\},v_\delta)}{\sum_{k\in S}a_k} \\ &= \tfrac{1}{\delta^{|N|-|S|}}P^{a,b,\alpha}(S,v_\delta). \end{aligned}$$

The second equality follows from the induction assumption and the definition of the discounted game v_δ . Hence, we have

$$\begin{aligned} \psi^{a,\delta\cdot b,\alpha}(N,v) &= a_i P^{a,\delta\cdot b,\alpha}(N,v) - \delta\cdot b_i P^{a,\delta\cdot b,\alpha}(N\setminus\{i\},v) + \alpha\tfrac{v(N)}{|N|} \\ &= a_i P^{a,b,\alpha}(N,v_\delta) - \delta\cdot\tfrac{1}{\delta}b_i P^{a,b,\alpha}(N\setminus\{i\},v_\delta) + \alpha\tfrac{v(N)}{|N|} \\ &= a_i P^{a,b,\alpha}(N,v_\delta) - b_i P^{a,b,\alpha}(N\setminus\{i\},v_\delta) + \alpha\tfrac{v(N)}{|N|} \\ &= \psi^{a,b,\alpha}(N,v_\delta). \end{aligned}$$

Proof of Proposition 7: We prove the part for $\alpha \neq 1$.**(ii) implies (i):** Straightforward. **(i) implies (ii):** Take $i,j \in Z$, then by symmetry of $\psi^{a,b,\alpha}$

$$0 = \psi_i^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) - \psi_j^{a,b,\alpha}(\{i,j\},u_{\{i,j\}})$$

$$
\begin{aligned}
&= a_i P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) - b_i P^{a,b,\alpha}(\{j\},u_{\{i,j\}}) + \alpha\tfrac{u_{\{i,j\}}(\{i,j\})}{2} - \\
&\quad \left[a_j P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) - b_j P^{a,b,\alpha}(\{i\},u_{\{i,j\}}) + \alpha\tfrac{u_{\{i,j\}}(\{i,j\})}{2}\right] \\
&= (a_i - a_j) P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}).
\end{aligned}
$$

Since $P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) \neq 0$, we obtain $a_i = a_j$. So take τ_1 such that $\tau_1 = a_i$ for all $i \in Z$. Take $(\{i,j\},v)$ with $v(\{i\}) = v(\{j\}) \neq 0$. Then, by symmetry of $\psi^{a,b,\alpha}$

$$
\begin{aligned}
0 &= \psi_i^{a,b,\alpha}(\{i,j\},v) - \psi_j^{a,b,\alpha}(\{i,j\},v) \\
&= \tau_1 P^{a,b,\alpha}(\{i,j\},v) - b_i P^{a,b,\alpha}(\{j\},v) + \alpha\tfrac{v(\{i,j\})}{2} - \\
&\quad \left[\tau_1 P^{a,b,\alpha}(\{i,j\},v) - b_j P^{a,b,\alpha}(\{i\},v) + \alpha\tfrac{v(\{i,j\})}{2}\right] \\
&= -b_i P^{a,b,\alpha}(\{j\},v) + b_j P^{a,b,\alpha}(\{i\},v) \\
&= -b_i \tfrac{(1-\alpha)v(\{j\})}{\tau_1} + b_j \tfrac{(1-\alpha)v(\{i\})}{\tau_1} = (b_j - b_i)\tfrac{(1-\alpha)}{\tau_1} v(\{i\}).
\end{aligned}
$$

Hence, $b_i = b_j$ for all $i, j \in Z$. Take τ_2 such that $b_i = \tau_2$ for all $i \in Z$.
Now, homogeneity in vectors (a,b) of linear potential values implies that one may divide both τ_1 and τ_2 by τ_1 and subsequently substituting $\delta = \frac{\tau_2}{\tau_1}$ in Definition 15 yields the mathematical reformulation $\psi^{a,b,\alpha} = Sh^{\alpha}_{\frac{\tau_2}{\tau_1}}$ whenever $a_i = \tau_1 \in \backslash\{0\}$ and $b_i = \tau_2 \in \backslash\{0\}$ for all $i \in Z$.

Proof of Proposition 8: For $\alpha = 1$, HM-consistency follows immediately as $\psi^{a,b,1} = \eta$.
We now prove the part for $\alpha = 0$. Let $(N,v) \in G$. For $U \subset N$ satisfying $|U| = |N| - 1$, the proposition follows immediately. Now fix $U \subset N$ satisfying $0 \neq |U| < |N| - 1$. To prove the proposition for $\alpha = 0$, we show

$$
P^{a,b,0}(N,v) = P^{a,b,0}\left(N\backslash U, v^{U,\psi^{a,b,0}}\right) + \textstyle\prod_{k\in N\backslash U} \frac{b_k}{a_k} P^{a,b,0}(U,v). \qquad (2)
$$

We do this by induction on the cardinality of nonempty subsets S of $N\backslash U$. Note that for all $N \subseteq Z$, and $\emptyset \neq U \subset N$, we have

$$
P^{a,b,0}(N,v) = \frac{v(N) - \sum_{k\in U}\psi_k^{a,b,0}(N,v) + \sum_{k\in N\backslash U} b_k P^{a,b,0}(N\backslash\{k\},v)}{\sum_{k\in N\backslash U} a_k}.
$$

Hence, taking $i \in N\backslash U$, we have

$$
\begin{aligned}
P^{a,b,0}(\{i\}\cup U,v) &= \frac{v(\{i\}\cup U) - \sum_{k\in U}\psi_k^{a,b,0}(\{i\}\cup U,v) + b_i P^{a,b,0}(U,v)}{a_i} \\
&= \frac{v^{U,\psi^{a,b,0}}(\{i\})}{a_i} + \frac{b_i}{a_i} P^{a,b,0}(U,v) \\
&= P^{a,b,0}\left(\{i\}, v^{U,\psi^{a,b,0}}\right) + \frac{b_i}{a_i} P^{a,b,0}(U,v).
\end{aligned}
$$

Let $S \subseteq N\backslash U$, $|S| \geq 2$, and assume for all $D \subset S$:

$$
P^{a,b,0}(D\cup U,v) = P^{a,b,0}\left(D, v^{U,\psi^{a,b,0}}\right) + \textstyle\prod_{i\in D} \frac{b_i}{a_i} P^{a,b,0}(U,v). \qquad (3)
$$

Then,

$$\begin{aligned}
P^{a,b,0}(S\cup U,v) &= \frac{v(S\cup U)-\sum_{k\in U}\psi^{a,b,0}(S\cup U,v)+\sum_{k\in S}b_kP^{a,b,0}((S\cup U)\backslash\{k\},v)}{\sum_{k\in S}a_k}\\
&= \frac{v^{U,\psi^{a,b,0}}(S)+\sum_{k\in S}b_kP^{a,b,0}((S\cup U)\backslash\{k\},v)}{\sum_{k\in S}a_k}\\
&= \frac{v^{U,\psi^{a,b,0}}(S)+\sum_{k\in S}b_kP^{a,b,0}\left(S\backslash\{k\},v^{U,\psi^{a,b,0}}\right)}{\sum_{k\in S}a_k}+\\
&\quad \frac{\sum_{k\in S}b_k\left[\prod_{i\in S\backslash\{k\}}\frac{b_i}{a_i}P^{a,b,0}(U,v)\right]}{\sum_{k\in S}a_k}\\
&= P^{a,b,0}\left(S,v^{U,\psi^{a,b,0}}\right)+\frac{\sum_{k\in S}b_k\left[\prod_{i\in S\backslash\{k\}}\frac{b_i}{a_i}P^{a,b,0}(U,v)\right]}{\sum_{k\in S}a_k}\\
&= P^{a,b,0}\left(S,v^{U,\psi^{a,b,0}}\right)+\frac{\sum_{k\in S}a_k\left[\prod_{i\in S}\frac{b_i}{a_i}P^{a,b,0}(U,v)\right]}{\sum_{k\in S}a_k}\\
&= P^{a,b,0}\left(S,v^{U,\psi^{a,b,0}}\right)+\prod_{i\in S}\frac{b_i}{a_i}P^{a,b,0}(U,v).
\end{aligned}$$

the second equality follows from Def. 10, the third one follows from (3). Now, let $k\in N\backslash U$, then by definition of $\psi^{a,b,0}$ applied to the game (N,v)

$$\begin{aligned}
&\psi_k^{a,b,0}(N,v)\\
=\ & a_kP^{a,b,0}(N,v)-b_kP^{a,b,0}(N\backslash\{k\},v)\\
=\ & a_kP^{a,b,0}\left(N\backslash U,v^{U,\psi^{a,b,0}}\right)-b_kP^{a,b,0}\left(N\backslash(U\cup\{k\}),v^{U,\psi^{a,b,0}}\right)+\\
& a_k\left[\prod_{i\in N\backslash U}\frac{b_i}{a_i}P^{a,b,0}(U,v)\right]-b_k\left[\prod_{i\in N\backslash(U\cup\{k\})}\frac{b_i}{a_i}P^{a,b,0}(U,v)\right]\\
=\ & \psi_k^{a,b,0}\left(N\backslash U,v^{U,\psi^{a,b,0}}\right)+\\
& \left[\frac{\prod_{i\in N\backslash U}b_i}{\prod_{i\in N\backslash(U\cup\{k\})}a_i}-\frac{\prod_{i\in N\backslash U}b_i}{\prod_{i\in N\backslash(U\cup\{k\})}a_i}\right]P^{a,b,0}(U,v)\\
=\ & \psi_k^{a,b,0}\left(N\backslash U,v^{U,\psi^{a,b,0}}\right).
\end{aligned}$$

The second equality follows from (3), the third one follows by definition of the value $\psi^{a,b,0}$ applied to the game $\left(N\backslash U,v^{U,\psi^{a,b,0}}\right)$.

Proof of Proposition 9: The part **(ii) implies (i)** is confirmed easily by writing out the equations for two-person games. We now prove that **(i) implies (ii)**. Let $\alpha\neq 1$ and let $\psi^{a,b,\alpha}$ satisfy λ-standardness. Then, apply $\psi^{a,b,\alpha}$ to the two-person game $(\{i,j\},v)$. Because $P^{a,b,\alpha}(\varnothing,v)=0$, $P^{a,b,\alpha}(\{i\},v)=\frac{(1-\alpha)v(\{i\})}{a_i}$, $P^{a,b,\alpha}(\{j\},v)=\frac{(1-\alpha)v(\{j\})}{a_j}$ and

$$P^{a,b,\alpha}(\{i,j\},v)=\frac{(1-\alpha)v(\{i,j\})+b_iP^{a,b,\alpha}(\{j\},v)+b_jP^{a,b,\alpha}(\{i\},v)}{a_i+a_j},$$

it follows that

$$\begin{aligned}
\psi_i^{a,b,\alpha}(\{i,j\},v) &= a_i\frac{(1-\alpha)v(\{i,j\})+b_iP^{a,b,\alpha}(\{j\},v)+b_jP^{a,b,\alpha}(\{i\},v)}{a_i+a_j}\\
&\quad -b_iP^{a,b,\alpha}(\{j\},v)+\frac{\alpha v(\{i,j\})}{2}.
\end{aligned}$$

As λ-standardness implies $\psi_i^{a,b,\alpha}(\{i,j\},v) = \frac{v(\{i,j\})+(1-\lambda)v(\{i\})-(1-\lambda)v(\{j\})}{2}$ and isolating parts depending on $v(\{i,j\})$ we obtain

$$\frac{a_i}{a_i+a_j}(1-\alpha)\,v(\{i,j\}) + \frac{\alpha}{2}v(\{i,j\}) = \frac{1}{2}v(\{i,j\}).$$

This in turn implies $\frac{a_i}{a_i+a_j}(1-\alpha)\,v(\{i,j\}) = \frac{1}{2}(1-\alpha)\,v(\{i,j\})$. As $(\{i,j\},v)$ was chosen arbitrarily and $\alpha \neq 1$, we may conclude $a_i = a_j$ for all $i,j \in Z$.

Let τ denote the real number satisfying $a_i = \tau$ for all $i \in Z$. Then, substituting τ and isolating those parts depending on $v(\{i\})$, we obtain:

$$\frac{\tau}{\tau+\tau}b_j\frac{(1-\alpha)v(\{i\})}{\tau} = \frac{(1-\lambda)v(\{i\})}{2}.$$

Since $(\{i,j\},v)$ was chosen arbitrarily and $\alpha \neq 1$, $b_j = \tau\frac{1-\lambda}{1-\alpha}$ for all $j \in Z$. Finally, isolating the part depending on $v(\{j\})$ and substituting the real numbers obtained yields that the following equation must hold:

$$\left[\frac{\tau}{\tau+\tau}-1\right]\tau\frac{1-\lambda}{1-\alpha}P^{a,b,\alpha}(\{j\},v) = -\frac{(1-\lambda)}{2}v(\{j\}).$$

This in turn implies that it must hold that

$$\begin{aligned} 0 &= -\frac{1}{2}\tau\frac{1-\lambda}{1-\alpha}P^{a,b,\alpha}(\{j\},v) + \frac{(1-\lambda)}{2}v(\{j\}) \\ &= -\frac{1}{2}\tau\frac{1-\lambda}{1-\alpha}\frac{(1-\alpha)v(\{j\})}{\tau} + \frac{(1-\lambda)}{2}v(\{j\}) \\ &= -\frac{(1-\lambda)}{2}\frac{\tau}{\tau}\frac{(1-\alpha)}{1-\alpha}v(\{j\}) + \frac{(1-\lambda)}{2}v(\{j\}) = 0. \end{aligned}$$

Since the latter did not yield a contradiction this completes the proof.

Proof of Proposition 10 : Part **(ii) implies (i)**: Straightforward. **(i) implies (ii)**: Observe that for all (N,v), $T \subseteq N$, $P^{a,b,\alpha}(N\backslash\{k\},cu_T) = 0$ for all $k \in T$, as $(N\backslash\{k\},cu_T)$ is a zero-game. So we obtain $\psi_k^{a,b,\alpha}(N,cu_T) = a_k P^{a,b,\alpha}(N,cu_T) + \frac{\alpha c}{|N|}$. Let $i,j \in Z$, clearly $\psi_i^{a,b,\alpha}(\{i,j\},v_0) = \psi_j^{a,b,\alpha}(\{i,j\},v_0) = 0$, and $(\{i,j\},u_{\{i,j\}}) = (\{i,j\},v_0+u_{\{i,j\}})$. By equal coalitional improvement $\psi_i^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) - \psi_j^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) = 0$, hence,

$$a_i P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) + \frac{\alpha}{|N|} = a_j P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) + \frac{\alpha}{|N|}$$

which proves $a_i = a_j$.

(iii) is equivalent to (i): Observe that for $(N,v) \in G$, and nonempty $U \subseteq N$,

$$v^{U,\psi^{a,b,\alpha},\alpha} = v^{U,\psi^{a,b,0}} + \alpha\left[\sum_{k\in U}\psi_k^{a,b,0}(N,v) - \frac{|U|}{|N|}v(N)\right]u_{N\backslash U}.$$

Let $\alpha \notin \{0,1\}$, let $\emptyset \neq U \subset N$, then for $i \in N\backslash U$

$$\begin{aligned} &\psi_i^{a,b,0}\left(N\backslash U, v^{U,\psi^{a,b,0},\alpha}\right) \\ =\;& \psi_i^{a,b,0}\left(N\backslash U, v^{U,\psi^{a,b,0}} + \alpha\left[\sum_{k\in U}\psi_k^{a,b,0}(N,v) - \frac{|U|}{|N|}v(N)\right]u_{N\backslash U}\right) \\ =\;& \psi_i^{a,b,0}(N,v) + \alpha\left[\sum_{k\in U}\psi_k^{a,b,0}(N,v) - \frac{|U|}{|N|}v(N)\right]\psi_i^{a,b,0}\left(N\backslash U, u_{N\backslash U}\right). \end{aligned}$$

The second equality follows from linearity, and from HM-consistency of $\psi^{a,b,0}$ applied to the first part between the brackets. Furthermore,

$$\eta_i\left(N\backslash U, v^{U,\psi^{a,b,\alpha},\alpha}\right) = \eta_i(N,v) - (1-\alpha)\frac{\left[\sum_{k\in U}\psi_k^{a,b,0}(N,v) - \frac{|U|}{|N|}v(N)\right]}{|N\backslash U|}.$$

It is now a matter of calculation to find that $\psi_i^{a,b,\alpha}\left(N\backslash U, v^{U,\psi^{a,b,\alpha},\alpha}\right) = \psi_i^{a,b,\alpha}(N,v)$ if and only if $\psi_i^{a,b,0}\left(N\backslash U, u_{N\backslash U}\right) = \frac{1}{|N\backslash U|}$. The latter holds if and only if $\psi^{a,b,0}$ (and hence, $\psi^{a,b,\alpha}$) satisfies equal coalitional improvement.

Proof of Proposition 12: Lemma 6 implies that for all $(N,v) \in G$, $\varphi^{\alpha}_{\frac{1-\lambda}{1-\alpha}}(N,v) = \varphi^{\alpha}\left(N, v_{\frac{1-\lambda}{1-\alpha}}\right)$. By Prop. 4, φ^{α} is characterized by α-consistency and α-standardness, or by Sobolev-consistency and α-standardness. So, α-consistency and Sobolev-consistency of $\varphi^{\alpha}_{\frac{1-\lambda}{1-\alpha}}$ are immediate and λ-standard-ness follows by writing out α-standardness for a 2-player $\frac{1-\lambda}{1-\alpha}$-discounted game.

Proof of Proposition 13: Clearly, $\varphi^{w,0}_{1-\lambda}(N,v) = \varphi^{w}(N, v_{1-\lambda})$ for all games (N,v). Since φ^{w} is uniquely determined by HM-consistency and w-proportionality, this implies that $\varphi^{w,0}_{1-\lambda}$ is uniquely determined by HM-consistency and (w,λ)-proportionality.

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