SIGNAL RECOVERY AND SYNTHESIS WITH INCOMPLETE INFORMATION AND PARTIAL CONSTRAINTS

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TOPICAL MEETING ON SIGNAL RECOVERY AND SYNTHESIS WITH INCOMPLETE INFORMATION AND PARTIAL CONSTRAINTS

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A technique for the calculation of the global extremum of a function of several variables.

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Introduction

The determination of the global extremum of a function is a notorious numerical problem when there are local extrema present. The numerical algorithm which has to determine the global extremum iteratively is in this case very likely to produce a local extremum in the neighborhood of the initial guess of the solution. Moreover, usually one cannot be sure not to have missed an extremum, i.e. various procedures together with various initial values might still overlook the global extremum.

It would therefore be of great value if the total number of stationary points of a function in a certain domain could be calculated easily and exactly: This information would tell us whether or not the numerical algorithm had determined all stationary points.

The stationary points of a function $f(\underline{x})$ are the zeros of the set of eqs. $\nabla f(\underline{x})=0$, and we are therefore looking for a simple procedure determining the number of zeros of a set of eqs. in a certain domain D in the $\underline{x}$-space.

Such a procedure is provided by an integral derived by Picard [1] from previous work by Kronecker [2]. The integration is over the domain of interest, whereas the integrand only contains simple algebraic functions (viz.eqs.(1), (2), (3)). An extensive discussion of this integral is given by Hoenders and Slump [3].

This so-called Picard-Kronecker integral (P.K.integral) is equal to the number of zeros of a set of eqs. in a certain domain provided that all these zeros are simple. The case of multiple zeros will be discussed in a forthcoming paper. (See also Davidoglou [4] and Tzitzéica [5]).

We will illustrate the use of this elegant formalism with the following problem: Estimate the position a and width σ of a Gaussian wave in a Poissonian pulse train using the maximum likelihood method.

The estimation boils down to the determination of the absolute maximum of the likelihood function in a certain domain of the two parameter space $\underline{x} = (a,\sigma)$ (We neglect the possibility of extreme values occurring at the boundary which is only an inessential complication).

The P.K. integral *then provides* an excellent tool for the calculation of the total number of stationary points of the likelihood function $L(\underline{x})$ in a domain. This number is very useful for the calculation of these points.

We determine to this end with the P.K. integral the total number of roots of the set of eqs. $\nabla L(\underline{x}) = 0$ in the domain of interest and subdivide this domain (nesting) till only one or no root is located in the domain. The position of a stationary point can then eventually be determined with zero locating numerical procedures, e.g. Powell [6].

Theory

The basic idea of the P.K. integral is the generalisation of the concept of the solid angle into higher dimensional space. It is then intuitively clear that the solid angle is a measure for the number of zeros of a set of functions in a certain domain by the following argument: Suppose that in two dimensions *the transformation*: $x = x(u,v)$ and $y=y(u,v)$ admits n zeros in a certain domain D with boundary S of the uv-plane. One would then expect that the solid angle connected with the surface S is equal to $0, \pm 2\pi, \pm 4\pi, \ldots \pm 2\pi n$ depending on the orientation of the mapping $(x,y)\rightarrow(u,v)$ in the neighborhood of zeros. This idea was put into an *exact* analytical form by Kronecker [2]. (See for a modern formulation and application of this concept in algebraic topology and functional analysis Schwartz [7]). The Kronecker integral is not conclusive for the calculation of the number of zeros as it is proportional to the number of zeros with positive Jacobian minus the number *of zeros* with negative Jacobian.

Picard [1] showed how to use this idea to obtain the exact number of zeros of a set of eqs. in a certain domain by a very simple extension of the original set of eqs. He derived that the number n of simple zeros of the function $y = f(x)$ in the interval $a \leq x \leq b$ is equal to:

$$n = -(\pi)^{-1} \varepsilon \int_a^b \frac{f(x)\ f''(x) - f'^2(x)}{f^2(x) + \left(\varepsilon\ f'(x)\right)^2}\, dx + (\pi)^{-1} \operatorname{arctg} \left(\frac{\varepsilon f'(y)}{f(y)}\right)\Bigg|_a^b , \tag{1}$$

where ε denotes an arbitrary constant.

The number of simple zeros n *of the* equations: $f(x,y)=0$; $g(x,y)=0$ in the domain D with boundary S and Jacobian J can be shown to be equal to (no zeros are allowed on S):

$$n = (2\pi)^{-1} \int_S (P dx + Q dy) + \varepsilon (2\pi)^{-1} \iint_D R(f^2+g^2+\varepsilon^2 J^2)^{-\frac{3}{2}} dx dy , \tag{2}$$

where $P = (f \frac{\partial g}{\partial x} - g \frac{\partial f}{\partial x})(f^2 + g^2)^{-1}(f^2 + g^2 + \varepsilon^2 J^2)^{-\frac{1}{2}}$, (3)

and Q is obtained from P changing $\frac{\partial}{\partial x}$ into $\frac{\partial}{\partial y}$. R is the determinant $\frac{\partial(zf,zg,zJ)}{\partial(z,x,y)}$ evaluated at z = 1.

The r.h.s. of eqs. (1) and (2) can be shown to be independent of ε,Picard [1].

Example.

In e.g. optical communication practice and in the processing of seismic signals one encounters the problem how to detect a partially known wave shape in measurements corrupted by noise. We consider the problem of the estimation of the position a and (half)-width σ of a Gaussian wave shape in a Poissonian pulse train,(see fig.1).The measurements are assumed to consist of $\underline{n} = (n_1, n_2, ..n_N)$ Poisson - distributed random variables with parameter

$$\lambda_{t_i}(a,\sigma) = \alpha\left[1 + \beta(2\pi)^{-\frac{1}{2}} \sigma^{-1}\exp\left\{-(2\sigma^2)^{-1}(t_i-a)^2\right\}\right]+\lambda_0, \quad (4)$$

$i = 1,...N$, λ_0 denotes the dark current of the detector, α and β are constants (see fig.1), a and σ are estimated by the absolute maximum of the likelihood function:

$$L(\underline{n};a,\sigma) = \prod_{i=1}^{N} \exp(-\lambda_{t_i}(a,\sigma))\lambda_{t_i}(a,\sigma)^{n_i}/n_i!. \quad (5)$$

Applying the method described above to the set of eqs. $\frac{\partial}{\partial a} \ln L = 0$, $\frac{\partial}{\partial \sigma} \ln L = 0$ reveals that there are 4 zeros located in the region $50.0 \leqq a \leqq 80.0$; $2.5 \leqq \sigma \leqq 12.5$. Subdividing the domain and applying [6] leads to the stationary points in table 1.

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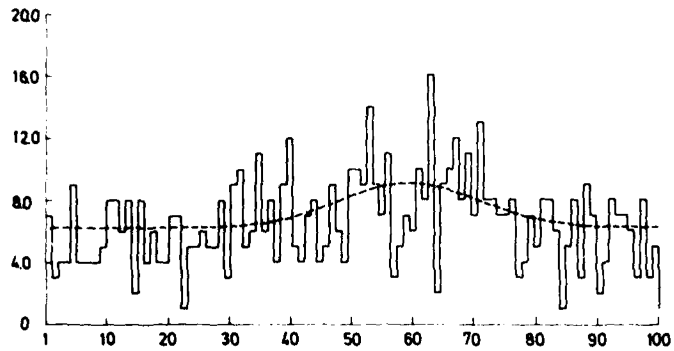

fig.1. The simulated Poissonian pulse train, (drawn line) and λ_t (dotted line), with the values: α =1.75 β=18.0 a =60.0 σ= 8.0 λ_0= 4.0 N=100

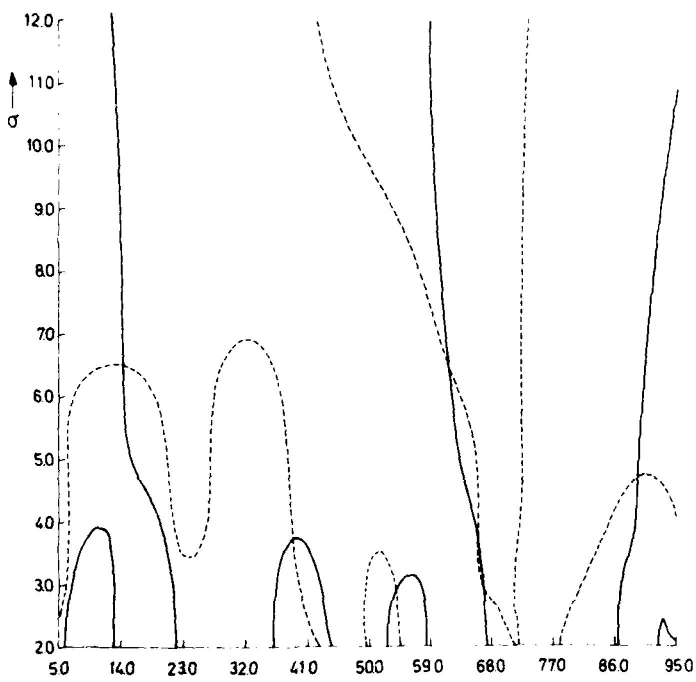

fig.2. $\frac{\partial}{\partial a} \ln L() = 0$ (drawn line) $\frac{\partial}{\partial \sigma} \ln L() = 0$ (dotted line)

Table 1.

a	σ	Magnitude sequence
53.74	4.12	4
62.88	7.59	2
65.47	5.78	3
66.87	4.26	1