

Estimation of a regular conditional functional by conditional U-statistics regression

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Abstract

U-statistics constitute a large class of estimators, generalizing the empirical mean of a random variable \mathbf{X} to sums over every k -tuple of distinct observations of \mathbf{X} . They may be used to estimate a regular functional $\theta(\mathbb{P}_{\mathbf{X}})$ of the law of \mathbf{X} . When a vector of covariates \mathbf{Z} is available, a conditional U-statistic may describe the effect of \mathbf{z} on the conditional law of \mathbf{X} given $\mathbf{Z} = \mathbf{z}$, by estimating a regular conditional functional $\theta(\mathbb{P}_{\mathbf{X}|\mathbf{Z}=\cdot})$. We prove concentration inequalities for conditional U-statistics. Assuming a parametric model of the conditional functional of interest, we propose a regression-type estimator based on conditional U-statistics. Its theoretical properties are derived, first in a non-asymptotic framework and then in two different asymptotic regimes. Some examples are given to illustrate our methods.

Keywords: U-statistics, regression-type models, conditional distribution, penalized regression

2010 MSC: 62F12, 62G05, 62J99

1. Introduction

Let \mathbf{X} be a random element with values in a measurable space $(\mathcal{X}, \mathcal{A})$, and denote by $\mathbb{P}_{\mathbf{X}}$ its law. A natural framework is $\mathcal{X} = \mathbb{R}^{p_{\mathbf{X}}}$, for a fixed dimension $p_{\mathbf{X}} > 0$. Often, we are interested in estimating a regular functional $\theta(\mathbb{P}_{\mathbf{X}})$ of the law of \mathbf{X} , of the form

$$\theta(\mathbb{P}_{\mathbf{X}}) = \mathbb{E}[g(\mathbf{X}_1, \dots, \mathbf{X}_k)] = \int g(\mathbf{x}_1, \dots, \mathbf{x}_k) d\mathbb{P}_{\mathbf{X}}(\mathbf{x}_1) \cdots d\mathbb{P}_{\mathbf{X}}(\mathbf{x}_k),$$

for a fixed $k > 0$, a function $g : \mathcal{X}^k \rightarrow \mathbb{R}$ and $\mathbf{X}_1, \dots, \mathbf{X}_k \stackrel{\text{i.i.d.}}{\sim} \mathbb{P}_{\mathbf{X}}$. Following Hoeffding [6], a natural estimator of $\theta(\mathbb{P}_{\mathbf{X}})$ is the U-statistics $\hat{\theta}(\mathbb{P}_{\mathbf{X}})$, defined by

$$\hat{\theta}(\mathbb{P}_{\mathbf{X}}) := |\mathcal{J}_{k,n}|^{-1} \sum_{\sigma \in \mathcal{J}_{k,n}} g(\mathbf{X}_{\sigma(1)}, \dots, \mathbf{X}_{\sigma(k)}),$$

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where $\mathcal{J}_{k,n}$ is the set of injective functions from $\{1, \dots, k\}$ to $\{1, \dots, n\}$. For an introduction to the theory of U-statistics, we refer to Koroljuk and Borovskich [8] and Serfling [10, Chapter 5]

In our framework, we assume that we actually observe (\mathbf{X}, \mathbf{Z}) where \mathbf{Z} is a p -dimensional covariate. We are now interested in regular functionals of the conditional law $\mathbb{P}_{\mathbf{X}|\mathbf{Z}}$. For each $\mathbf{z}_1, \dots, \mathbf{z}_k \in \mathcal{Z}$, where \mathcal{Z} is a compact subset of \mathbb{R}^p , we can define such a functional $\theta_{\mathbf{z}_1, \dots, \mathbf{z}_k}$ by

$$\begin{aligned} \theta_{\mathbf{z}_1, \dots, \mathbf{z}_k}(\mathbb{P}_{\mathbf{X}|\mathbf{Z}=\cdot}) &:= \theta(\mathbb{P}_{\mathbf{X}|\mathbf{Z}=\mathbf{z}_1}, \dots, \mathbb{P}_{\mathbf{X}|\mathbf{Z}=\mathbf{z}_k}) \\ &= \mathbb{E}_{\otimes_{i=1}^k \mathbb{P}_{\mathbf{X}|\mathbf{Z}=\mathbf{z}_i}} [g(\mathbf{X}_1, \dots, \mathbf{X}_k)] = \mathbb{E}[g(\mathbf{X}_1, \dots, \mathbf{X}_k) | \mathbf{Z}_i = \mathbf{z}_i, \forall i = 1, \dots, k] \\ &= \int g(\mathbf{x}_1, \dots, \mathbf{x}_k) d\mathbb{P}_{\mathbf{X}|\mathbf{Z}=\mathbf{z}_1}(\mathbf{x}_1) \cdots d\mathbb{P}_{\mathbf{X}|\mathbf{Z}=\mathbf{z}_k}(\mathbf{x}_k). \end{aligned}$$

This can be seen as a generalization of $\theta(\mathbb{P}_{\mathbf{X}})$ to the conditional case. Indeed, when \mathbf{X} and \mathbf{Z} are independent, the new functional $\theta_{\mathbf{z}_1, \dots, \mathbf{z}_k}(\mathbb{P}_{\mathbf{X}|\mathbf{Z}=\cdot})$ is equal to the unconditional functional $\theta(\mathbb{P}_{\mathbf{X}})$. For convenience, we will use the notation $\theta(\mathbf{z}_1, \dots, \mathbf{z}_k) := \theta_{\mathbf{z}_1, \dots, \mathbf{z}_k}(\mathbb{P}_{\mathbf{X}|\mathbf{Z}=\cdot})$, treating the law of (\mathbf{X}, \mathbf{Z}) as fixed (but unknown). Stute [11] defined a kernel-based estimator $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$ of the conditional functional $\theta(\mathbf{z}_1, \dots, \mathbf{z}_k)$ by

$$\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k) := \frac{\sum_{\sigma \in \mathcal{J}_{k,n}} K_h(\mathbf{Z}_{\sigma(1)} - \mathbf{z}_1) \cdots K_h(\mathbf{Z}_{\sigma(k)} - \mathbf{z}_k) g(\mathbf{X}_{\sigma(1)}, \dots, \mathbf{X}_{\sigma(k)})}{\sum_{\sigma \in \mathcal{J}_{k,n}} K_h(\mathbf{Z}_{\sigma(1)} - \mathbf{z}_1) \cdots K_h(\mathbf{Z}_{\sigma(k)} - \mathbf{z}_k)}, \quad (1)$$

where $h > 0$ is the bandwidth, $K(\cdot)$ a kernel on \mathbb{R}^p , $K_h(\cdot) := h^{-p}K(\cdot/h)$, and $(\mathbf{X}_i, \mathbf{Z}_i) \stackrel{\text{i.i.d.}}{\sim} \mathbb{P}_{\mathbf{X}, \mathbf{Z}}$. Stute [11] proved the asymptotic normality of $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$ and its weak and strong consistency. Dony and Mason [5] derived its uniform in bandwidth consistency under VC-type conditions over a class of possible functions g .

Nevertheless, the estimator (1) has several weaknesses. First, the interpretation of the whole hypersurface $(\mathbf{z}_1, \dots, \mathbf{z}_k) \mapsto \hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$ can be difficult. Indeed, the latter curve is of dimension $1 + p \times k$, and it is rather challenging to visualize it even for small values of p and k . Second, for each new k -uplet $(\mathbf{z}_1, \dots, \mathbf{z}_k)$, the computation of $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$ has a cost of $O(n^k)$. Then, if we want to estimate $\hat{\theta}(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)})$ for every $i = 1, \dots, N$, where $(\mathbf{z}_1^{(1)}, \dots, \mathbf{z}_k^{(1)}, \dots, \mathbf{z}_1^{(N)}, \dots, \mathbf{z}_k^{(N)}) \in \mathcal{Z}^{k \times N}$, then the total cost is $O(Nn^k)$. Third, it is well-known that kernel estimators are not very smooth, in the sense that they usually present many spurious local minima and maxima, and this can be a problem in some applications. Therefore, we may want to build estimators which are more regular with respect to the conditioning variables $\mathbf{z}_1, \dots, \mathbf{z}_k$, and have a simple functional form.

Another idea is to decompose the function $(\mathbf{z}_1, \dots, \mathbf{z}_k) \mapsto \theta(\mathbf{z}_1, \dots, \mathbf{z}_k)$ on a basis $(\psi_i)_{i \geq 0}$, generalizing the work of Derumigny and Fermanian [3]. This may not be always easy if the range of the function $\theta(\cdot, \dots, \cdot)$ is a strict subset of \mathbb{R} . In that case, it is always possible to use a ‘‘link function’’ Λ , that would be strictly

increasing and continuously differentiable and such that the range $\Lambda \circ \theta(\cdot, \dots, \cdot)$ is exactly \mathbb{R} . Whatever the choice of Λ (including the identity function), we can decompose the latter function on any basis $(\psi_i)_{i \geq 0}$. If only a finite number $r > 0$ of elements of this basis are necessary to represent the whole function $\Lambda \circ \theta(\cdot, \dots, \cdot)$ over \mathcal{Z}^k , then we have the following parametric model:

$$\forall (\mathbf{z}_1, \dots, \mathbf{z}_k) \in \mathcal{Z}^k, \Lambda(\theta(\mathbf{z}_1, \dots, \mathbf{z}_k)) = \boldsymbol{\psi}(\mathbf{z}_1, \dots, \mathbf{z}_k)^T \boldsymbol{\beta}^*, \quad (2)$$

where $\boldsymbol{\beta}^* \in \mathbb{R}^r$ is the true parameter and $\boldsymbol{\psi}(\cdot) := (\psi_1(\cdot), \dots, \psi_r(\cdot))^T \in \mathbb{R}^r$. In most applications, finding an appropriate basis $\boldsymbol{\psi}$ is not easy. This will depend on the choice of the (conditional) functional θ . Therefore, the most simple solution consists in choosing a concatenation of several well-known basis such as polynomials, exponentials, sinuses and cosinuses, indicator functions, etc... They allow to take into account potential non-linearities and even discontinuities of the function $\Lambda \circ \theta(\cdot, \dots, \cdot)$. For the sake of inference, a necessary condition is the linear independence of such functions, as seen in the following proposition (whose straightforward proof is omitted).

Proposition 1. *The parameter $\boldsymbol{\beta}^*$ is identifiable in Model (2) if and only if the functions $(\psi_1(\cdot), \dots, \psi_r(\cdot))$ are linearly independent $\mathbb{P}_{\mathbf{Z}}^{\otimes n}$ -almost everywhere in the sense that, for all vectors $\mathbf{t} = (t_1, \dots, t_r) \in \mathbb{R}^r$, $\mathbb{P}_{\mathbf{Z}}^{\otimes n}(\boldsymbol{\psi}(\mathbf{Z}_1, \dots, \mathbf{Z}_n)^T \mathbf{t} = 0) = 1 \implies \mathbf{t} = 0$.*

With such a choice of a wide and flexible class of functions, it is likely that not all these functions are relevant. This is known as sparsity, i.e. the number of non-zero coefficients of $\boldsymbol{\beta}^*$, denoted by $|\mathcal{S}| = |\boldsymbol{\beta}^*|_0$ is less than s , for some $s \in \{1, \dots, r\}$. Here, $|\cdot|_0$ denotes the number of non-zero components of a vector of \mathbb{R}^r and \mathcal{S} is the set of non-zero components of $\boldsymbol{\beta}^*$. Note that, in this framework, r can be moderately large, for example 30 or 50, while the original dimension p is small, for example $p = 1$ or 2. This corresponds to the decomposition of a function, defined on a small-dimension domain, in a mildly large basis.

Remark 2. *At first sight, in Model (2), there seems to be no noise perturbing the variable of interest. In fact, this can be seen as a simple consequence of our formulation of the model. In the same way, the classical linear model $Y = \mathbf{X}^T \boldsymbol{\beta}^* + \varepsilon$ can be rewritten as $\mathbb{E}[Y | \mathbf{X} = \mathbf{x}] = \mathbf{x}^T \boldsymbol{\beta}^*$ without any explicit noise. By definition, $\mathbb{E}[Y | \mathbf{X} = \mathbf{x}]$ is a deterministic function of a given \mathbf{x} . In our case, the corresponding fact is: $\Lambda(\theta(\mathbf{z}_1, \dots, \mathbf{z}_k))$ is a deterministic function of the variables $(\mathbf{z}_1, \dots, \mathbf{z}_k)$. This means that we cannot write formally a model with noise, such as $\Lambda(\theta(\mathbf{z}_1, \dots, \mathbf{z}_k)) = \boldsymbol{\psi}(\mathbf{z}_1, \dots, \mathbf{z}_k)^T \boldsymbol{\beta}^* + \varepsilon$ where ε is independent of the choice of $(\mathbf{z}_1, \dots, \mathbf{z}_k)$ since the left-hand side of the latter equality is a $(\mathbf{z}_1, \dots, \mathbf{z}_k)$ -measurable quantity, unless ε is constant almost surely.*

Contrary to more usual models, the explained variable $\Lambda(\theta(\mathbf{z}_1, \dots, \mathbf{z}_k))$, is not observed in Model (2).

Therefore, a direct estimation of the parameter β^* (for example, by the ordinary least squares, or by the Lasso) is unfeasible. In other words, even if the function $(\mathbf{z}_1, \dots, \mathbf{z}_k) \mapsto \Lambda(\theta(\mathbf{z}_1, \dots, \mathbf{z}_k))$ is deterministic (by definition of conditional probabilities), finding the best β in Model (2) is far from being a numerical analysis problem since the function to be decomposed is unknown. Nevertheless, we will replace $\Lambda(\theta(\mathbf{z}_1, \dots, \mathbf{z}_k))$ by the nonparametric estimate $\Lambda(\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k))$, and use it as an approximation of the explained variable.

More precisely, we fix a finite collection of points $\mathbf{z}'_1, \dots, \mathbf{z}'_{n'} \in \mathcal{Z}^{n'}$ and a collection $\mathcal{J}_{k,n'}$ of injective functions $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, n'\}$. Note that we are not forced to include *all* the injective functions in $\mathcal{J}_{k,n'}$, reducing its number of elements. This will allow us to decrease the computational cost of the procedure. For every $\sigma \in \mathcal{J}_{k,n'}$, we estimate $\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})$. Finally, the estimator $\hat{\beta}$ is defined as the minimizer of the following l_1 -penalized criteria

$$\hat{\beta} := \arg \min_{\beta \in \mathbb{R}^r} \left[\frac{(n' - k)!}{n'^!} \sum_{\sigma \in \mathcal{J}_{k,n'}} \left(\Lambda(\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})) - \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T \beta \right)^2 + \lambda |\beta|_1 \right], \quad (3)$$

where λ is a positive tuning parameter (that may depend on n and n'), and $|\cdot|_q$ denotes the l_q norm, for $1 \leq q \leq \infty$. This procedure is summed up in the following Algorithm 1. Note that even if we study the general case with any $\lambda \geq 0$, the corresponding properties of the unpenalized estimator can be derived by choosing the particular case $\lambda = 0$.

Algorithm 1: Two-step estimation of β and prediction of the conditional parameters $\theta(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)})$, for $i = 1, \dots, N$

Input: A dataset $(X_{i,1}, X_{i,2}, \mathbf{Z}_i)$, $i = 1, \dots, n$

Input: A finite collection of points $\mathbf{z}'_1, \dots, \mathbf{z}'_{n'} \in \mathcal{Z}^{n'}$, selected for estimation

Input: A collection of N k -tuples for prediction $(\mathbf{z}_1^{(1)}, \dots, \mathbf{z}_k^{(1)}, \dots, \mathbf{z}_1^{(N)}, \dots, \mathbf{z}_k^{(N)}) \in \mathcal{Z}^{k \times N}$

for $\sigma \in \mathcal{J}_{k,n'}$ **do**

 | Compute the estimator $\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})$ using the sample $(\mathbf{X}_i, \mathbf{Z}_i)$, $i = 1, \dots, n$;

end

Compute the minimizer $\hat{\beta}$ of (3) using the $\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})$, $j = 1, \dots, n'$, estimated in the above step ;

for $i \leftarrow 1$ **to** N **do**

 | Compute the prediction $\tilde{\theta}(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)}) := \Lambda^{(-1)}(\boldsymbol{\psi}(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)})^T \hat{\beta})$;

end

Output: An estimator $\hat{\beta}$ and N predictions $\tilde{\theta}(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)})$, $i = 1, \dots, N$.

Once an estimator $\hat{\beta}$ of β^* has been computed, the prediction of all the conditional functionals is reduced to the computation of $\Lambda^{(-1)}(\boldsymbol{\psi}(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)})^T \hat{\beta}) := \tilde{\theta}(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)})$, for every $i = 1, \dots, N$. The total computational cost of this new method is therefore $O(|\mathcal{J}_{k,n'}|n'^k + |\mathcal{J}_{k,n'}|r + Ns)$ operations. The first term corresponds to the cost of evaluating each non-parametric estimator (1). The second term corresponds to

the minimization of the convex optimization program (3), and the last one is the prediction cost. Note that the procedure described in Algorithm 1 can provide a huge improvement compared to the previously available estimator with a cost in $O(Nn^k)$ when $N \rightarrow \infty$, i.e. when we want to recover the full function $\theta(\cdot, \dots, \cdot)$. Moreover, the speed-up given by Algorithm 1 compared to the original conditional U-statistics (1) even increases with the sample size n , for moderate choices of n' .

A similar model, called *functional response*, has already been studied: see, e.g. Kowalski and Tu [9, Chapter 6.2]. They provide a method to estimate the parameter β^* , using generalized estimating equations. However, they only provide asymptotic results for their estimator, and their algorithm needs to solve a multi-dimensional equation which has no reason to be convex.

In Section 2, we provide non-asymptotic bounds for the non-parametric estimator $\hat{\theta}$. Then Section 3 is devoted to the statement of corresponding bounds, as well as asymptotic properties for the parametric estimator $\hat{\beta}$. Finally, a few examples are presented in Section 4. All proofs have been postponed to the Appendix.

2. Theoretical properties of the nonparametric estimator $\hat{\theta}(\cdot)$

2.1. Non-asymptotic bounds for N_k

We remark that the estimator $\hat{\theta}$ is well-defined if and only if $N_k(\mathbf{z}_1, \dots, \mathbf{z}_k) > 0$, where

$$N_k(\mathbf{z}_1, \dots, \mathbf{z}_k) := \frac{k!(n-k)!}{n!} \sum_{\sigma \in \mathfrak{J}_{k,n}^\uparrow} K_h(\mathbf{Z}_{\sigma(1)} - \mathbf{z}_1) \cdots K_h(\mathbf{Z}_{\sigma(k)} - \mathbf{z}_k). \quad (4)$$

To prove that our estimator $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$ exists with a probability that tends to 1, we will therefore study the behavior of N_k . We will need the following assumptions to control the kernel K and the density of \mathbf{Z} .

Assumption 1. *The kernel $K(\cdot)$ is bounded, i.e. there exists a finite constant C_K such that $K(\cdot) \leq C_K$ and $\int K(\mathbf{u}) d\mathbf{u} = 1$. The kernel is of order α for some $\alpha > 0$, i.e. for all $j = 1, \dots, \alpha - 1$ and all $1 \leq i_1, \dots, i_\alpha \leq p$, $\int K(\mathbf{u}) u_{i_1} \dots u_{i_j} d\mathbf{u} = 0$.*

Assumption 2. *$f_{\mathbf{Z}}$ is α -times continuously differentiable on \mathcal{Z} and there exists a finite constant $C_{K,\alpha}$ such that, for all $\mathbf{z}_1, \dots, \mathbf{z}_k$,*

$$\int \left| K(\mathbf{u}_1) \cdots K(\mathbf{u}_k) \right| \sum_{m_1 + \dots + m_k = \alpha} \binom{\alpha}{m_{1:k}} \cdot \prod_{i=1}^k \sum_{j_1, \dots, j_{m_i}=1}^p |u_{i,j_1} \cdots u_{i,j_{m_i}}| \sup_{t \in [0,1]} \left| \frac{\partial^{m_i} f_{\mathbf{Z}}}{\partial z_{j_1} \cdots \partial z_{j_{m_i}}}(\mathbf{z}_i + t\mathbf{u}_i) \right| d\mathbf{u}_1 \cdots d\mathbf{u}_k \leq C_{K,\alpha}$$

where $\binom{\alpha}{m_1, \dots, m_k} := \alpha! / (\prod_{i=1}^k m_i!)$ is the multinomial coefficient.

Assumption 3. $f_{\mathbf{Z}}(\cdot) \leq f_{\mathbf{Z}, \max}$ for some finite constant $f_{\mathbf{Z}, \max}$.

Lemma 3. Under Assumptions 1, 2 and 3, we have for any $t > 0$,

$$\mathbb{P}\left(\left|N_k(\mathbf{z}_1, \dots, \mathbf{z}_k) - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)\right| \leq \frac{C_{K, \alpha}}{\alpha!} h^\alpha + t\right) \geq 1 - 2 \exp\left(-\frac{[n/k]t^2}{h^{-kp}C_1 + h^{-kp}C_2 t}\right),$$

where $C_1 := 2f_{\mathbf{Z}, \max}^k \|K\|_2^{2k}$, and $C_2 := (4/3)C_K^k$ and $\|K\|_2^2 := \int K^2$.

This Lemma is proved in Appendix C.1. More can be said if the density $f_{\mathbf{Z}}$ is bounded below. Therefore, we will use the following assumption.

Assumption 4. There exists a constant $f_{\mathbf{Z}, \min} > 0$ such that for every $\mathbf{z} \in \mathcal{Z}$, $f_{\mathbf{Z}}(\mathbf{z}) > f_{\mathbf{Z}, \min}$.

If for some $\epsilon > 0$, we have $C_{K, \alpha} h^\alpha / \alpha! + t \leq f_{\mathbf{Z}, \min} - \epsilon$, then $\hat{f}(\mathbf{z}) \geq \epsilon > 0$ with probability larger than on the event whose probability is bound in Lemma 3. We should therefore choose the largest t possible, which yields the following corollary.

Corollary 4. Under Assumptions 1-4, if $C_{K, \alpha} h^\alpha / \alpha! < f_{\mathbf{Z}, \min}$, then the random variable $N_k(\mathbf{z}_1, \dots, \mathbf{z}_k)$ is strictly positive with a probability larger than $1 - 2 \exp\left(-\frac{[n/k]h^{kp}(f_{\mathbf{Z}, \min} - C_{K, \alpha} h^\alpha / \alpha!)^2}{C_1 + C_2(f_{\mathbf{Z}, \min} - C_{K, \alpha} h^\alpha / \alpha!)}\right)$, guaranteeing the existence of the estimator $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$ on this event.

2.2. Non-asymptotic bounds in probability for $\hat{\theta}$

In this section, we generalize the bounds given in [4] for the conditional Kendall's tau to any conditional U-statistics. To establish bounds on $\hat{\theta}$ for every fixed n , we will need some assumptions on the joint law of (\mathbf{X}, \mathbf{Z}) .

Assumption 5. There exists a measure μ on $(\mathcal{X}, \mathcal{A})$ such that $\mathbb{P}_{\mathbf{X}, \mathbf{Z}}$ is absolutely continuous with respect to $\mu \otimes \text{Leb}_p$, where Leb_p is the Lebesgue measure on \mathbb{R}^p .

Assumption 6. For every $\mathbf{x} \in \mathcal{X}$, $\mathbf{z} \mapsto f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}, \mathbf{z})$ is differentiable almost everywhere up to the order α . Moreover, there exists a finite constant $C_{g, f, \alpha} > 0$, such that, for every positive integers m_1, \dots, m_k such that $\sum_{i=1}^k m_i = \alpha$, for every $0 \leq j_1, \dots, j_{m_i} \leq p$,

$$\int \prod_{i=1}^k \left| \left(g(\mathbf{x}_1, \dots, \mathbf{x}_k) - \mathbb{E}[g(\mathbf{X}_1, \dots, \mathbf{X}_k) | \mathbf{Z}_i = \mathbf{z}_i, \forall i = 1, \dots, k] \right) \cdot \left(\frac{\partial^{m_i} f_{\mathbf{X}, \mathbf{Z}}}{\partial z_{j_1} \dots \partial z_{j_{m_i}}}(\mathbf{x}_i, \mathbf{z}_i + \mathbf{u}_i) - \frac{\partial^{m_i} f_{\mathbf{X}, \mathbf{Z}}}{\partial z_{j_1} \dots \partial z_{j_{m_i}}}(\mathbf{x}_i, \mathbf{z}_i) \right) \right| d\mu(\mathbf{x}_1) \dots d\mu(\mathbf{x}_k) \leq C_{g, f, \alpha} \prod_{i=1}^k \|\mathbf{u}_i\|_\infty,$$

for every choices of $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathcal{X}$ and $\mathbf{z}_1, \dots, \mathbf{z}_k \in \mathcal{Z}$, $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{R}^p$ such that $\mathbf{z}_i + \mathbf{u}_i \in \mathcal{Z}$. There exists a constant $C'_{K,\alpha}$ such that $\sum_{m_1 + \dots + m_k = \alpha} \binom{\alpha}{m_1:k} \int \prod_{i=1}^k K(\mathbf{u}_i) \sum_{j_1, \dots, j_{m_i}=1}^p u_{i,j_1} \dots u_{i,j_{m_i}} \prod_{i=1}^k \|\mathbf{u}_i\|_\infty d\mathbf{u}_1 \dots d\mathbf{u}_k \leq C'_{K,\alpha}$.

An easy situation is the case when g is bounded, i.e. when the following assumption hold.

Assumption 7. *There exists a constant C_g such that $\|g\|_\infty \leq C_g < +\infty$.*

When g is not bounded, a weaker result can still be proved under a ‘‘conditional Bernstein’’ assumption. This assumption will help us to control the tail behavior of g so that exponential concentration bounds are available.

Assumption 8 (conditional Bernstein assumption). *There exists a positive function B_g such that, for all $l \geq 1$ and $\mathbf{z}_1, \dots, \mathbf{z}_k \in \mathbb{R}^{kp}$, $\mathbb{E} \left[|g(\mathbf{X}_1, \dots, \mathbf{X}_k)|^l \mid \mathbf{Z}_1 = \mathbf{z}_1, \dots, \mathbf{Z}_k = \mathbf{z}_k \right] \leq B_g(\mathbf{z}_1, \dots, \mathbf{z}_k)^l l!$, such that $B_g(\mathbf{Z}_1, \dots, \mathbf{Z}_k) \leq \tilde{B}_g$ almost surely, for some finite positive constant \tilde{B}_g .*

As a shortcut notation, we will define also $B_{g,\mathbf{z}} := B_g(\mathbf{z}_1, \dots, \mathbf{z}_k)$. The following proposition is proved in Appendix C.2.

Proposition 5 (Exponential bound for the estimator $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$, with fixed $\mathbf{z}_1, \dots, \mathbf{z}_k \in \mathcal{Z}^k$). *Assume either Assumption 7 or the weaker Assumption 8. Under Assumptions 1-6, for every $t, t' > 0$ such that $C_{K,\alpha} h^\alpha / \alpha! + t < f_{\mathbf{Z},\min} / 2$, we have*

$$\begin{aligned} \mathbb{P} \left(\left| \hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k) - \theta(\mathbf{z}_1, \dots, \mathbf{z}_k) \right| < (1 + C_3 h^\alpha + C_4 t) \times (C_5 h^{k+\alpha} + t') \right) \\ \geq 1 - 2 \exp \left(- \frac{[n/k] t^2 h^{kp}}{C_1 + C_2 t} \right) - 2 \exp \left(- \frac{[n/k] t'^2 h^{kp}}{C_6 + C_7 t'} \right), \end{aligned}$$

where $C_3 := 4f_{\mathbf{Z},\max}^k f_{\mathbf{Z},\min}^{-2k} C_{K,\alpha} / \alpha!$, $C_4 := 4f_{\mathbf{Z},\max}^k f_{\mathbf{Z},\min}^{-2k}$ and $C_5 := C_{g,f,\alpha} C'_{K,\alpha} f_{\mathbf{Z},\min}^{-k} / \alpha!$.

If Assumption 7 is satisfied, the result holds with the following values: $C_6 := 2C_g^2 f_{\mathbf{Z},\max}^k f_{\mathbf{Z},\min}^{-2k} \|K\|_2^{2k}$, $C_7 := (8/3)C_K^k C_g^k f_{\mathbf{Z},\min}^{-k}$; in the case of Assumption 8, the result holds with the following alternative values: $\tilde{C}_6 := 128(B_{g,\mathbf{z}} + \tilde{B}_g)^2 C_K^{2k-1} f_{\mathbf{Z},\min}^{-2k}$, $\tilde{C}_7 := 2(B_{g,\mathbf{z}} + \tilde{B}_g) C_K^k f_{\mathbf{Z},\min}^{-k}$.

3. Theoretical properties of the estimator $\hat{\beta}$

Let us define the matrix \mathbb{Z} of dimension $|\mathcal{J}_{k,n'}| \times r$ by $[\mathbb{Z}']_{i,j} := \psi_j(\mathbf{z}'_{\sigma_i(1)}, \dots, \mathbf{z}'_{\sigma_i(k)})$, where $1 \leq i \leq |\mathcal{J}_{k,n'}|$, $1 \leq j \leq r$ and σ_i is the i -th element of $\mathcal{J}_{k,n'}$. The chosen order of $\mathcal{J}_{k,n'}$ is arbitrary and has no impact in practice. In the same way, we define the vector \mathbf{Y} of dimension $|\mathcal{J}_{k,n'}|$ defined by $Y_i := \Lambda \left(\hat{\theta}(\mathbf{z}'_{\sigma_i(1)}, \dots, \mathbf{z}'_{\sigma_i(k)}) \right)$, such that the criterion (3) is in the standard Lasso form $\hat{\beta} := \arg \min_{\beta \in \mathbb{R}^r} \left[\|\mathbf{Y} - \mathbb{Z}'\beta\|^2 + \lambda|\beta|_1 \right]$, where for

any vector \mathbf{v} of size $|\mathcal{J}_{k,n'}|$, its scaled norm is defined by $\|\mathbf{v}\| := |\mathbf{v}|_2 / \sqrt{|\mathcal{J}_{k,n'}|}$. Following [3], we define $\xi_{i,n}$, for $1 \leq i \leq |\mathcal{J}_{k,n'}|$, by $\xi_{i,n} = \xi_{\sigma_i,n} := \Lambda\left(\hat{\theta}(\mathbf{z}'_{\sigma_i(1)}, \dots, \mathbf{z}'_{\sigma_i(k)})\right) - \psi(\mathbf{z}'_{\sigma_i(1)}, \dots, \mathbf{z}'_{\sigma_i(k)})^T \beta^*$.

3.1. Non-asymptotic bounds on $\hat{\beta}$

We will also use the *Restricted Eigenvalue* (RE) condition, introduced by Bickel, Ritov and Tsybakov [2]. For $c_0 > 0$ and $s \in \{1, \dots, p\}$, it is defined as follows:

RE(s, c_0) condition : *The design matrix \mathbb{Z}' satisfies*

$$\kappa(s, c_0) := \min \left\{ \frac{\|\mathbb{Z}'\delta\|}{|\delta|_2} : \delta \neq 0, |\delta_{J_0^c}|_1 \leq c_0 |\delta_{J_0}|_1, J_0 \subset \{1, \dots, r\}, |J_0| \leq s \right\} > 0.$$

Note that this condition is very mild, and is satisfied with a high probability for a large class of random matrices: see Bellec et al. [1, Section 8.1] for references and a discussion. We will also need the following regularity assumption on the function $\Lambda(\cdot)$.

Assumption 9. *The function $\mathbf{z} \mapsto \psi(\mathbf{z})$ are bounded on \mathcal{Z} by a constant C_ψ . Moreover, $\Lambda(\cdot)$ is continuously differentiable. Let \mathcal{T} be the range of θ , from \mathcal{Z}^k towards \mathbb{R} . On an open neighborhood of \mathcal{T} , the derivative of $\Lambda(\cdot)$ is bounded by a constant $C_{\Lambda'}$.*

The following theorem is proved in Appendix C.3.

Theorem 6. *Assume either Assumption 7 or the weaker Assumption 8. Suppose that Assumptions 1-6 and 9 hold and that the design matrix \mathbb{Z}' satisfies the RE($s, 3$) condition. Choose the tuning parameter as $\lambda = \gamma t$, with $\gamma \geq 4$ and $t > 0$, and assume that we choose h small enough such that*

$$h \leq \min \left(\left(\frac{f_{\mathbf{Z}, \min} \alpha!}{4C_{K, \alpha}} \right)^{1/\alpha}, \left(\frac{t}{2C_5 C_8} \right)^{1/(k+\alpha)} \right), \quad (5)$$

where $C_8 := C_\psi C_{\Lambda'} (1 + C_4 f_{\mathbf{Z}, \min} / 2)$. Then, we have

$$\begin{aligned} \mathbb{P} \left(\|\mathbb{Z}'(\hat{\beta} - \beta^*)\| \leq \frac{4(\gamma+1)t\sqrt{s}}{\kappa(s, 3)} \text{ and } |\hat{\beta} - \beta^*|_q \leq \frac{4^{2/q}(\gamma+1)ts^{1/q}}{\kappa^2(s, 3)}, \text{ for every } 1 \leq q \leq 2 \right) \\ \geq 1 - 2 \sum_{\sigma \in \mathcal{J}_{k,n'}} \left[\exp \left(- \frac{[n/k] f_{\mathbf{Z}, \min}^2 h^{kp}}{16C_1 + 4C_2 f_{\mathbf{Z}, \min}} \right) + \exp \left(- \frac{[n/k] t^2 h^{kp}}{4C_8^2 C_{6,\sigma} + 2C_8 C_{7,\sigma} t} \right) \right]. \end{aligned} \quad (6)$$

If Assumption 7 is satisfied, the result holds with $C_{6,\sigma}$ and $C_{7,\sigma}$ constant, respectively to C_6 and C_7 defined in Proposition 5. In the case of Assumption 8, the result holds with the following alternative values: $C_{6,\sigma} := 128(B_g(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) + \tilde{B}_g)^2 C_K^{2k} f_{\mathbf{Z}, \min}^{-2k}$ and $C_{7,\sigma} := 2(B_g(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) + \tilde{B}_g) C_K^k f_{\mathbf{Z}, \min}^{-k}$.

The latter theorem gives some bounds that hold in probability for the prediction error $\|\mathbf{Z}'(\hat{\beta} - \beta^*)\|_{n'}$ and for the estimation error $|\hat{\beta} - \beta^*|_q$ with $1 \leq q \leq 2$ under the specification (2). Note that the influence of n' and r is hidden through the Restricted Eigenvalue number $\kappa(s, 3)$.

3.2. Asymptotic properties of $\hat{\beta}$ when $n \rightarrow \infty$ and for fixed n'

In this part, n' is still assumed to be fixed and we state the consistency and the asymptotic normality of $\hat{\beta}$ as $n \rightarrow \infty$. As above, we adopt a fixed design: the \mathbf{z}'_i are arbitrarily fixed or, equivalently, our reasoning are made conditionally on the second sample. In this section, we follow Section 3 of Derumigny and Fermanian [3] which gives similar results for the conditional Kendall's tau, a particular conditional U-statistic of order 2. Proofs are identical and therefore omitted. Nevertheless, asymptotic properties of $\hat{\beta}$ require corresponding results on the first-step estimators $\hat{\theta}$. These results are state in Stute [11] and recalled for convenience in Appendix B. For $n, n' > 0$, denote by $\hat{\beta}_{n, n'}$ the estimator (3) with $h = h_n$ and $\lambda = \lambda_{n, n'}$.

Lemma 7. *We have $\hat{\beta}_{n, n'} = \arg \min_{\beta \in \mathbb{R}^{p'}} \mathbb{G}_{n, n'}(\beta)$, where*

$$\begin{aligned} \mathbb{G}_{n, n'}(\beta) &:= \frac{2(n' - k)!}{n'^!} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \xi_{\sigma, n} \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T (\beta^* - \beta) \\ &+ \frac{(n' - k)!}{n'^!} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \left\{ \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T (\beta^* - \beta) \right\}^2 + \lambda_{n, n'} |\beta|_1. \end{aligned} \quad (7)$$

Theorem 8 (Consistency of $\hat{\beta}$). *Under Assumption 10, if n' is fixed and $\lambda = \lambda_{n, n'} \rightarrow \lambda_0$, then, given $\mathbf{z}'_1, \dots, \mathbf{z}'_{n'}$ and as n tends to the infinity, $\hat{\beta}_{n, n'} \xrightarrow{\mathbb{P}} \beta^{**} := \inf_{\beta} \mathbb{G}_{\infty, n'}(\beta)$, where*

$$\mathbb{G}_{\infty, n'}(\beta) := \frac{1}{n'} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \left(\boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T (\beta^* - \beta) \right)^2 + \lambda_0 |\beta|_1.$$

In particular, if $\lambda_0 = 0$ and $\langle \{\boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) : \sigma \in \mathfrak{J}_{k, n'}\} \rangle = \mathbb{R}^r$, then $\hat{\beta}_{n, n'} \xrightarrow{\mathbb{P}} \beta^$.*

Theorem 9 (Asymptotic law of the estimator). *Under Assumption 11, and if $\lambda_{n, n'} (nh_{n, n'}^p)^{1/2}$ tends to ℓ when $n \rightarrow \infty$, we have $(nh_{n, n'}^p)^{1/2} (\hat{\beta}_{n, n'} - \beta^*) \xrightarrow{D} \mathbf{u}^* := \arg \min_{\mathbf{u} \in \mathbb{R}^r} \mathbb{F}_{\infty, n'}(\mathbf{u})$, given $\mathbf{z}'_1, \dots, \mathbf{z}'_{n'}$, where*

$$\begin{aligned} \mathbb{F}_{\infty, n'}(\mathbf{u}) &:= \frac{2(n' - k)!}{n'^!} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \sum_{j=1}^r W_{\sigma} \psi_j(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) u_j + \frac{(n' - k)!}{n'^!} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \left(\boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T \mathbf{u} \right)^2 \\ &+ \ell \sum_{i=1}^r (|u_i| \mathbb{1}_{\{\beta_i^* = 0\}} + u_i \text{sign}(\beta_i^*) \mathbb{1}_{\{\beta_i^* \neq 0\}}), \end{aligned}$$

with $\mathbf{W} = (W_\sigma)_{\sigma \in \mathcal{J}_{k,n'}} \sim \mathcal{N}(0, \tilde{\mathbb{H}})$ where

$$\begin{aligned} [\tilde{\mathbb{H}}]_{\sigma, \varsigma} &:= \sum_{j,l=1}^k \mathbb{1}_{\{\mathbf{z}'_{\sigma(j)} = \mathbf{z}'_{\varsigma(l)}\}} \frac{\|K\|_2^2}{f_{\mathbf{Z}}(\mathbf{z}'_{\sigma(j)})} \Lambda' \left(\theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \right) \Lambda' \left(\theta(\mathbf{z}'_{\varsigma(1)}, \dots, \mathbf{z}'_{\varsigma(k)}) \right) \\ &\cdot \left(\tilde{\theta}_{j,l}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}, \mathbf{z}'_{\varsigma(1)}, \dots, \mathbf{z}'_{\varsigma(k)}) - \theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \theta(\mathbf{z}'_{\varsigma(1)}, \dots, \mathbf{z}'_{\varsigma(k)}) \right), \end{aligned}$$

and $\tilde{\theta}_{j,l}$ is as defined in Equation (B.1).

Moreover, $\limsup_{n \rightarrow \infty} \mathbb{P}(\mathcal{S}_n = \mathcal{S}) = c < 1$, where $\mathcal{S}_n := \{j : \hat{\beta}_j \neq 0\}$ and $\mathcal{S} := \{j : \beta_j \neq 0\}$.

A usual way of obtaining the oracle property is to modify our estimator in an ‘‘adaptive’’ way. Following Zou [12], consider a preliminary ‘‘rough’’ estimator of β^* , denoted by $\tilde{\beta}_n$, or more simply $\tilde{\beta}$. Moreover $\nu_n(\tilde{\beta}_n - \beta^*)$ is assumed to be asymptotically normal, for some deterministic sequence (ν_n) that tends to the infinity. Now, let us consider the same optimization program as in (3) but with a random tuning parameter given by $\lambda_{n,n'} := \tilde{\lambda}_{n,n'} / |\tilde{\beta}_n|^\delta$, for some constant $\delta > 0$ and some positive deterministic sequence $(\tilde{\lambda}_{n,n'})$. The corresponding adaptive estimator (solution of the modified Equation (3)) will be denoted by $\check{\beta}_{n,n'}$, or simply $\check{\beta}$. Hereafter, we still set $\mathcal{S}_n = \{j : \check{\beta}_j \neq 0\}$.

Theorem 10 (Asymptotic law of the adaptive estimator of β). *Under Assumption 11, if $\tilde{\lambda}_{n,n'}(nh_{n,n'}^p)^{1/2} \rightarrow \ell \geq 0$ and $\tilde{\lambda}_{n,n'}(nh_{n,n'}^p)^{1/2}\nu_n^\delta \rightarrow \infty$ when $n \rightarrow \infty$, we have $(nh_{n,n'}^p)^{1/2}(\check{\beta}_{n,n'} - \beta^*)_{\mathcal{S}} \xrightarrow{D} \mathbf{u}_{\mathcal{S}}^* := \arg \min_{\mathbf{u}_{\mathcal{S}} \in \mathbb{R}^{\mathcal{S}}} \check{\mathbb{F}}_{\infty,n'}(\mathbf{u}_{\mathcal{S}})$, where*

$$\check{\mathbb{F}}_{\infty,n'}(\mathbf{u}_{\mathcal{S}}) := \frac{2(n' - k)!}{n'!} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{j \in \mathcal{S}} W_\sigma \psi_j(\mathbf{z}'_i) u_j + \frac{(n' - k)!}{n'!} \sum_{\sigma \in \mathcal{J}_{k,n'}} \left(\sum_{j \in \mathcal{S}} \psi_j(\mathbf{z}'_i) u_j \right)^2 + \ell \sum_{i \in \mathcal{S}} \frac{u_i}{|\beta_i^*|^\delta} \text{sign}(\beta_i^*),$$

and $\mathbf{W} = (W_\sigma)_{\sigma \in \mathcal{J}_{k,n'}} \sim \mathcal{N}(0, \tilde{\mathbb{H}})$.

Moreover, when $\ell = 0$, the oracle property is fulfilled: $\mathbb{P}(\mathcal{S}_n = \mathcal{S}) \rightarrow 1$ as $n \rightarrow \infty$.

3.3. Asymptotic properties of $\hat{\beta}$ jointly in (n, n')

Now, we consider the framework in which both n and n' are going to infinity, while the dimensions p and r stay fixed. We now provide a consistency result for $\hat{\beta}_{n,n'}$.

Theorem 11 (Consistency of $\hat{\beta}_{n,n'}$, jointly in (n, n')). *Assume that Assumptions 1-6, 8 and 9 are satisfied. Assume that $\sum_{\sigma \in \mathcal{J}_{k,n'}} \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T / n'$ converges to a matrix $M_{\boldsymbol{\psi}, \mathbf{z}'}$, as $n' \rightarrow \infty$. Assume that $\lambda_{n,n'} \rightarrow \lambda_0$ and $n' \exp(-Anh^{2kp}) \rightarrow 0$ for every $A > 0$, when $(n, n') \rightarrow \infty$. Then $\hat{\beta}_{n,n'} \xrightarrow{\mathbb{P}} \arg \min_{\beta \in \mathbb{R}^r} \mathbb{G}_{\infty, \infty}(\beta)$, as $(n, n') \rightarrow \infty$, where $\mathbb{G}_{\infty, \infty}(\beta) := (\beta^* - \beta) M_{\boldsymbol{\psi}, \mathbf{z}'} (\beta^* - \beta)^T + \lambda_0 |\beta|_1$. Moreover, if $\lambda_0 = 0$ and $M_{\boldsymbol{\psi}, \mathbf{z}'}$ is invertible, then $\hat{\beta}_{n,n'}$ is consistent and tends to the true value β^* .*

Note that, since the sequence (\mathbf{z}'_i) is deterministic, we only assume the convergence of the sequence of deterministic matrices $\sum_{\sigma \in \mathcal{J}_{k,n'}} \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T / n'$ in \mathbb{R}^{r^2} . Moreover, if the “second subset” $(\mathbf{z}'_i)_{i=1, \dots, n'}$ were a random sample (drawn along the law $\mathbb{P}_{\mathbf{Z}}$), the latter convergence would be understood “in probability”. And if $\mathbb{P}_{\mathbf{Z}}$ satisfies the identifiability condition (Proposition 1), then $M_{\boldsymbol{\psi}, \mathbf{z}'}$ would be invertible and $\hat{\beta}_{n,n'} \rightarrow \beta^*$ in probability. Now, we want to go one step further and derive the asymptotic law of the estimator $\hat{\beta}_{n,n'}$.

Theorem 12 (Asymptotic law of $\hat{\beta}_{n,n'}$, jointly in (n, n')). *Under Assumptions 1-5 and under Assumption 12, we have*

$$(n \times n' \times h_{n,n'}^p)^{1/2} (\hat{\beta}_{n,n'} - \beta^*) \xrightarrow{D} \mathcal{N}(0, \tilde{V}_{as}),$$

where $\tilde{V}_{as} := V_1^{-1} V_2 V_1^{-1}$, V_1 is the matrix defined in Assumption 12(iv), and V_2 in Assumption 12(v).

This theorem is proved in Appendix D where we state Assumption 12.

4. Applications and examples

Following Example 4.4 in Stute [11], we consider the function $g(x_1, x_2) := \mathbb{1}\{x_1 \leq x_2\}$, with $k = 2$. In this case $\theta(\mathbf{z}_1, \mathbf{z}_2) = \mathbb{P}(X_1 \leq X_2 | \mathbf{Z}_1 = \mathbf{z}_1, \mathbf{Z}_2 = \mathbf{z}_2)$. The parameter $\theta(\mathbf{z}_1, \mathbf{z}_2)$ quantifies the probability that the quantity of interest X be smaller if we knew that $\mathbf{Z} = \mathbf{z}_1$ than if we knew that $\mathbf{Z} = \mathbf{z}_2$.

To illustrate our methods, we choose a simple example, with the Epanechnikov kernel, defined by $K(x) := (3/4)(1 - u^2)\mathbb{1}\{|u| \leq 1\}$. It is a kernel of order $\alpha = 2$, with $\int K^2 = 3/5$. Assumption 1 is then satisfied with $C_K := 3/4$. Fix $p = 1$, $\mathcal{Z} = [-1, 1]$, $\mathcal{X} = \mathbb{R}$, $f_Z(z) = \phi(z)\mathbb{1}\{|z| \leq 1\} / (1 - 2\Phi(-1))$, where Φ and ϕ are respectively the cdf and the density of the standard Gaussian distribution and $X|Z = z \sim \mathcal{N}(z, 1)$, for every $z \in \mathcal{Z}$.

Assumption 2 is then satisfied with $C_{K,\alpha} = 0.2$. Assumption 3 is easily satisfied with $f_{Z,\max} = 1 / (\sqrt{2\pi}(1 - 2\Phi(-1))) \leq 0.59$. Therefore, we can apply Lemma 3. We compute the constants $C_1 := 2f_{Z,\max}^k \|K\|_2^{2k} = 2 \times 0.59^2 \times (3/5)^2 \leq 0.26$ and $C_2 := (4/3)C_K^k = (4/3) \times (3/4)^2 = 3/4$. Therefore, for any $n \geq 0$, $h, t > 0$, $z_1, z_2 \in \mathcal{Z}$, we have

$$\mathbb{P}\left(|N_2(z_1, z_2) - f_Z(z_1)f_Z(z_2)| \leq 0.1h^\alpha + t\right) \geq 1 - 2 \exp\left(-\frac{[n/2]t^2}{0.26h^2 + 0.75h^2t}\right),$$

Assumption 4 is satisfied with $f_{Z,\min} = \phi(1)/(1 - 2\Phi(-1)) > 0.35$, so that we can apply Corollary 4. Therefore, the estimator $\hat{\theta}(z_1, z_2)$ exists with probability greater than $1 - 2 \exp\left(-\frac{(n-1)h^2(0.35-0.1h^2)^2}{0.52+1.5 \times (0.35-0.1h^2)}\right)$. Note that this probability is greater than 0.99 as soon as $n \geq 3(0.52 + 1.5 \times (0.35 - 0.1h^2)) / (h^2(0.35 - 0.1h^2)^2)$. For

example, with $h = 0.2$, it means that the estimator $\hat{\theta}(z_1, z_2)$ exists with a probability greater than 99% as soon as n is greater than 651.

We list below other possible examples of applications. Conditional moments constitute also a natural class of U-statistics. They include the conditional variance ($p_{\mathbf{X}} = 1, k = 2, g(X_1, X_2) = X_1^2 - X_1 \cdot X_2$) and the conditional covariance ($p_{\mathbf{X}} = 2, k = 2, g(\mathbf{X}_1, \mathbf{X}_2) := X_{1,1} \times X_{2,1} - X_{1,1} \times X_{2,2}$). The conditional variance gives information about the volatility of X given the variable \mathbf{Z} . Conditional covariances can be used to describe how the dependence moves as a function of the conditioning variables \mathbf{Z} . Higher-order conditional moments (skewness, kurtosis, and so on) can also be estimated by higher-order conditional U-statistics, and they described respectively how the asymmetry and the behavior of the tails of X change as function of Z .

Gini's mean difference, an indicator of dispersion, can also be used in this framework. Formally, it is defined as the U-statistic with $p_{\mathbf{X}} = 1, k = 2$ and $g(X_1, X_2) := |X_1 - X_2|$. Its conditional version describes how two variables are far away, on average, given their conditioning variables \mathbf{Z} . for example, X could be the income of an individual, \mathbf{Z} could be the position of its home, and $\theta(\mathbf{z}_1, \mathbf{z}_2)$ represent the average inequality between the income of two persons, one at point \mathbf{z}_1 and the other at point \mathbf{z}_2 .

Other conditional dependence measures can also be written as conditional U-statistics, see e.g. Example 1.1.7 of Koroljuk and Borovskisch [8]. They show how a U-statistic of order $k = 5$ can be used to estimated the dependence parameter

$$\theta = \iint (F_{1,2}(x, y) - F_{1,2}(x, \infty)F_{1,2}(\infty, y))dF_{1,2}(x, y).$$

In our framework, we could consider a conditional version, given by

$$\theta(\mathbf{z}_1, \mathbf{z}_2) = \iint (F_{1,2|\mathbf{Z}=\mathbf{z}}(x, y) - F_{1,2|\mathbf{Z}=\mathbf{z}}(x, \infty)F_{1,2|\mathbf{Z}=\mathbf{z}}(\infty, y))dF_{1,2|\mathbf{Z}=\mathbf{z}}(x, y),$$

where \mathbf{X} is of dimension $p_{\mathbf{X}} = 2$.

Acknowledgements: This work is supported by the GENES and by the Labex Ecodec under the grant ANR-11-LABEX-0047 from the French Agence Nationale de la Recherche. The author thanks Professor Jean-David Fermanian for helpful comments and discussions.

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Appendix A. Notations

In the proofs, we will use the following shortcut notation. First, $\mathbf{x}_{1:k}$ denotes the k -tuple $(\mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathcal{X}^k$. Similarly, for a function σ , $\sigma(1:k)$ denotes the tuple $(\sigma(1), \dots, \sigma(k))$, and $\mathbf{X}_{\sigma(1:k)}$ is the k -tuple $(\mathbf{X}_{\sigma(1)}, \dots, \mathbf{X}_{\sigma(k)})$. For any variable Y and any collection of given points $(\mathbf{z}_1, \dots, \mathbf{z}_k)$, the conditional expectation $\mathbb{E}[Y|\mathbf{Z}_{1:k} = \mathbf{z}_{1:k}]$ denotes $\mathbb{E}[Y|\mathbf{Z}_1 = \mathbf{z}_1, \dots, \mathbf{Z}_k = \mathbf{z}_k]$. We denote by $\int \phi(\mathbf{z}_{1:k}) d\mathbf{z}_{1:k}$ the integral $\int \phi(\mathbf{z}_1, \dots, \mathbf{z}_k) d\mathbf{z}_1 \cdots d\mathbf{z}_k$ for any integrable function $\phi : \mathbb{R}^{k \times p} \rightarrow \mathbb{R}$, and by $\int g(\mathbf{x}_{1:k}) d\mu^{\otimes k}(\mathbf{x}_{1:k})$ the integral $\int g(\mathbf{z}_1, \dots, \mathbf{z}_k) d\mu(\mathbf{x}_1) \cdots d\mu(\mathbf{x}_k)$ for any μ -integrable function $g : \mathcal{X}^k \rightarrow \mathbb{R}$.

Appendix B. Asymptotic results for $\hat{\theta}$

The estimator $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$ has been first studied by Stute (1991) [11]. He proved the consistency and the asymptotic normality of $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k)$. We recall his results.

Assumption 10. (i) $h_n \rightarrow 0$ and $nh_n^p \rightarrow \infty$;

(ii) $K(\mathbf{z}) \geq C_{K,1} \mathbb{1}_{\{|\mathbf{z}|_\infty \leq C_{K,2}\}}$ for some $C_{K,1}, C_{K,2} > 0$;

(iii) there exists a decreasing function $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and positive constants c_1, c_2 such that $H(t) \underset{t \rightarrow \infty}{=} o(t^{-1})$ and $c_1 H(|\mathbf{z}|_\infty) \leq K(\mathbf{z}) \leq c_2 H(|\mathbf{z}|_\infty)$.

Proposition 13 (Consistency of $\hat{\theta}$, Theorem 2 in Stute [11]). *Under Assumption 10, for $\mathbb{P}_{\mathbf{Z}}^{\otimes k}$ -almost all $(\mathbf{z}_1, \dots, \mathbf{z}_k)$, $\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k) \xrightarrow{\mathbb{P}} \theta(\mathbf{z}_1, \dots, \mathbf{z}_k)$ as $n \rightarrow \infty$.*

We introduce now a few more notations to state the asymptotic normality of $\hat{\theta}$. For $1 \leq j, l, m \leq k$ and $\mathbf{z}_1, \dots, \mathbf{z}_{3k} \in \mathcal{Z}^{3k}$, define

$$\begin{aligned}
\theta_{j,l}(\mathbf{z}_1, \dots, \mathbf{z}_k) &:= \mathbb{E} \left[g(\mathbf{X}_1, \dots, \mathbf{X}_{j-1}, \mathbf{X}, \mathbf{X}_{j+1}, \dots, \mathbf{X}_k) g(\mathbf{X}_{k+1}, \dots, \mathbf{X}_{k+l-1}, \mathbf{X}, \mathbf{X}_{k+l+1}, \dots, \mathbf{X}_{2k}) \right. \\
&\quad \left. | \mathbf{Z} = \mathbf{z}_j ; \mathbf{Z}_i = \mathbf{z}_i, \forall i = 1, \dots, k, i \neq j ; \mathbf{Z}_{k+i} = \mathbf{z}_i, \forall i = 1, \dots, k, i \neq l \right], \\
\tilde{\theta}_{j,l}(\mathbf{z}_1, \dots, \mathbf{z}_{2k}) &:= \mathbb{E} \left[g(\mathbf{X}_1, \dots, \mathbf{X}_{j-1}, \mathbf{X}, \mathbf{X}_{j+1}, \dots, \mathbf{X}_k) g(\mathbf{X}_{k+1}, \dots, \mathbf{X}_{k+l-1}, \mathbf{X}, \mathbf{X}_{k+l+1}, \dots, \mathbf{X}_{2k}) \right. \\
&\quad \left. | \mathbf{Z} = \mathbf{z}_j ; \mathbf{Z}_i = \mathbf{z}_i, \forall i = 1, \dots, 2k, i \notin \{j, k+l\} \right]. \tag{B.1} \\
\theta_{j,l,m}(\mathbf{z}_1, \dots, \mathbf{z}_{3k}) &:= \mathbb{E} \left[g(\mathbf{X}_1, \dots, \mathbf{X}_{j-1}, \mathbf{X}, \mathbf{X}_{j+1}, \dots, \mathbf{X}_k) \right. \\
&\quad \left. g(\mathbf{X}_{k+1}, \dots, \mathbf{X}_{k+l-1}, \mathbf{X}, \mathbf{X}_{k+l+1}, \dots, \mathbf{X}_{2k}) g(\mathbf{X}_{2k+1}, \dots, \mathbf{X}_{2k+m-1}, \mathbf{X}, \mathbf{X}_{2k+m+1}, \dots, \mathbf{X}_{3k}) \right. \\
&\quad \left. | \mathbf{Z} = \mathbf{z}_j ; \mathbf{Z}_i = \mathbf{z}_i, \forall i = 1, \dots, 3k, i \notin \{j, k+l, 2k+m\} \right].
\end{aligned}$$

Assumption 11. (i) $h_n \rightarrow 0$ and $nh_n^p \rightarrow \infty$;

(ii) K is symmetric at 0, bounded and compactly supported ;

(iii) $\theta_{j,l}$ is continuous at $(\mathbf{z}_1, \dots, \mathbf{z}_k)$ for all $1 \leq j, l \leq k$;

(iv) θ is two times continuously differentiable in a neighborhood of $(\mathbf{z}_1, \dots, \mathbf{z}_k)$;

(v) $\theta_{j,l,m}$ is bounded in a neighborhood of $(\mathbf{z}_1, \dots, \mathbf{z}_k, \mathbf{z}_1, \dots, \mathbf{z}_k, \mathbf{z}_1, \dots, \mathbf{z}_k) \in \mathcal{Z}^{3k}$, for all $1 \leq j, l, m \leq k$;

(vi) $f_{\mathbf{Z}}$ is twice differentiable in neighborhoods of $\mathbf{z}_i, 1 \leq i \leq k$.

Proposition 14 (Asymptotic normality of $\hat{\theta}$, Corollary 2.4 in Stute [11]). *Under Assumption 11, we have*

$$\sqrt{nh_n^p}(\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k) - \theta(\mathbf{z}_1, \dots, \mathbf{z}_k)) \xrightarrow{D} \mathcal{N}(0, \rho^2),$$

where $\rho^2 := \sum_{j,l=1}^k \mathbb{1}_{\{\mathbf{z}_j = \mathbf{z}_l\}} (\theta_{j,l}(\mathbf{z}_1, \dots, \mathbf{z}_k) - \theta^2(\mathbf{z}_1, \dots, \mathbf{z}_k)) \|K\|_2^2 / f_{\mathbf{Z}}(\mathbf{z}_j)$.

Moreover, let N be a positive integer, and $(\mathbf{z}_1^{(1)}, \dots, \mathbf{z}_k^{(1)}, \dots, \mathbf{z}_1^{(N)}, \dots, \mathbf{z}_k^{(N)}) \in \mathcal{Z}^{k \times N}$. Then under similar regularity conditions, $\sqrt{nh_n^p}(\hat{\theta}(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)}) - \theta(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)}))_{i=1, \dots, N} \xrightarrow{D} \mathcal{N}(0, \mathbb{H})$, where, for $1 \leq \tilde{j}, \tilde{l} \leq N$,

$$[\mathbb{H}]_{\tilde{j}, \tilde{l}} := \sum_{j,l=1}^k \mathbb{1}_{\{\mathbf{z}_j^{(\tilde{j})} = \mathbf{z}_l^{(\tilde{l})}\}} \left(\tilde{\theta}_{j,l}(\mathbf{z}_1^{(\tilde{j})}, \dots, \mathbf{z}_k^{(\tilde{j})}, \mathbf{z}_1^{(\tilde{l})}, \dots, \mathbf{z}_k^{(\tilde{l})}) - \theta(\mathbf{z}_1^{(\tilde{j})}, \dots, \mathbf{z}_k^{(\tilde{j})}) \theta(\mathbf{z}_1^{(\tilde{l})}, \dots, \mathbf{z}_k^{(\tilde{l})}) \right) \frac{\|K\|_2^2}{f_{\mathbf{Z}}(\mathbf{z}_j^{(\tilde{j})})}.$$

Note that the second part of Proposition 14 above is a consequence of the first one. Indeed, for every $(c_1, \dots, c_N) \in \mathbb{R}^N$, we can define $\theta(\mathbf{z}_1^{(1)}, \dots, \mathbf{z}_k^{(1)}, \dots, \mathbf{z}_1^{(N)}, \dots, \mathbf{z}_k^{(N)}) := \sum_{i=1}^N c_i \theta(\mathbf{z}_1^{(i)}, \dots, \mathbf{z}_k^{(i)})$ and corresponding versions of g , $\hat{\theta}$ and ρ^2 . Finally, the conclusion follows from the Cramér-Wold device.

Appendix C. Finite distance proofs for $\hat{\theta}$ and $\hat{\beta}$

For convenience, we recall Berk's (1970) inequality (see Theorem A in Serfling [10, p.201]). Note that, if $m = 1$, this reduces to Bernstein's inequality.

Lemma 15. *Let $k > 0$, $n \geq k$, $\mathbf{X}_1, \dots, \mathbf{X}_n$ i.i.d. random vectors with values in a measurable space \mathcal{X} and $g : \mathcal{X}^k \rightarrow [a, b]$ be a real bounded function. Set $\theta := \mathbb{E}[g(\mathbf{X}_{1:k})]$ and $\sigma^2 := \text{Var}[g(\mathbf{X}_{1:k})]$. Then, for any $t > 0$,*

$$\mathbb{P} \left(\binom{n}{k}^{-1} \sum_{\sigma \in \mathfrak{J}_{k,n}^\uparrow} g(\mathbf{X}_{\sigma(1:k)}) - \theta \geq t \right) \leq \exp \left(- \frac{[n/k]t^2}{2\sigma^2 + (2/3)(b-\theta)t} \right),$$

where $\mathfrak{J}_{k,n}$ is the set of bijective functions from $\{1, \dots, k\}$ to $\{1, \dots, n\}$ and $\mathfrak{J}_{k,n}^\uparrow$ is the subset of $\mathfrak{J}_{k,n}$ made of increasing functions.

Note that g does not need to be symmetric for this bound to hold. Indeed, if g is not symmetric, we can nonetheless apply this lemma to the symmetrized version \tilde{g} defined as $\tilde{g}(\mathbf{x}_{1:k}) := (k!)^{-1} \sum_{\sigma \in \mathfrak{J}_{k,k}} g(\mathbf{x}_{\sigma(1:k)})$, and we get the result.

Appendix C.1. Proof of Lemma 3

We decompose the quantity to bound into a stochastic part and a bias as follows:

$$N_k(\mathbf{z}_{1:k}) - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i) = (N_k(\mathbf{z}_{1:k}) - \mathbb{E}[N_k(\mathbf{z}_{1:k})]) + (\mathbb{E}[N_k(\mathbf{z}_{1:k})] - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)).$$

We first bound the bias.

$$\begin{aligned}
\left| \mathbb{E}[N_k(\mathbf{z}_{1:k})] - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i) \right| &= \left| \mathbb{E} \left[\binom{n}{k}^{-1} \sum_{\sigma \in \mathcal{J}_{k,n}} \prod_{i=1}^k K_h(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i) \right] - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i) \right| \\
&= \left| \int \left(\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i + h\mathbf{u}_i) - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i) \right) \prod_{i=1}^k K(\mathbf{u}_i) d\mathbf{u}_i \right| \\
&= \left| \int \left(\phi_{\mathbf{z},\mathbf{u}}(1) - \phi_{\mathbf{z},\mathbf{u}}(0) \right) \prod_{i=1}^k K(\mathbf{u}_i) d\mathbf{u}_i \right|,
\end{aligned}$$

where $\phi_{\mathbf{z},\mathbf{u}}(t) := \prod_{j=1}^k f_{\mathbf{Z}}(\mathbf{z}_j + t h \mathbf{u}_j)$ for $t \in [-1, 1]$. Note that this function has at least the same regularity as $f_{\mathbf{Z}}$, so it is α -differentiable, and by a Taylor-Lagrange expansion, we get

$$\left| \mathbb{E}[N_k(\mathbf{z}_{1:k})] - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i) \right| = \left| \int_{\mathbb{R}^{kp}} \left(\sum_{i=1}^{\alpha-1} \frac{1}{i!} \phi_{\mathbf{z},\mathbf{u}}^{(i)}(0) + \frac{1}{\alpha!} \phi_{\mathbf{z},\mathbf{u}}^{(\alpha)}(t_{\mathbf{z},\mathbf{u}}) \right) \prod_{i=1}^k K(\mathbf{u}_i) d\mathbf{u}_i \right|.$$

For $l > 0$, we have

$$\begin{aligned}
\phi_{\mathbf{z},\mathbf{u}}^{(l)}(0) &= \sum_{m_1 + \dots + m_k = l} \binom{\alpha}{m_{1:k}} \prod_{i=1}^k \frac{\partial^{m_i} (f_{\mathbf{Z}}(\mathbf{z}_i + h t \mathbf{u}_i))}{\partial t^{m_i}}(0) \\
&= \sum_{m_1 + \dots + m_k = l} \binom{\alpha}{m_{1:k}} \prod_{i=1}^k \sum_{j_1, \dots, j_{m_i} = 1}^p h^{m_i} u_{i,j_1} \dots u_{i,j_{m_i}} \frac{\partial^{m_i} f_{\mathbf{Z}}}{\partial z_{j_1} \dots \partial z_{j_{m_i}}}(\mathbf{z}_i + t_{\mathbf{z},\mathbf{u}} h \mathbf{u}_i),
\end{aligned}$$

where $\binom{\alpha}{m_{1:k}} := \alpha! / (\prod_{i=1}^k m_i!)$ is the multinomial coefficient. Using Assumption 1, for every $i = 1, \dots, \alpha-1$, we get $\int K(\mathbf{u}_1) \dots K(\mathbf{u}_k) \phi_{\mathbf{z},\mathbf{u}}^{(i)}(0) d\mathbf{u}_1 \dots d\mathbf{u}_k = 0$. Therefore, only the last term remains and we have

$$\left| \mathbb{E}[N_k(\mathbf{z}_{1:k})] - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i) \right| = \left| \int \left(\frac{1}{\alpha!} \phi_{\mathbf{z},\mathbf{u}}^{(\alpha)}(t_{\mathbf{z},\mathbf{u}}) \right) \prod_{i=1}^k K(\mathbf{u}_i) d\mathbf{u}_i \right| \leq \frac{C_{K,\alpha}}{\alpha!} h^\alpha,$$

using Assumption 2.

Second, we bound the stochastic part. We have

$$N_k(\mathbf{z}_{1:k}) - \mathbb{E}[N_k(\mathbf{z}_{1:k})] = \frac{k!(n-k)!}{n!} \sum_{\sigma \in \mathcal{J}_{k,n}^\dagger} \prod_{i=1}^k K_h(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i) - \prod_{i=1}^k \mathbb{E}[K_h(\mathbf{Z}_i - \mathbf{z}_i)].$$

Then, we can apply Lemma 15 to the function g defined by $g(\tilde{\mathbf{z}}_1, \dots, \tilde{\mathbf{z}}_k) := \prod_{i=1}^k K_h(\tilde{\mathbf{z}}_i - \mathbf{z}_i)$. Here, we have $b = -a = h^{-kp} C_K^k$, and

$$\text{Var}[g(\mathbf{Z}_1, \dots, \mathbf{Z}_k)^2] \leq \mathbb{E}[g(\mathbf{Z}_1, \dots, \mathbf{Z}_k)^2] = \prod_{i=1}^k \mathbb{E}[K_h(\mathbf{Z}_i - \mathbf{z}_i)^2] \leq h^{-kp} f_{\mathbf{Z}, \max}^k \|K\|_2^{2k}.$$

Finally, we get

$$\mathbb{P} \left(\binom{n}{k}^{-1} N_k(\mathbf{z}_{1:k}) - \mathbb{E}[N_k(\mathbf{z}_{1:k})] \geq t \right) \leq \exp \left(- \frac{[n/k]t^2}{2h^{-kp} f_{\mathbf{Z},max}^k \|K\|_2^{2k} + (4/3)h^{-kp} C_K^k t} \right),$$

□

Appendix C.2. Proof of Proposition 5

We have the following decomposition

$$\begin{aligned} & |\hat{\theta}(\mathbf{z}_{1:k}) - \theta(\mathbf{z}_{1:k})| \\ &= \left| N_k(\mathbf{z}_{1:k})^{-1} \frac{(n-k)!}{n!} \sum_{\sigma \in \mathfrak{J}_{k,n}} \prod_{i=1}^k K_h(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i) \left(g(\mathbf{X}_{\sigma(1:k)}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \right| \\ &= \frac{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)}{N_k(\mathbf{z}_1, \dots, \mathbf{z}_k)} \cdot \left| \frac{(n-k)!}{n!} \sum_{\sigma \in \mathfrak{J}_{k,n}} \prod_{i=1}^k \frac{K_h(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i)}{f_{\mathbf{Z}}(\mathbf{z}_i)} \left(g(\mathbf{X}_{\sigma(1:k)}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \right| \\ &=: \frac{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)}{N_k(\mathbf{z}_1, \dots, \mathbf{z}_k)} \cdot \left| \sum_{\sigma \in \mathfrak{J}_{k,n}} S_{\sigma} \right|. \end{aligned}$$

The conclusion will follow from the next three lemmas, where we will bound separately $\prod_{i=1}^k f_{\mathbf{Z}}/N_k$, the bias term $|\sum_{\sigma \in \mathfrak{J}_{k,n}} \mathbb{E}[S_{\sigma}]|$ and the stochastic component $|\sum_{\sigma \in \mathfrak{J}_{k,n}} (S_{\sigma} - \mathbb{E}[S_{\sigma}])|$.

Lemma 16 (Bound for $\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)/N_k$). *Under Assumptions 1, 2, 3, and 4 and if for some $t > 0$, $C_{K,\alpha} h^{\alpha}/\alpha! + t < f_{\mathbf{Z},min}^k/2$, we have*

$$\begin{aligned} \mathbb{P} \left(\left| \frac{1}{N_k(\mathbf{z}_{1:k})} - \frac{1}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)} \right| \leq \frac{4}{f_{\mathbf{Z},min}^{2k}} \left(\frac{C_{K,\alpha} h^{\alpha}}{\alpha!} + t \right) \right) \\ \geq 1 - 2 \exp \left(- \frac{[n/k]t^2}{2h^{-kp} f_{\mathbf{Z},max}^k \|K\|_2^{2k} + (4/3)h^{-kp} C_K^k t} \right), \end{aligned}$$

and on the same event, $N_k(\mathbf{z}_{1:k})$ is strictly positive and

$$\frac{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)}{N_k(\mathbf{z}_{1:k})} \leq 1 + \frac{4f_{\mathbf{Z},max}^k}{f_{\mathbf{Z},min}^{2k}} \left(\frac{C_{K,\alpha} h^{\alpha}}{\alpha!} + t \right).$$

Proof : Using the mean value inequality for the function $x \mapsto 1/x$, we get

$$\left| \frac{1}{N_k(\mathbf{z}_{1:k})} - \frac{1}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)} \right| \leq \frac{1}{N_*^2} \left| N_k(\mathbf{z}_{1:k}) - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i) \right|,$$

where N_* lies between $N_k(\mathbf{z}_{1:k})$ and $\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)$. By Lemma 3, we get

$$\mathbb{P}\left(\left|N_k(\mathbf{z}_{1:k}) - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)\right| \leq \frac{C_{K,\alpha} h^\alpha}{\alpha!} + t\right) \geq 1 - 2 \exp\left(-\frac{[n/k]t^2}{2h^{-kp} f_{\mathbf{Z},max}^k \|K\|_2^{2k} + (4/3)h^{-kp} C_{K,\alpha}^k t}\right).$$

On this event, $|N_k(\mathbf{z}_{1:k}) - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)| \leq (1/2) \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)$ by assumption, so that $f_{\mathbf{Z},min}^k/2 \leq N_k(\mathbf{z}_{1:k})$. We have also $f_{\mathbf{Z},min}^k/2 \leq \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)$. Thus, we have $f_{\mathbf{Z},min}^k/2 \leq N_*$. Combining the previous inequalities, we finally get

$$\left|\frac{1}{N_k(\mathbf{z}_{1:k})} - \frac{1}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)}\right| \leq \frac{1}{N_*^2} |N_k(\mathbf{z}_{1:k}) - \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)| \leq \frac{4}{f_{\mathbf{Z},min}^{2k}} \left(\frac{C_{K,\alpha} h^\alpha}{\alpha!} + t\right).$$

□

Now, we provide a bound on the bias.

Lemma 17. *Under Assumptions 1 and 6, we have $|\mathbb{E}[S_\sigma]| \leq C_{g,f,\alpha} C_{K,\alpha} h^{k\alpha} / (f_{\mathbf{Z},min}^k \alpha!)$.*

Proof : We remark that

$$\begin{aligned} 0 &= \int \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}]\right) f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}_1}(\mathbf{x}_1) \cdots f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}_k}(\mathbf{x}_k) d\mu^{\otimes k}(\mathbf{x}_{1:k}) \\ &= \int \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}]\right) \frac{f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_1, \mathbf{z}_1) \cdots f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_k, \mathbf{z}_k)}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)} d\mu^{\otimes k}(\mathbf{x}_{1:k}). \end{aligned} \quad (\text{C.1})$$

We have

$$\begin{aligned} \mathbb{E}[S_\sigma] &= \mathbb{E}\left[\frac{K_h(\mathbf{Z}_{\sigma(1)} - \mathbf{z}_1) \cdots K_h(\mathbf{Z}_{\sigma(k)} - \mathbf{z}_k)}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)} \left(g(\mathbf{X}_{\sigma(1)}, \dots, \mathbf{X}_{\sigma(k)}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}]\right)\right] \\ &= \int \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}]\right) \prod_{i=1}^k \frac{K(\mathbf{u}_i)}{f_{\mathbf{Z}}(\mathbf{z}_i)} f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i + h\mathbf{u}_i) d\mu(\mathbf{x}_i) d\mathbf{u}_i \\ &= \int \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}]\right) \left(\prod_{i=1}^k f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i + h\mathbf{u}_i) - \prod_{i=1}^k f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i)\right) \prod_{i=1}^k \frac{K(\mathbf{u}_i)}{f_{\mathbf{Z}}(\mathbf{z}_i)} d\mu(\mathbf{x}_i) d\mathbf{u}_i. \end{aligned}$$

We apply now the Taylor-Lagrange formula to the function

$$\phi_{\mathbf{x}_{1:k}, \mathbf{u}_{1:k}}(t) := \prod_{i=1}^k f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i + h\mathbf{u}_i),$$

and get

$$\begin{aligned}
\mathbb{E}[S_\sigma] &= \int \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \left(\phi_{\mathbf{x}_{1:k}, \mathbf{u}_{1:k}}(t)(1) - \phi_{\mathbf{x}_{1:k}, \mathbf{u}_{1:k}}(t)(0) \right) \prod_{i=1}^k \frac{K(\mathbf{u}_i)}{f_{\mathbf{Z}}(\mathbf{z}_i)} d\mu(\mathbf{x}_i) d\mathbf{u}_i \\
&= \int \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \\
&\quad \cdot \left(\sum_{j=1}^{\alpha-1} \frac{1}{j!} \phi_{\mathbf{x}_{1:k}, \mathbf{u}_{1:k}}(t)^{(j)}(0) + \frac{1}{\alpha!} \phi_{\mathbf{x}_{1:k}, \mathbf{u}_{1:k}}(t)^{(\alpha)}(t_{\mathbf{x}, \mathbf{u}}) \right) \prod_{i=1}^k \frac{K(\mathbf{u}_i)}{f_{\mathbf{Z}}(\mathbf{z}_i)} d\mu(\mathbf{x}_i) d\mathbf{u}_i \\
&= \int \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \\
&\quad \cdot \left(\frac{1}{\alpha!} \phi_{\mathbf{x}_{1:k}, \mathbf{u}_{1:k}}(t)^{(\alpha)}(t_{\mathbf{x}, \mathbf{u}}) \right) \prod_{i=1}^k \frac{K(\mathbf{u}_i)}{f_{\mathbf{Z}}(\mathbf{z}_i)} d\mu(\mathbf{x}_i) d\mathbf{u}_i \\
&= \int \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \\
&\quad \cdot \frac{1}{\alpha!} \left(\phi_{\mathbf{x}_{1:k}, \mathbf{u}_{1:k}}(t)^{(\alpha)}(t_{\mathbf{x}, \mathbf{u}}) - \phi_{\mathbf{x}_{1:k}, \mathbf{u}_{1:k}}(t)^{(\alpha)}(0) \right) \prod_{i=1}^k \frac{K(\mathbf{u}_i)}{f_{\mathbf{Z}}(\mathbf{z}_i)} d\mu(\mathbf{x}_i) d\mathbf{u}_i.
\end{aligned}$$

For every real t , we have

$$\begin{aligned}
\phi^{(\alpha)}(t) &= \sum_{m_1 + \dots + m_k = \alpha} \binom{n}{m_{1:k}} \prod_{i=1}^k \frac{\partial^{m_i} (f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i + ht\mathbf{u}_i))}{\partial t^{m_i}} \\
&= \sum_{m_1 + \dots + m_k = \alpha} \binom{n}{m_{1:k}} \prod_{i=1}^k \sum_{j_1, \dots, j_{m_i}=1}^p h^{m_i} u_{i,j_1} \dots u_{i,j_{m_i}} \frac{\partial^{m_i} f_{\mathbf{X}, \mathbf{Z}}}{\partial z_{j_1} \dots \partial z_{j_{m_i}}}(\mathbf{x}_i, \mathbf{z}_i + ht\mathbf{u}_i) \\
&= h^\alpha \sum_{m_1 + \dots + m_k = \alpha} \binom{n}{m_{1:k}} \prod_{i=1}^k \sum_{j_1, \dots, j_{m_i}=1}^p u_{i,j_1} \dots u_{i,j_{m_i}} \frac{\partial^{m_i} f_{\mathbf{X}, \mathbf{Z}}}{\partial z_{j_1} \dots \partial z_{j_{m_i}}}(\mathbf{x}_i, \mathbf{z}_i + ht\mathbf{u}_i). \tag{C.2}
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\mathbb{E}[S_\sigma] &= \sum_{m_1 + \dots + m_k = \alpha} \binom{n}{m_{1:k}} \int \prod_{i=1}^k \frac{K(\mathbf{u}_i)}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)} \sum_{j_1, \dots, j_{m_i}=1}^p u_{i,j_1} \dots u_{i,j_{m_i}} \\
&\quad \cdot \left(g(\mathbf{x}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \\
&\quad \cdot \left(\frac{\partial^{m_i} f_{\mathbf{X}, \mathbf{Z}}}{\partial z_{j_1} \dots \partial z_{j_{m_i}}}(\mathbf{x}_i, \mathbf{z}_i + ht\mathbf{u}_i) - \frac{\partial^{m_i} f_{\mathbf{X}, \mathbf{Z}}}{\partial z_{j_1} \dots \partial z_{j_{m_i}}}(\mathbf{x}_i, \mathbf{z}_i) \right) d\mu(\mathbf{x}_1) d\mathbf{u}_1 \dots d\mu(\mathbf{x}_k) d\mathbf{u}_k,
\end{aligned}$$

and, using Assumption 6, this yields

$$|\mathbb{E}[S_\sigma]| \leq \frac{C_{g,f,\alpha} C_{K,\alpha} h^{\alpha+k}}{f_{\mathbf{Z}, \min}^k \alpha!}.$$

□

Now we bound the stochastic component. We have the following equality

$$\left| \sum_{\sigma \in \mathfrak{J}_{k,n}} (S_\sigma - \mathbb{E}[S_\sigma]) \right| = \left| \frac{(n-k)!}{n!} \sum_{\sigma \in \mathfrak{J}_{k,n}} g((\mathbf{X}_{\sigma(1)}, \mathbf{Z}_{\sigma(1)}), \dots, (\mathbf{X}_{\sigma(k)}, \mathbf{Z}_{\sigma(k)})) \right|$$

with the function \tilde{g} defined by

$$\begin{aligned} & \tilde{g}((\mathbf{X}_1, \mathbf{Z}_1), \dots, (\mathbf{X}_k, \mathbf{Z}_k)) \\ &= \frac{K_h(\mathbf{Z}_1 - \mathbf{z}_1) \cdots K_h(\mathbf{Z}_k - \mathbf{z}_k)}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)} \left(g(\mathbf{X}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \\ & - \mathbb{E} \left[\frac{K_h(\mathbf{Z}_1 - \mathbf{z}_1) \cdots K_h(\mathbf{Z}_k - \mathbf{z}_k)}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}_i)} \left(g(\mathbf{X}_{1:k}) - \mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}] \right) \right] \end{aligned}$$

By construction, $\mathbb{E}[\tilde{g}((\mathbf{X}_1, \mathbf{Z}_1), \dots, (\mathbf{X}_k, \mathbf{Z}_k))] = 0$. If \tilde{g} is bounded, we can derive an immediate bound for this stochastic component. Indeed, we would have $\|\tilde{g}\|_\infty \leq 4C_K^k h^{-kp} C_g^k / f_{\mathbf{Z}, \min}^k$. Moreover, we have

$$\begin{aligned} \text{Var}[\tilde{g}((\mathbf{X}_1, \mathbf{Z}_1), \dots, (\mathbf{X}_k, \mathbf{Z}_k))] &\leq \mathbb{E} \left[\frac{K_h^2(\mathbf{Z}_1 - \mathbf{z}_1) \cdots K_h^2(\mathbf{Z}_k - \mathbf{z}_k)}{\prod_{i=1}^k f_{\mathbf{Z}}^2(\mathbf{z}_i)} g^2(\mathbf{X}_1, \dots, \mathbf{X}_k) \right] \\ &\leq C_g^2 f_{\mathbf{Z}, \max}^k f_{\mathbf{Z}, \min}^{-2k} h^{-kp} \|K\|_2^{2k}. \end{aligned}$$

Therefore, we can apply Lemma 15, and we get

$$\mathbb{P} \left(\left| \sum_{\sigma \in \mathfrak{J}_{k,n}} (S_\sigma - \mathbb{E}[S_\sigma]) \right| > t \right) \leq 2 \exp \left(- \frac{[n/k] t^2}{2C_g^2 f_{\mathbf{Z}, \max}^k f_{\mathbf{Z}, \min}^{-2k} h^{-kp} \|K\|_2^{2k} + (8/3) C_K^k h^{-kp} C_g^k f_{\mathbf{Z}, \min}^{-k} t} \right).$$

In the following Lemma 18, our goal will be to bound the stochastic component using only Assumption 8 on the conditional moments of g .

Lemma 18. *Under Assumptions 1, 4 and 8, for every $t > 0$, we have*

$$\mathbb{P} \left(\sum_{\sigma \in \mathfrak{J}_{k,n}} S_\sigma - \mathbb{E}[S_\sigma] > t \right) \leq \exp \left(- \frac{t^2 f_{\mathbf{Z}, \min}^{2k} h^{kp} [n/k]}{128 (B_{g, \mathbf{z}} + \tilde{B}_g)^2 C_K^{2k-1} + 2t (B_{g, \mathbf{z}} + \tilde{B}_g) C_K^k f_{\mathbf{Z}, \min}^k} \right).$$

Proof: Using the same decomposition for U-statistics as in Hoeffding [7], we obtain

$$\sum_{\sigma \in \mathfrak{J}_{k,n}} S_\sigma - \mathbb{E}[S_\sigma] = \frac{1}{n!} \sum_{\sigma \in \mathfrak{J}_{k,n}} \frac{1}{[n/k]} \sum_{i=1}^{[n/k]} V_{n,i,\sigma},$$

where

$$V_{n,i,\sigma} := \tilde{g}\left(\left(\mathbf{X}_{\sigma(1+(i-1)k)}, \mathbf{Z}_{\sigma(2+(i-1)k)}\right), \dots, \left(\mathbf{X}_{\sigma(ik)}, \mathbf{Z}_{\sigma(jk)}\right)\right).$$

For any $\lambda > 0$, we have

$$\begin{aligned} \mathbb{P}\left(\sum_{\sigma \in \mathcal{J}_{k,n}} S_\sigma - \mathbb{E}[S_\sigma] > t\right) &\leq e^{-\lambda t} \mathbb{E}\left[\exp\left(\lambda \sum_{\sigma \in \mathcal{J}_{k,n}} S_\sigma - \mathbb{E}[S_\sigma]\right)\right] \\ &\leq e^{-\lambda t} \mathbb{E}\left[\exp\left(\lambda \frac{1}{n!} \sum_{\sigma \in \mathcal{J}_{n,n}} \frac{1}{[n/k]} \sum_{i=1}^{[n/k]} V_{n,i,\sigma}\right)\right] \\ &\leq e^{-\lambda t} \frac{1}{n!} \sum_{\sigma \in \mathcal{J}_{n,n}} \mathbb{E}\left[\exp\left(\lambda \frac{1}{[n/k]} \sum_{i=1}^{[n/k]} V_{n,i,\sigma}\right)\right] \\ &\leq e^{-\lambda t} \frac{1}{n!} \sum_{\sigma \in \mathcal{J}_{n,n}} \prod_{i=1}^{[n/k]} \mathbb{E}\left[\exp\left(\lambda \frac{1}{[n/k]} V_{n,i,\sigma}\right)\right] \\ &\leq e^{-\lambda t} \left(\sup_{\sigma \in \mathcal{J}_{n,n}, i=1, \dots, [n/k]} \mathbb{E}\left[\exp\left(\lambda [n/k]^{-1} V_{n,i,\sigma}\right)\right]\right)^{[n/k]}. \end{aligned} \quad (\text{C.3})$$

Let $l \geq 2$. Using the inequality $(a + b + c + d)^l \leq 4^l(a^l + b^l + c^l + d^l)$, we get

$$\begin{aligned} \mathbb{E}[|V_{n,i,\sigma}|^l] &= \mathbb{E}[|V_{n,1,\sigma}|^l] \leq 4^l \mathbb{E}\left[|g(\mathbf{X}_{\sigma(1)}, \dots, \mathbf{X}_{\sigma(k)})|^l \prod_{i=1}^k \frac{|K_h|^l(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i)}{f_{\mathbf{Z}}^l(\mathbf{z}_i)}\right] \\ &\quad + 4^l \mathbb{E}\left[|\mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}]|^l \prod_{i=1}^k \frac{|K_h|^l(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i)}{f_{\mathbf{Z}}^l(\mathbf{z}_i)}\right] \\ &\quad + 4^l \left|\mathbb{E}\left[g(\mathbf{X}_{\sigma(1)}, \dots, \mathbf{X}_{\sigma(k)}) \prod_{i=1}^k \frac{K_h(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i)}{f_{\mathbf{Z}}^l(\mathbf{z}_i)}\right]\right|^l \\ &\quad + 4^l \left|\mathbb{E}\left[|\mathbb{E}[g(\mathbf{X}_{1:k}) | \mathbf{Z}_{1:k} = \mathbf{z}_{1:k}]| \prod_{i=1}^k \frac{K_h(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i)}{f_{\mathbf{Z}}^l(\mathbf{z}_i)}\right]\right|^l \end{aligned}$$

Using Jensen's inequality for the function $x \mapsto |x|^p$ with the second, third and fourth terms, and the law of iterated expectations for the first and the third terms, we get

$$\begin{aligned} \mathbb{E}[|V_{n,i,\sigma}|^l] &\leq 4^l \cdot 2 \mathbb{E}\left[\mathbb{E}[|g(\mathbf{X}_{\sigma(1)}, \dots, \mathbf{X}_{\sigma(k)})|^l | \mathbf{Z}_{\sigma(1)}, \dots, \mathbf{Z}_{\sigma(k)}] \prod_{i=1}^k \frac{|K_h|^l(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i)}{f_{\mathbf{Z}}^l(\mathbf{z}_i)}\right] \\ &\quad + 4^l \cdot 2 \mathbb{E}\left[\mathbb{E}[|g(\mathbf{X}_{1:k})|^l | \mathbf{Z}_i = \mathbf{z}_i, \forall i = 1, \dots, k] \prod_{i=1}^k \frac{|K_h|^l(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i)}{f_{\mathbf{Z}}^l(\mathbf{z}_i)}\right] \\ &\leq 4^l \cdot 2 \mathbb{E}\left[\left(B_g^l(\mathbf{Z}_1, \dots, \mathbf{Z}_k) + B_g^l(\mathbf{z}_1, \dots, \mathbf{z}_k)\right)^l l! \prod_{i=1}^k \frac{|K_h|^l(\mathbf{Z}_{\sigma(i)} - \mathbf{z}_i)}{f_{\mathbf{Z}}^l(\mathbf{z}_i)}\right] \end{aligned}$$

$$\begin{aligned}
&\leq 4^l \cdot 2 (\tilde{B}_g^l + B_g^l(\mathbf{z}_1, \dots, \mathbf{z}_k))! (h^{-kp} C_K^k f_{\mathbf{z}, \min}^{-k})^{l-1} f_{\mathbf{z}, \min}^{-k} \\
&\leq 2 \left(4(\tilde{B}_g + B_{g, \mathbf{z}}) h^{-kp} C_K^k f_{\mathbf{z}, \min}^{-k} \right)^l l! h^{kp} C_K^{-1},
\end{aligned}$$

where $B_{g, \mathbf{z}} := B_g(\mathbf{z}_1, \dots, \mathbf{z}_k)$. Remarking that $\mathbb{E}[V_{n, i, \sigma}] = 0$ by construction of \tilde{g} , we obtain

$$\begin{aligned}
\mathbb{E} \left[\exp(\lambda [n/k]^{-1} V_{n, i, \sigma}) \right] &= 1 + \sum_{l=2}^{\infty} \frac{\mathbb{E}[(\lambda [n/k]^{-1} V_{n, i, \sigma})^l]}{l!} \\
&\leq 1 + 2C_K^{-1} h^{kp} \sum_{l=2}^{\infty} (4\lambda [n/k]^{-1} (B_{g, \mathbf{z}} + \tilde{B}_g) h^{-kp} C_K^k f_{\mathbf{z}, \min}^{-k})^l \\
&\leq 1 + 2C_K^{-1} h^{kp} \cdot \frac{\left(4\lambda [n/k]^{-1} (B_{g, \mathbf{z}} + \tilde{B}_g) h^{-kp} C_K^k f_{\mathbf{z}, \min}^{-k} \right)^2}{1 - 4\lambda [n/k]^{-1} (B_{g, \mathbf{z}} + \tilde{B}_g) h^{-kp} C_K^k f_{\mathbf{z}, \min}^{-k}} \\
&\leq \exp \left(\frac{32\lambda^2 [n/k]^{-2} (B_{g, \mathbf{z}} + \tilde{B}_g)^2 h^{-kp} C_K^{2k-1} f_{\mathbf{z}, \min}^{-2k}}{1 - 4\lambda [n/k]^{-1} (B_{g, \mathbf{z}} + \tilde{B}_g) h^{-kp} C_K^k f_{\mathbf{z}, \min}^{-k}} \right),
\end{aligned}$$

where the last statement follows from the inequality $1 + x \leq \exp(x)$. Combining the latter bound with Equation (C.3), we get

$$\mathbb{P} \left(\sum_{\sigma \in \mathcal{J}_{k, n}} S_{\sigma} - \mathbb{E}[S_{\sigma}] > t \right) \leq \exp \left(-\lambda t + \frac{32\lambda^2 (B_{g, \mathbf{z}} + \tilde{B}_g)^2 C_K^{2k-1}}{f_{\mathbf{z}, \min}^{2k} h^{kp} [n/k] - 4\lambda (B_{g, \mathbf{z}} + \tilde{B}_g) C_K^k f_{\mathbf{z}, \min}^k} \right). \quad (\text{C.4})$$

Remarking that the right-hand side term inside the exponential is of the form $-\lambda t + \frac{a\lambda^2}{b-c\lambda}$, we choose the value

$$\lambda_* = \frac{tb}{2a + tc} = \frac{t f_{\mathbf{z}, \min}^{2k} h^{kp} [n/k]}{64(B_{g, \mathbf{z}} + \tilde{B}_g)^2 C_K^{2k-1} + t(B_{g, \mathbf{z}} + \tilde{B}_g) C_K^k f_{\mathbf{z}, \min}^k} \quad (\text{C.5})$$

such that $-\lambda_* t + \frac{a\lambda_*^2}{b-c\lambda_*} = -\frac{t^2 b}{4a+2ct} = -\frac{t}{2}\lambda_*$. Therefore, the right-hand side term of Equation (C.4) can be simplified, and combining this with Equation (C.5), we obtain

$$\mathbb{P} \left(\sum_{\sigma \in \mathcal{J}_{k, n}} S_{\sigma} - \mathbb{E}[S_{\sigma}] > t \right) \leq \exp \left(-\frac{t^2 f_{\mathbf{z}, \min}^{2k} h^{kp} [n/k]}{128(B_{g, \mathbf{z}} + \tilde{B}_g)^2 C_K^{2k-1} + 2t(B_{g, \mathbf{z}} + \tilde{B}_g) C_K^k f_{\mathbf{z}, \min}^k} \right).$$

□

Appendix C.3. Proof of Theorem 6

By Proposition 5, for every $t_1, t_2 > 0$ such that $C_{K,\alpha}h^\alpha/\alpha! + t < f_{\mathbf{z},\min}/2$, we have

$$\begin{aligned} \mathbb{P}\left(\left|\hat{\theta}(\mathbf{z}_1, \dots, \mathbf{z}_k) - \theta(\mathbf{z}_1, \dots, \mathbf{z}_k)\right| < (1 + C_3h^\alpha + C_4t_1) \times (C_5h^{k+\alpha} + t_2)\right) \\ \geq 1 - 2 \exp\left(-\frac{[n/k]t_1^2h^{kp}}{C_1 + C_2t_1}\right) - 2 \exp\left(-\frac{[n/k]t_2^2h^{kp}}{C_6 + C_7t_2}\right), \end{aligned}$$

We apply this proposition to every k -tuple $(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})$ where $\sigma \in \mathfrak{J}_{k,n'}$. Combining it with Assumption 9, we get

$$\begin{aligned} \mathbb{P}\left(\sup_i |\xi_{i,n}| < C_{\Lambda'}(1 + C_3h^\alpha + C_4t_1) \times (C_5h^{k+\alpha} + t_2)\right) \\ \geq 1 - 2 \sum_{i=1}^{|\mathfrak{J}_{k,n'}|} \left[\exp\left(-\frac{[n/k]t_1^2h^{kp}}{C_1 + C_2t_1}\right) + \exp\left(-\frac{[n/k]t_2^2h^{kp}}{C_6 + C_7t_2}\right) \right], \end{aligned}$$

Choosing $t_1 := f_{\mathbf{z},\min}/4$ and using the bound (5) on h , we get

$$\begin{aligned} \mathbb{P}\left(\sup_i |\xi_{i,n}| < C_{\Lambda'}\left(1 + C_3\frac{f_{\mathbf{z},\min}\alpha!}{4C_{K,\alpha}} + C_4\frac{f_{\mathbf{z},\min}}{4}\right) \times (C_5h^{k+\alpha} + t_2)\right) \\ \geq 1 - 2 \sum_{i=1}^{|\mathfrak{J}_{k,n'}|} \left[\exp\left(-\frac{[n/k]f_{\mathbf{z},\min}^2h^{kp}}{16C_1 + 4C_2f_{\mathbf{z},\min}}\right) + \exp\left(-\frac{[n/k]t_2^2h^{kp}}{C_6 + C_7t_2}\right) \right]. \end{aligned}$$

Choosing $t_2 = t/(2C_8) = t/(2C_\psi C_{\Lambda'}(1 + C_3\frac{f_{\mathbf{z},\min}\alpha!}{4C_{K,\alpha}} + C_4\frac{f_{\mathbf{z},\min}}{4}))$, and using the bound (5) on h^α , we get

$$\mathbb{P}\left(\sup_i |\xi_{i,n}| < t/C_\psi\right) \geq 1 - 2 \sum_{i=1}^{|\mathfrak{J}_{k,n'}|} \left[\exp\left(-\frac{[n/k]f_{\mathbf{z},\min}^2h^{kp}}{16C_1 + 4C_2f_{\mathbf{z},\min}}\right) + \exp\left(-\frac{[n/k]t^2h^{kp}}{4C_8^2C_6 + 2C_8C_7t}\right) \right].$$

On the same event, we have $\max_{j=1,\dots,p'} \left| \frac{1}{n'} \sum_{i=1}^{n'} Z'_{i,j} \xi_{i,n} \right| \leq t$, by Assumption 9. The conclusion results from the following lemma.

Lemma 19 (From [3, Lemma 25]). *Assume that $\max_{j=1,\dots,p'} \left| \frac{1}{n'} \sum_{i=1}^{n'} Z'_{i,j} \xi_{i,n} \right| \leq t$, for some $t > 0$, that the assumption $RE(s, 3)$ is satisfied, and that the tuning parameter is given by $\lambda = \gamma t$, with $\gamma \geq 4$. Then, $\|Z'(\hat{\beta} - \beta^*)\| \leq \frac{4(\gamma + 1)t\sqrt{s}}{\kappa(s, 3)}$ and $|\hat{\beta} - \beta^*|_q \leq \frac{4^{2/q}(\gamma + 1)ts^{1/q}}{\kappa^2(s, 3)}$, for every $1 \leq q \leq 2$.*

□

Appendix D. Proof of Theorem 12

We detail the assumption which we will use to prove Theorem 12.

Assumption 12. (i) The support of the kernel $K(\cdot)$ is included into $[-1, 1]^p$. Moreover, for all n, n' and every $(i, j) \in \{1, \dots, n'\}^2$, $i \neq j$, we have $\|\mathbf{z}'_i - \mathbf{z}'_j\|_\infty > 2h_{n, n'}$.

(ii) (a) $n'(nh_{n, n'}^{p+4\alpha} + h_{n, n'}^{2\alpha} + h_{n, n'}^p + (nh_{n, n'}^p)^{-1}) \rightarrow 0$, (b) $\lambda_{n, n'}(n' n h_{n, n'}^p)^{1/2} \rightarrow 0$, (c) $n' n h_{n, n'}^p \rightarrow \infty$ and $n h_{n, n'}^{p+2\alpha-\epsilon} / \ln n' \rightarrow \infty$ for some $\epsilon \in [0, 2\alpha[$.

(iii) The distribution $\mathbb{P}_{\mathbf{z}', n'} := |\mathfrak{J}_{k, n'}|^{-1} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \delta_{(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})}$ weakly converges as $n' \rightarrow \infty$, to a distribution $\mathbb{P}_{\mathbf{z}', k, \infty}$ on \mathbb{R}^{kp} . There exists a distribution $\mathbb{P}_{\mathbf{z}', \infty}$ on \mathbb{R}^{kp} , with a density $f_{\mathbf{z}', \infty}$ with respect to the p -dimensional Lebesgue measure such that $\mathbb{P}_{\mathbf{z}', k, \infty} = \mathbb{P}_{\mathbf{z}', \infty}^{\otimes k}$.

(iv) The matrix $V_1 := \int \boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k) \boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k)^T f_{\mathbf{z}', \infty}(\mathbf{z}'_1) \cdots f_{\mathbf{z}', \infty}(\mathbf{z}'_k) d\mathbf{z}'_1 \cdots d\mathbf{z}'_k$ is non-singular.

(v) $\Lambda(\cdot)$ is two times continuously differentiable. Let \mathcal{T} be the range of θ , from \mathcal{Z}^k towards \mathbb{R} . On an open neighborhood of \mathcal{T} , the second derivative of $\Lambda(\cdot)$ is bounded by a constant $C_{\Lambda'}$.

(vi) Several integrals exist and are finite, including

$$\begin{aligned} \tilde{V}_1 &:= \int \theta(\mathbf{z}'_1, \dots, \mathbf{z}'_k) \Lambda'(\theta(\mathbf{z}'_1, \dots, \mathbf{z}'_k)) \boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k) f_{\mathbf{z}', \infty}(\mathbf{z}'_1) \cdots f_{\mathbf{z}', \infty}(\mathbf{z}'_k) d\mathbf{z}'_1 \cdots d\mathbf{z}'_k \text{ and} \\ V_2 &:= \int \frac{\|K\|_2^2}{f_{\mathbf{z}}(\mathbf{z}'_1)} g(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) g(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k) \Lambda'^2(\theta(\mathbf{z}'_1, \dots, \mathbf{z}'_k)) \boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k) \boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k)^T \\ &\quad \times f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_1}(\mathbf{x}_1) d\mu(\mathbf{x}_1) d\mu(\mathbf{z}'_1) \prod_{i=2}^k f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_i}(\mathbf{y}_i) f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_i}(\mathbf{x}_i) f_{\mathbf{z}', \infty}(\mathbf{z}'_i) d\mu(\mathbf{x}_i) d\mu(\mathbf{y}_i) d\mathbf{z}'_i. \end{aligned}$$

Define $\tilde{r}_{n, n'} := (n \times n' \times h_{n, n'}^p)^{1/2}$, $\mathbf{u} := \tilde{r}_{n, n'}(\beta - \beta^*)$ and $\hat{\mathbf{u}}_{n, n'} := \tilde{r}_{n, n'}(\hat{\beta}_{n, n'} - \beta^*)$, so that $\hat{\beta}_{n, n'} = \beta^* + \hat{\mathbf{u}}_{n, n'} / \tilde{r}_{n, n'}$. We define for every $\mathbf{u} \in \mathbb{R}^{p'}$,

$$\begin{aligned} \mathbb{F}_{n, n'}(\mathbf{u}) &:= \frac{-2\tilde{r}_{n, n'}}{|\mathfrak{J}_{k, n'}|} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \xi_{\sigma, n} \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T \mathbf{u} \\ &\quad + \frac{1}{|\mathfrak{J}_{k, n'}|} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \left\{ \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T \mathbf{u} \right\}^2 + \lambda_{n, n'} \tilde{r}_{n, n'}^2 \left(\left| \beta^* + \frac{\mathbf{u}}{\tilde{r}_{n, n'}} \right|_1 - |\beta^*|_1 \right), \end{aligned} \quad (\text{D.1})$$

and we obtain $\hat{\mathbf{u}}_{n, n'} = \arg \min_{\mathbf{u} \in \mathbb{R}^{p'}} \mathbb{F}_{n, n'}(\mathbf{u})$ applying Lemma 7.

Lemma 20. Under the same assumptions as in Theorem 12,

$$T_1 := \frac{\tilde{r}_{n, n'}}{|\mathfrak{J}_{k, n'}|} \sum_{\sigma \in \mathfrak{J}_{k, n'}} \xi_{\sigma, n} \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \xrightarrow{D} \mathcal{N}(0, V_2).$$

This lemma is proved in Appendix D.1. It will help to control the first term of Equation (D.1), which is simply $-2T_1^T \mathbf{u}$.

Concerning the second term of Equation (D.1), using Assumption 12(iii), we have for every $\mathbf{u} \in \mathbb{R}^{p'}$

$$\begin{aligned} & \frac{1}{|\mathcal{J}_{k,n'}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \{ \boldsymbol{\psi}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})^T \mathbf{u} \}^2 \\ & \rightarrow \int (\boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k)^T \mathbf{u})^2 f_{\mathbf{z}', \infty}(\mathbf{z}'_1) \cdots f_{\mathbf{z}', \infty}(\mathbf{z}'_k) d\mathbf{z}'_1 \cdots d\mathbf{z}'_k. \end{aligned} \quad (\text{D.2})$$

This has to be read as a convergence of a sequence of real numbers indexed by \mathbf{u} , because the design points \mathbf{z}'_i are deterministic. We also have, for any $\mathbf{u} \in \mathbb{R}^{p'}$ and when n is large enough,

$$\left| \beta^* + \frac{\mathbf{u}}{\tilde{r}_{n,n'}} \Big|_1 - |\beta^*|_1 \right| = \sum_{i=1}^{p'} \left(\frac{|u_i|}{\tilde{r}_{n,n'}} \mathbb{1}_{\{\beta_i^* = 0\}} + \frac{u_i}{\tilde{r}_{n,n'}} \text{sign}(\beta_i^*) \mathbb{1}_{\{\beta_i^* \neq 0\}} \right).$$

Therefore, by Assumption 12(ii)(b), for every $\mathbf{u} \in \mathbb{R}^{p'}$,

$$\lambda_{n,n'} \tilde{r}_{n,n'}^2 \left(\left| \beta^* + \frac{\mathbf{u}}{\tilde{r}_{n,n'}} \Big|_1 - |\beta^*|_1 \right| \right) \rightarrow 0, \quad (\text{D.3})$$

when (n, n') tends to the infinity. Combining Lemma 20 and Equations (D.1-D.3), and defining the function $\mathbb{F}_{\infty, \infty}$ by

$$\mathbb{F}_{\infty, \infty}(\mathbf{u}) := 2\tilde{\mathbf{W}}^T \mathbf{u} + \int (\boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k)^T \mathbf{u})^2 f_{\mathbf{z}', \infty}(\mathbf{z}'_1) \cdots f_{\mathbf{z}', \infty}(\mathbf{z}'_k) d\mathbf{z}'_1 \cdots d\mathbf{z}'_k,$$

where $\mathbf{u} \in \mathbb{R}^r$ and $\tilde{\mathbf{W}} \sim \mathcal{N}(0, V_2)$, we obtain that every finite-dimensional margin of $\mathbb{F}_{n,n'}$ weakly converges to the corresponding margin of $\mathbb{F}_{\infty, \infty}$. Now, applying the convexity lemma, we get

$$\hat{\mathbf{u}}_{n,n'} \xrightarrow{D} \mathbf{u}_{\infty, \infty}, \text{ where } \mathbf{u}_{\infty, \infty} := \arg \min_{\mathbf{u} \in \mathbb{R}^r} \mathbb{F}_{\infty, \infty}(\mathbf{u}).$$

Since $\mathbb{F}_{\infty, \infty}(\mathbf{u})$ is a continuously differentiable convex function, apply the first-order condition $\nabla \mathbb{F}_{\infty, \infty}(\mathbf{u}) = 0$, which yields

$$2\tilde{\mathbf{W}} + 2 \int \boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k) \boldsymbol{\psi}(\mathbf{z}'_1, \dots, \mathbf{z}'_k)^T \mathbf{u}_{\infty, \infty} f_{\mathbf{z}', \infty}(\mathbf{z}'_1) \cdots f_{\mathbf{z}', \infty}(\mathbf{z}'_k) d\mathbf{z}'_1 \cdots d\mathbf{z}'_k = 0.$$

As a consequence $\mathbf{u}_{\infty, \infty} = -V_1^{-1} \tilde{\mathbf{W}} \sim \mathcal{N}(0, \tilde{V}_{as})$, using Assumption 12(iv). We finally obtain $\tilde{r}_{n,n'} (\hat{\beta}_{n,n'} - \beta^*) \xrightarrow{D} \mathcal{N}(0, \tilde{V}_{as})$, as claimed. \square

Appendix D.1. Proof of Lemma 20

Using a Taylor expansion yields

$$\begin{aligned}
T_1 &:= \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \xi_{\sigma,n} \psi(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \\
&= \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \left(\Lambda(\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})) - \Lambda(\theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})) \right) \psi(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \\
&= T_2 + T_3,
\end{aligned}$$

where the main term is

$$T_2 := \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \Lambda'(\theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})) \left(\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) - \theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \right) \psi(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}),$$

and the remainder is

$$T_3 := \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \alpha_{3,\sigma} \cdot \left(\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) - \theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \right)^2 \psi(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}),$$

with $\forall \sigma \in \mathfrak{J}_{k,n'}$, $|\alpha_{3,\sigma}| \leq C_{\Lambda''}/2$, by Assumption 12(v).

Let us define $\bar{\psi}_\sigma := \Lambda'(\theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})) \psi(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})$, for every $\sigma \in \mathfrak{J}_{k,n'}$. Using the definition (1), we rewrite $T_2 := T_4 + T_5$ where

$$\begin{aligned}
T_4 &:= \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}| \cdot |\mathfrak{J}_{k,n}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \sum_{\varsigma \in \mathfrak{J}_{k,n}} \frac{\prod_{i=1}^k K_h(\mathbf{Z}_{\varsigma(i)} - \mathbf{z}'_{\sigma(i)})}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}'_{\sigma(i)})} \left(g(\mathbf{X}_{\varsigma(1)}, \dots, \mathbf{X}_{\varsigma(k)}) - \theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \right) \bar{\psi}_\sigma, \\
T_5 &:= \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}| \cdot |\mathfrak{J}_{k,n}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \sum_{\varsigma \in \mathfrak{J}_{k,n}} \prod_{i=1}^k K_h(\mathbf{Z}_{\varsigma(i)} - \mathbf{z}'_{\sigma(i)}) \left(g(\mathbf{X}_{\varsigma(1)}, \dots, \mathbf{X}_{\varsigma(k)}) - \theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \right) \\
&\quad \times \left(\frac{1}{N_k(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})} - \frac{1}{\prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}'_{\sigma(i)})} \right) \bar{\psi}_\sigma.
\end{aligned}$$

To lighten the notations, we will define $K_{\sigma,\varsigma} := \prod_{i=1}^k K_h(\mathbf{Z}_{\varsigma(i)} - \mathbf{z}'_{\sigma(i)})$, $g_\varsigma := g(\mathbf{X}_{\varsigma(1)}, \dots, \mathbf{X}_{\varsigma(k)})$, $\theta_\sigma := \theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})$, $f_{\mathbf{Z}',\sigma} := \prod_{i=1}^k f_{\mathbf{Z}}(\mathbf{z}'_{\sigma(i)})$, and $N_\sigma := N_k(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)})$, for every $\sigma \in \mathfrak{J}_{k,n'}$ and $\varsigma \in \mathfrak{J}_{k,n}$, so that

$$T_4 := \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}| \cdot |\mathfrak{J}_{k,n}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \sum_{\varsigma \in \mathfrak{J}_{k,n}} \frac{K_{\sigma,\varsigma}}{f_{\mathbf{Z}',\sigma}} (g_\varsigma - \theta_\sigma) \bar{\psi}_\sigma, \quad (\text{D.4})$$

$$T_5 := \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}| \cdot |\mathfrak{J}_{k,n}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \sum_{\varsigma \in \mathfrak{J}_{k,n}} K_{\sigma,\varsigma} (g_\varsigma - \theta_\sigma) \left(\frac{1}{N_\sigma} - \frac{1}{f_{\mathbf{Z}',\sigma}} \right) \bar{\psi}_\sigma. \quad (\text{D.5})$$

Using α -order limited expansions, we get

$$\begin{aligned}
\mathbb{E}[T_4] &= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \int \frac{\prod_{i=1}^k K_h(\mathbf{z}_i - \mathbf{z}'_{\sigma(i)})}{f_{\mathbf{Z}',\sigma}} (g(\mathbf{x}_{1:k}) - \theta_\sigma) \prod_{i=1}^k f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i) d\mu^{\otimes k}(\mathbf{x}_{1:k}) d\mathbf{z}_{1:k} \quad (\text{D.6}) \\
&= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \int \frac{\prod_{i=1}^k K(\mathbf{t}_i)}{f_{\mathbf{Z}',\sigma}} (g(\mathbf{x}_{1:k}) - \theta_\sigma) \prod_{i=1}^k f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}'_{\sigma(i)} + h\mathbf{t}_i) d\mu^{\otimes k}(\mathbf{x}_{1:k}) d\mathbf{t}_{1:k} \\
&= \frac{\tilde{r}_{n,n'} h^{k\alpha}}{|\mathcal{J}_{k,n'}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \int \frac{\prod_{i=1}^k K(\mathbf{t}_i)}{f_{\mathbf{Z}',\sigma}} (g(\mathbf{x}_{1:k}) - \theta_\sigma) \prod_{i=1}^k d_{\mathbf{Z}}^{(\alpha)} f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}^*_{\sigma(i)}) d\mu^{\otimes k}(\mathbf{x}_{1:k}) d\mathbf{t}_{1:k} \\
&= O\left(\tilde{r}_{n,n'} h^{k\alpha}\right) = O\left((n \times n' \times h_{n,n'}^{p+2k\alpha})^{1/2}\right) = o(1),
\end{aligned}$$

where above, \mathbf{z}_i^* denote some vectors in \mathbb{R}^p such that $|\mathbf{z}'_i - \mathbf{z}_i^*|_\infty \leq 1$, depending on \mathbf{z}'_i and \mathbf{x}_i .

We can therefore use the centered version of T_4 , defined as

$$\begin{aligned}
T_4 - \mathbb{E}[T_4] &= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} g_{\sigma,\varsigma}, \\
g_{\sigma,\varsigma} &:= \frac{\bar{\psi}_\sigma}{f_{\mathbf{Z}',\sigma}} \left(K_{\sigma,\varsigma}(g_\varsigma - \theta_\sigma) - \mathbb{E}[K_{\sigma,\varsigma}(g_\varsigma - \theta_\sigma)] \right).
\end{aligned}$$

Computation of the limit of the variance matrix $\text{Var}[T_4]$.

We have $\text{Var}[T_4] = \mathbb{E}[T_4 T_4^T] + o(1)$.

$$\text{Var}[T_4] = \frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} \mathbb{E}[g_{\sigma,\varsigma} g_{\bar{\sigma},\bar{\varsigma}}^T] + o(1).$$

By independence, $\mathbb{E}[g_{\sigma,\varsigma} g_{\bar{\sigma},\bar{\varsigma}}^T] = 0$ as soon as $\varsigma \cap \bar{\varsigma} = \emptyset$, where we identify a permutation ς and its image $\varsigma(\{1, \dots, k\})$. Therefore, we get

$$\begin{aligned}
\text{Var}[T_4] &\simeq \frac{nn' h_{n,n'}^p}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\substack{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n} \\ \varsigma \cap \bar{\varsigma} \neq \emptyset}} \mathbb{E}[g_{\sigma,\varsigma} g_{\bar{\sigma},\bar{\varsigma}}^T] \\
&= \frac{nn' h_{n,n'}^p}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\substack{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n} \\ \varsigma \cap \bar{\varsigma} \neq \emptyset}} g_{\sigma,\varsigma, \bar{\sigma}, \bar{\varsigma}} - \tilde{g}_\sigma \tilde{g}_{\bar{\sigma}}^T,
\end{aligned}$$

where $\tilde{g}_\sigma := \bar{\psi}_\sigma \mathbb{E}[K_{\sigma,\varsigma}(g_\varsigma - \theta_\sigma)] / f_{\mathbf{Z}',\sigma}$ and

$$g_{\sigma,\varsigma, \bar{\sigma}, \bar{\varsigma}} := \frac{\bar{\psi}_\sigma \bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}',\sigma} f_{\mathbf{Z}',\bar{\sigma}}} \mathbb{E} \left[K_{\sigma,\varsigma} K_{\bar{\sigma},\bar{\varsigma}} (g_\varsigma - \theta_\sigma) (g_{\bar{\varsigma}} - \theta_{\bar{\sigma}}) \right].$$

Assume now that $\varsigma \cap \bar{\varsigma}$ is of cardinality 1, i.e. there exists only one couple $(j, \bar{j}) \in \{1, \dots, k\}^2$ such that $\varsigma(j) = \bar{\varsigma}(\bar{j})$. Then,

$$\begin{aligned}
g_{\sigma, \varsigma, \bar{\sigma}, \bar{\varsigma}} &= \frac{\bar{\psi}_\sigma \bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}'_\sigma} f_{\mathbf{Z}'_{\bar{\sigma}}}} \int (g(\mathbf{X}_{1:k}) - \theta_\sigma) (g(\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k+\bar{j}-1}, \mathbf{x}_j, \mathbf{x}_{k+\bar{j}+1}, \dots, \mathbf{x}_{2k}) - \theta_{\bar{\sigma}}) \\
&\quad \cdot \prod_{i=1}^k K_h(\mathbf{z}_i - \mathbf{z}'_{\sigma(i)}) f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i) d\mu(\mathbf{x}_i) d\mathbf{z}_i \cdot K_h(\mathbf{z}_j - \mathbf{z}'_{\bar{\sigma}(\bar{j})}) \\
&\quad \cdot \prod_{\bar{i}=1, \bar{i} \neq \bar{j}}^k K_h(\mathbf{z}_{k+i} - \mathbf{z}'_{\bar{\sigma}(\bar{i})}) f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}_{k+i}, \mathbf{z}_{k+i}) d\mu(\mathbf{x}_{k+i}) d\mathbf{z}_{k+i} \\
&= \frac{\bar{\psi}_\sigma \bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}}(\mathbf{z}_j)} \int (g(\mathbf{X}_{1:k}) - \theta_\sigma) (g(\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k+\bar{j}-1}, \mathbf{x}_j, \mathbf{x}_{k+\bar{j}+1}, \dots, \mathbf{x}_{2k}) - \theta_{\bar{\sigma}}) \\
&\quad \cdot \prod_{i=1}^k K(\mathbf{t}_i) \frac{f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}_i, \mathbf{z}'_{\sigma(i)} + h\mathbf{t}_i)}{f_{\mathbf{Z}}(\mathbf{z}'_{\sigma(i)})} d\mu(\mathbf{x}_i) d\mathbf{t}_i \cdot h^{-p} K\left(\mathbf{t}_i + \frac{\mathbf{z}'_{\sigma(j)} - \mathbf{z}'_{\bar{\sigma}(\bar{j})}}{h}\right) \\
&\quad \cdot \prod_{\bar{i}=1, \bar{i} \neq \bar{j}}^k K(\mathbf{t}_{k+i}) \frac{f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}_{k+i}, \mathbf{z}'_{\bar{\sigma}(\bar{i})} + h\mathbf{t}_{k+i})}{f_{\mathbf{Z}}(\mathbf{z}_{k+i})} d\mu(\mathbf{x}_{k+i}) d\mathbf{t}_{k+i} \\
&\simeq \frac{\bar{\psi}_\sigma \bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}}(\mathbf{z}_j)} \int (g(\mathbf{X}_{1:k}) - \theta_\sigma) (g(\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k+\bar{j}-1}, \mathbf{x}_j, \mathbf{x}_{k+\bar{j}+1}, \dots, \mathbf{x}_{2k}) - \theta_{\bar{\sigma}}) \\
&\quad \cdot \prod_{i=1}^k K(\mathbf{t}_i) \frac{f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}_i, \mathbf{z}'_{\sigma(i)})}{f_{\mathbf{Z}}(\mathbf{z}_i)} d\mu(\mathbf{x}_i) d\mathbf{t}_i \cdot h^{-p} K\left(\mathbf{t}_i + \frac{\mathbf{z}'_{\sigma(j)} - \mathbf{z}'_{\bar{\sigma}(\bar{j})}}{h}\right) \\
&\quad \cdot \prod_{\bar{i}=1, \bar{i} \neq \bar{j}}^k K(\mathbf{t}_{k+i}) \frac{f_{\mathbf{X}, \mathbf{Z}}(\mathbf{x}_{k+i}, \mathbf{z}'_{\bar{\sigma}(\bar{i})})}{f_{\mathbf{Z}}(\mathbf{z}'_{\bar{\sigma}(\bar{i})})} d\mu(\mathbf{x}_{k+i}) d\mathbf{t}_{k+i}.
\end{aligned}$$

By assumption, this is zero unless $\sigma(j) = \bar{\sigma}(\bar{j})$. In this case, it can be simplified, giving

$$\begin{aligned}
g_{\sigma, \varsigma, \bar{\sigma}, \bar{\varsigma}} &\simeq \frac{\bar{\psi}_\sigma \bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}}(\mathbf{z}_j) h^p} \int K^2 \int (g(\mathbf{x}_{1:k}) - \theta_\sigma) (g(\mathbf{x}_{k:2k, \bar{j} \rightarrow j}) - \theta_{\bar{\sigma}}) \\
&\quad \cdot \prod_{i=1}^k f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_{\sigma(i)}}(\mathbf{x}_k) d\mu(\mathbf{x}_i) \prod_{\bar{i}=1, \bar{i} \neq \bar{j}}^k f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_{\bar{\sigma}(\bar{i})}}(\mathbf{x}_{k+i}) d\mu(\mathbf{x}_{k+i}) =: h^{-p} g_{\sigma, \bar{\sigma}, j, \bar{j}},
\end{aligned}$$

where $\mathbf{x}_{k:2k, \bar{j} \rightarrow j} := (\mathbf{x}_{k+1}, \dots, \mathbf{x}_{k+\bar{j}-1}, \mathbf{x}_j, \mathbf{x}_{k+\bar{j}+1}, \dots, \mathbf{x}_{2k})$.

Note that, if $\varsigma \cap \bar{\varsigma}$ is of cardinality strictly greater than 1, some supplementary powers of h^{-p} arise thanks to the repeated kernels in ς and $\bar{\varsigma}$. As a consequence, they are of lower order and therefore negligible. Using α -order expansions as in Equation (D.6), we get $\sup_\sigma |\tilde{g}_\sigma| = O(h^{k\alpha})$. Thus,

$$\text{Var}[T_4] \simeq O(nn' h_{n, n'}^{p+2k\alpha}) + \frac{nn' h_{n, n'}^p}{|\mathcal{J}_{k, n'}|^2 \cdot |\mathcal{J}_{k, n}|^2} \sum_{\varsigma \in \mathcal{J}_{k, n}} \sum_{j, \bar{j}=1}^k \sum_{\substack{\bar{\varsigma} \in \mathcal{J}_{k, n} \\ \varsigma(j) = \bar{\varsigma}(\bar{j}), |\varsigma \cap \bar{\varsigma}|=1}} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k, n'}, \sigma(j) = \bar{\sigma}(\bar{j})} h^{-p} g_{\sigma, \bar{\sigma}, j, \bar{j}}$$

$$\begin{aligned}
&\simeq \frac{n'}{|\mathcal{J}_{k,n'}|^2} \sum_{j,\bar{j}=1}^k \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}, \sigma(j)=\bar{\sigma}(\bar{j})} g_{\sigma, \bar{\sigma}, j, \bar{j}} \\
&\rightarrow \sum_{j,\bar{j}=1}^k g_{j,\bar{j},\infty} = V_2,
\end{aligned}$$

where

$$\begin{aligned}
g_{j,\bar{j},\infty} &:= \int \Lambda'(\theta(\mathbf{z}'_{1:k})) \Lambda'(\theta(\mathbf{z}'_{k:2k,\bar{j} \rightarrow j})) \boldsymbol{\psi}(\mathbf{z}'_{1:k}) \boldsymbol{\psi}^T(\mathbf{z}'_{k:2k,\bar{j} \rightarrow j}) \frac{\int K^2}{f_{\mathbf{Z}}(\mathbf{z}'_j)} \int (g(\mathbf{x}_{1:k}) - \theta(\mathbf{z}'_{1:k})) \\
&\quad \cdot (g(\mathbf{x}_{k:2k,\bar{j} \rightarrow j}) - \theta(\mathbf{z}'_{k:2k,\bar{j} \rightarrow j})) \prod_{i=1, i \neq k+\bar{j}}^{2k} f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_i}(\mathbf{x}_i) f_{\mathbf{Z}',\infty}(\mathbf{z}'_i) d\mu(\mathbf{x}_i) d\mathbf{z}'_i.
\end{aligned}$$

In Section Appendix D.2, we will prove that T_4 is asymptotically Gaussian ; therefore, its asymptotic variance will be given by V_2 .

Now, decompose the term T_5 , defined in Equation (D.5), using a Taylor expansion of the function $x \mapsto 1/(1+x)$ at 0.

$$\frac{1}{N_\sigma} - \frac{1}{f_{\mathbf{Z}',\sigma}} = \frac{1}{f_{\mathbf{Z}',\sigma}} \left(\frac{1}{1 + \frac{N_\sigma - f_{\mathbf{Z}',\sigma}}{f_{\mathbf{Z}',\sigma}}} - 1 \right) = -\frac{N_\sigma - f_{\mathbf{Z}',\sigma}}{f_{\mathbf{Z}',\sigma}^2} + T_{7,\sigma},$$

where

$$T_{7,\sigma} := \frac{1}{f_{\mathbf{Z}',\sigma}} (1 + \alpha_{7,\sigma})^{-3} \left(\frac{N_\sigma - f_{\mathbf{Z}',\sigma}}{f_{\mathbf{Z}',\sigma}} \right)^2, \text{ with } |\alpha_{7,\sigma}| \leq \left| \frac{N_\sigma - f_{\mathbf{Z}',\sigma}}{f_{\mathbf{Z}',\sigma}} \right|.$$

We have therefore the decomposition $T_5 = -T_6 + T_7$, where

$$T_6 := \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} K_{\sigma,\varsigma} (g_\varsigma - \theta_\sigma) \frac{N_\sigma - f_{\mathbf{Z}',\sigma}}{f_{\mathbf{Z}',\sigma}^2} \bar{\boldsymbol{\psi}}_\sigma, \tag{D.7}$$

$$T_7 := \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} K_{\sigma,\varsigma} (g_\varsigma - \theta_\sigma) T_{7,\sigma} \bar{\boldsymbol{\psi}}_\sigma. \tag{D.8}$$

Summing up all the previous equation, we get

$$T_1 = (T_4 - \mathbb{E}[T_4]) - T_6 + T_7 + T_3 + o(1).$$

Afterwards, we will prove that all the remainders terms T_6 , T_7 and T_3 are negligible, i.e. they tend to zero in probability. These results are respectively proved in Subsections Appendix D.3, Appendix D.4 and Appendix D.5. Combining all these elements with the asymptotic normality of T_4 (proved in Subsec-

tion Appendix D.2), we get $T_1 \xrightarrow{D} \mathcal{N}(0, V_2)$, as claimed. \square

Appendix D.2. Proof of the asymptotic normality of T_4

Using the Hájek projection of T_4 , we define

$$\begin{aligned} T_4 - \mathbb{E}[T_4] &= T_{4,1} + T_{4,2}, \text{ where} \\ T_{4,1} &:= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} \sum_{i=1}^k \mathbb{E}[g_{\sigma,\varsigma} | \varsigma(i)], \\ T_{4,2} &:= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} \left(g_{\sigma,\varsigma} - \sum_{i=1,\dots,k} \mathbb{E}[g_{\sigma,\varsigma} | \varsigma(i)] \right), \end{aligned}$$

denoting by $|i$ the conditioning with respect to $(\mathbf{X}_i, \mathbf{Z}_i)$, for $i \in \{1, \dots, n\}$. We will show that $T_{4,1}$ is asymptotically normal, and that $T_{4,2} = o(1)$.

Using the fact that the $(\mathbf{X}_i, \mathbf{Z}_i)_i$ are i.i.d., and denoting by Id the injective function $i \mapsto i$, we have

$$\begin{aligned} T_{4,1} &= \frac{k\tilde{r}_{n,n'}}{n|\mathcal{J}_{k,n'}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{i=1}^n \mathbb{E} \left[\frac{\bar{\psi}_\sigma}{f_{\mathbf{Z}',\sigma}} K_{\sigma,Id}(g_{Id} - \theta_\sigma) - \bar{g}_\sigma \middle| i \right] \\ &\simeq \frac{k\tilde{r}_{n,n'}}{n|\mathcal{J}_{k,n'}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{i=1}^n \mathbb{E} \left[\frac{\bar{\psi}_\sigma}{f_{\mathbf{Z}',\sigma}} K_{\sigma,Id}(g_{Id} - \theta_\sigma) \middle| i \right] =: \sum_{i=1}^n \alpha_{4,i,n}, \end{aligned}$$

because $\sup_\sigma |\bar{g}_\sigma| = O(h^{k\alpha})$, as proved in the previous section, hence negligible. The $\alpha_{4,i,n}$, for $1 \leq i \leq n$, form a triangular array of i.i.d. variables. To prove the asymptotic normality of $T_{4,1}$, it remains to check Lyapunov's condition, i.e. we will show that $\sum_{i=1}^n \mathbb{E}[|\alpha_{4,i,n}|_\infty^3] \rightarrow 0$. We have

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}[|\alpha_{4,i,n}|_\infty^3] &= n \mathbb{E}[|\alpha_{4,1,n}|_\infty^3] \\ &= \frac{k^3 n \tilde{r}_{n,n'}^3}{n^3 |\mathcal{J}_{k,n'}|^3} \sum_{\sigma, \nu, \vartheta \in \mathcal{J}_{k,n'}} \frac{\bar{\psi}_\sigma \otimes \bar{\psi}_\nu \otimes \bar{\psi}_\vartheta}{f_{\mathbf{Z}',\sigma} f_{\mathbf{Z}',\nu} f_{\mathbf{Z}',\vartheta}} \mathbb{E} \left[\mathbb{E} \left[K_{\sigma,Id}(g_{Id} - \theta_\sigma) \middle| 1 \right] \mathbb{E} \left[K_{\nu,Id}(g_{Id} - \theta_\nu) \middle| 1 \right] \mathbb{E} \left[K_{\vartheta,Id}(g_{Id} - \theta_\vartheta) \middle| 1 \right] \right] \\ &= \frac{k^3 \tilde{r}_{n,n'}^3}{n^2 |\mathcal{J}_{k,n'}|^3} \sum_{\sigma, \nu, \vartheta \in \mathcal{J}_{k,n'}} \frac{\bar{\psi}_\sigma \otimes \bar{\psi}_\nu \otimes \bar{\psi}_\vartheta}{f_{\mathbf{Z}'(\mathbf{z}'_{\nu(1)})} f_{\mathbf{Z}'(\mathbf{z}'_{\vartheta(1)})}} \int K_h(\mathbf{z}_1 - \mathbf{z}'_{\sigma(1)}) K_h(\mathbf{z}_1 - \mathbf{z}'_{\nu(1)}) K_h(\mathbf{z}_1 - \mathbf{z}'_{\vartheta(1)}) \\ &\quad \cdot \prod_{i=2}^k K_h(\mathbf{z}_i - \mathbf{z}'_{\sigma(i)}) K_h(\mathbf{z}_{k+i} - \mathbf{z}'_{\nu(i)}) K_h(\mathbf{z}_{2k+i} - \mathbf{z}'_{\vartheta(i)}) \\ &\quad \cdot (g(\mathbf{x}_{1:k}) - \theta_\sigma) (g(\mathbf{x}_1, \mathbf{x}_{(k+2):(2k)}) - \theta_\nu) (g(\mathbf{x}_1, \mathbf{x}_{(2k+2):(3k)}) - \theta_\vartheta) \\ &\quad \cdot \prod_{i=1}^k \frac{f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i)}{f_{\mathbf{Z}'(\mathbf{z}'_{\sigma(i)})}} d\mu(\mathbf{x}_i) d\mathbf{z}_i \prod_{i=2}^k \frac{f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_{k+i}, \mathbf{z}_{k+i})}{f_{\mathbf{Z}'(\mathbf{z}'_{\nu(i)})}} d\mu(\mathbf{x}_{k+i}) d\mathbf{z}_{k+i} \prod_{i=2}^k \frac{f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_{2k+i}, \mathbf{z}_{2k+i})}{f_{\mathbf{Z}'(\mathbf{z}'_{\vartheta(i)})}} d\mu(\mathbf{x}_{2k+i}) d\mathbf{z}_{2k+i} \\ &\simeq \frac{k^3 \tilde{r}_{n,n'}^3}{n^2 |\mathcal{J}_{k,n'}|^3} \sum_{\sigma, \nu, \vartheta \in \mathcal{J}_{k,n'}} \frac{\bar{\psi}_\sigma \otimes \bar{\psi}_\nu \otimes \bar{\psi}_\vartheta}{f_{\mathbf{Z}'(\mathbf{z}'_{\nu(1)})} f_{\mathbf{Z}'(\mathbf{z}'_{\vartheta(1)})}} \int h^{-2p} K(\mathbf{t}_1) K \left(\mathbf{t}_1 + \frac{\mathbf{z}'_{\sigma(1)} - \mathbf{z}'_{\nu(1)}}{h} \right) K \left(\mathbf{t}_1 + \frac{\mathbf{z}'_{\sigma(1)} - \mathbf{z}'_{\vartheta(1)}}{h} \right) \end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{i=2}^k K_h(\mathbf{t}_i) K_h(\mathbf{t}_{k+i}) K_h(\mathbf{t}_{2k+i}) (g(\mathbf{x}_{1:k}) - \theta_\sigma) (g(\mathbf{x}_1, \mathbf{x}_{(k+2):(2k)}) - \theta_\nu) (g(\mathbf{x}_1, \mathbf{x}_{(2k+2):(3k)}) - \theta_\vartheta) \\
& \cdot \prod_{i=1}^k f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_{\sigma(i)}}(\mathbf{x}_i) d\mu(\mathbf{x}_i) d\mathbf{z}_i \prod_{i=2}^k f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_{\nu(i)}}(\mathbf{x}_{k+i}) d\mu(\mathbf{x}_{k+i}) d\mathbf{z}_{k+i} \prod_{i=2}^k f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}'_{\vartheta(i)}}(\mathbf{x}_{2k+i}, \mathbf{t}_{2k+i}) d\mu(\mathbf{x}_{2k+i}) d\mathbf{t}_{2k+i},
\end{aligned}$$

where in the last equivalent, we use a change of variable from the \mathbf{z}_i to the \mathbf{t}_i , and then the continuity of the density $f_{\mathbf{X},\mathbf{Z}}$ with respect to \mathbf{z} , because $h = o(1)$.

Because of our assumptions, the terms of the sum for which $\sigma(1) \neq 1$ or $\nu(1) \neq 1$ are zero. Therefore, we get

$$\sum_{i=1}^n \mathbb{E}[|\alpha_{4,i,n}|^3_\infty] = \frac{\tilde{r}_{n,n'}^3 h^{-2p}}{n^2 |\mathcal{J}_{k,n'}|^3} \sum_{\sigma, \nu, \vartheta \in \mathcal{J}_{k,n'}, \sigma(1)=\nu(1)=1} O(1) = O\left(\frac{(nn'h^p)^{3/2}}{n^2 n'^2 h^{2p}}\right) = O\left(\frac{1}{(nn'h^p)^{1/2}}\right) = o(1).$$

We prove now that $T_{4,2} = o(1)$. Note first that, by construction, $\mathbb{E}[T_{4,2}] = 0$. Computing its variance, we get

$$\begin{aligned}
\mathbb{E}[T_{4,2} T_{4,2}^T] &= \mathbb{E}\left[\frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} \left(g_{\sigma,\varsigma} - \sum_{i=1,\dots,k} \mathbb{E}[g_{\sigma,\varsigma} | \varsigma(i)]\right) \left(g_{\bar{\sigma},\bar{\varsigma}} - \sum_{\bar{i}=1,\dots,k} \mathbb{E}[g_{\bar{\sigma},\bar{\varsigma}} | \bar{\varsigma}(\bar{i})]\right)^T\right] \\
&=: \frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} \mathbb{E}\left[\tilde{g}_{\sigma,\bar{\sigma},\varsigma,\bar{\varsigma}}\right]. \tag{D.9}
\end{aligned}$$

Because of $\mathbb{E}[g_{\sigma,\varsigma}] = 0$ and by independence, the terms in the latter sum for which $\varsigma \cap \bar{\varsigma} = \emptyset$ are zero. Otherwise, there exists $j_1, j_2 \in \{1, \dots, k\}$ such that $\varsigma(j_1) = \bar{\varsigma}(j_2)$. If $\varsigma \cap \bar{\varsigma}$ is of cardinal 1, meaning that there is no other identities between elements of ς and $\bar{\varsigma}$, then we will show that the corresponding term is zero as well. We place ourselves in this case, assuming that $|\varsigma \cap \bar{\varsigma}| = 1$, and we get

$$\begin{aligned}
\mathbb{E}\left[\tilde{g}_{\sigma,\bar{\sigma},\varsigma,\bar{\varsigma}}\right] &= \mathbb{E}\left[\left(g_{\sigma,\varsigma} - \sum_{i=1,\dots,k} \mathbb{E}[g_{\sigma,\varsigma} | \varsigma(i)]\right) \left(g_{\bar{\sigma},\bar{\varsigma}}^T - \sum_{\bar{i}=1,\dots,k} \mathbb{E}[g_{\bar{\sigma},\bar{\varsigma}}^T | \bar{\varsigma}(\bar{i})]\right)\right] \\
&= \mathbb{E}\left[\left(g_{\sigma,\varsigma} - \mathbb{E}[g_{\sigma,\varsigma} | \varsigma(j_1)]\right) \left(g_{\bar{\sigma},\bar{\varsigma}}^T - \mathbb{E}[g_{\bar{\sigma},\bar{\varsigma}}^T | \bar{\varsigma}(j_2)]\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(g_{\sigma,\varsigma} - \mathbb{E}[g_{\sigma,\varsigma} | \varsigma(j_1)]\right) \left(g_{\bar{\sigma},\bar{\varsigma}}^T - \mathbb{E}[g_{\bar{\sigma},\bar{\varsigma}}^T | \varsigma(j_1)]\right) \middle| \varsigma(j_1)\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[g_{\sigma,\varsigma} g_{\bar{\sigma},\bar{\varsigma}}^T \middle| \varsigma(j_1)\right]\right] - \mathbb{E}\left[\mathbb{E}[g_{\sigma,\varsigma} | \varsigma(j_1)] \mathbb{E}[g_{\bar{\sigma},\bar{\varsigma}}^T | \varsigma(j_1)]\right] = 0.
\end{aligned}$$

Therefore, non-zero terms in Equation (D.9) correspond to the case where there exists $j_3 \neq j_1, j_4 \neq j_1$ such that $\varsigma(j_3) = \bar{\varsigma}(j_4)$. It is equivalent to $|\varsigma \cap \bar{\varsigma}| \geq 2$. We will ignore higher-order terms, i.e. the ones for which

$|\varsigma \cap \bar{\varsigma}| > 2$, as they yield higher powers of h^p and are therefore negligible. Finally, Equation (D.9) becomes

$$\mathbb{E}[T_{4,2}T_{4,2}^T] \simeq \frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\substack{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n} \\ |\varsigma \cap \bar{\varsigma}|=2}} \left(\mathbb{E}[g_{\sigma, \varsigma} g_{\bar{\sigma}, \bar{\varsigma}}^T] - 2k \mathbb{E} \left[\mathbb{E}[g_{\sigma, \varsigma} | \varsigma(i)] \mathbb{E}[g_{\bar{\sigma}, \bar{\varsigma}}^T | \bar{\varsigma}(\bar{i})] \right] \right).$$

As before, using change of variables and limited expansions, we can prove that

$$\frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\substack{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n} \\ |\varsigma \cap \bar{\varsigma}|=2}} \mathbb{E}[g_{\sigma, \varsigma} g_{\bar{\sigma}, \bar{\varsigma}}^T] = o(1),$$

and similarly for the other term.

Appendix D.3. Convergence of T_6 to 0

Using Equation (D.7), we have $T_6 = T_{6,1} + T_{6,2}$, where

$$T_{6,1} := \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} K_{\sigma, \varsigma} (g_{\varsigma} - \theta_{\sigma}) \frac{N_{\sigma} - \mathbb{E}[N_{\sigma}]}{f_{\mathbf{Z}', \sigma}^2} \bar{\psi}_{\sigma}, \quad (\text{D.10})$$

$$T_{6,2} := \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} K_{\sigma, \varsigma} (g_{\varsigma} - \theta_{\sigma}) \frac{\mathbb{E}[N_{\sigma}] - f_{\mathbf{Z}', \sigma}}{f_{\mathbf{Z}', \sigma}^2} \bar{\psi}_{\sigma}. \quad (\text{D.11})$$

We first prove that $T_{6,1} = o(1)$. Using Equation (4), we have

$$\begin{aligned} T_{6,1} &= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} \frac{1}{f_{\mathbf{Z}', \sigma}^2} K_{\sigma, \varsigma} (g_{\varsigma} - \theta_{\sigma}) (N_k(\mathbf{z}'_{\sigma(1:k)}) - \mathbb{E}[N_k(\mathbf{z}'_{\sigma(1:k)})]) \bar{\psi}_{\sigma} \\ &= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} \frac{1}{f_{\mathbf{Z}', \sigma}^2} K_{\sigma, \varsigma} (g_{\varsigma} - \theta_{\sigma}) \sum_{\nu \in \mathcal{J}_{k,n}} \left(\prod_{i=1}^k K_h(\mathbf{z}_{\nu(i)} - \mathbf{z}'_{\sigma(i)}) - \mathbb{E} \left[\prod_{i=1}^k K_h(\mathbf{z}_{\nu(i)} - \mathbf{z}'_{\sigma(i)}) \right] \right) \bar{\psi}_{\sigma} \\ &= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\nu \in \mathcal{J}_{k,n}} \frac{1}{f_{\mathbf{Z}', \sigma}^2} K_{\sigma, \varsigma} (g_{\varsigma} - \theta_{\sigma}) \left(K_{\sigma, \nu} - \mathbb{E}[K_{\sigma, \nu}] \right) \bar{\psi}_{\sigma}. \end{aligned}$$

The terms for which $|\varsigma \cap \nu| \geq 1$ induce some powers of $(nh^p)^{-1}$, and are therefore negligible. We remove them to obtain an equivalent random vector $\bar{T}_{6,1}$, which is centered. Therefore it is sufficient to show that its second moment tends to 0.

$$\begin{aligned} \mathbb{E}[\bar{T}_{6,1} \bar{T}_{6,1}^T] &= \frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\substack{\varsigma, \nu \in \mathcal{J}_{k,n} \\ \varsigma \cap \nu = \emptyset}} \sum_{\substack{\bar{\varsigma}, \bar{\nu} \in \mathcal{J}_{k,n} \\ \bar{\varsigma} \cap \bar{\nu} = \emptyset}} \frac{\bar{\psi}_{\sigma}}{f_{\mathbf{Z}', \sigma}^2} \frac{\bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}', \bar{\sigma}}^2} g_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}, \nu, \bar{\nu}}, \\ g_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}, \nu, \bar{\nu}} &:= \mathbb{E} \left[K_{\sigma, \varsigma} (g_{\varsigma} - \theta_{\sigma}) \left(K_{\sigma, \nu} - \mathbb{E}[K_{\sigma, \nu}] \right) K_{\bar{\sigma}, \bar{\varsigma}} (g_{\bar{\varsigma}} - \theta_{\bar{\sigma}}) \left(K_{\bar{\sigma}, \bar{\nu}} - \mathbb{E}[K_{\bar{\sigma}, \bar{\nu}}] \right) \right]. \end{aligned}$$

The term $g_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}, \nu, \bar{\nu}}$ is 0 in two cases : if $\nu \cap (\varsigma \cup \bar{\varsigma} \cup \bar{\nu})$ or if $\bar{\nu} \cap (\varsigma \cup \bar{\varsigma} \cup \nu)$. This condition can be written as

$$\emptyset = [\nu \cap (\bar{\varsigma} \cup \bar{\nu})] \cup [\bar{\nu} \cap (\varsigma \cup \nu)] = (\nu \cup \bar{\nu}) \cap (\bar{\varsigma} \cup \bar{\nu}) \cap (\varsigma \cup \nu).$$

We deduce that non-zero terms arise only when there exists $j_1, j_2 \in \{1, \dots, k\}$ such that: $\nu(j_1) = \bar{\nu}(j_2)$ or $\nu(j_1) = \bar{\varsigma}(j_2)$ or $\bar{\nu}(j_1) = \varsigma(j_2)$. Therefore, we can write $\mathbb{E}[\bar{T}_{6,1} \bar{T}_{6,1}^T] = T_{6,1,1} + T_{6,1,2} + T_{6,1,3}$, where

$$\begin{aligned} T_{6,1,1} &= \frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{j_1, j_2=1}^k \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\substack{\varsigma, \nu \in \mathcal{J}_{k,n} \\ \varsigma \cap \nu = \emptyset}} \sum_{\substack{\bar{\varsigma}, \bar{\nu} \in \mathcal{J}_{k,n} \\ \bar{\varsigma} \cap \bar{\nu} = \emptyset, \bar{\nu}(j_2) = \nu(j_1)}} \frac{\bar{\psi}_\sigma}{f_{\mathbf{Z}', \sigma}^2} \frac{\bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}', \bar{\sigma}}^2} g_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}, \nu, \bar{\nu}}, \\ T_{6,1,2} &= \frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{j_1, j_2=1}^k \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\substack{\varsigma, \nu \in \mathcal{J}_{k,n} \\ \varsigma \cap \nu = \emptyset}} \sum_{\substack{\bar{\varsigma}, \bar{\nu} \in \mathcal{J}_{k,n} \\ \bar{\varsigma} \cap \bar{\nu} = \emptyset, \bar{\varsigma}(j_2) = \nu(j_1)}} \frac{\bar{\psi}_\sigma}{f_{\mathbf{Z}', \sigma}^2} \frac{\bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}', \bar{\sigma}}^2} g_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}, \nu, \bar{\nu}}, \\ T_{6,1,3} &= \frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{j_1, j_2=1}^k \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\substack{\varsigma, \nu \in \mathcal{J}_{k,n} \\ \varsigma \cap \nu = \emptyset}} \sum_{\substack{\bar{\varsigma}, \bar{\nu} \in \mathcal{J}_{k,n} \\ \bar{\varsigma} \cap \bar{\nu} = \emptyset, \bar{\nu}(j_1) = \varsigma(j_2)}} \frac{\bar{\psi}_\sigma}{f_{\mathbf{Z}', \sigma}^2} \frac{\bar{\psi}_{\bar{\sigma}}^T}{f_{\mathbf{Z}', \bar{\sigma}}^2} g_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}, \nu, \bar{\nu}}, \end{aligned}$$

We will prove that $T_{6,1,1} = o(1)$. The two other terms can be treated in a similar way. Because of our assumptions, the terms for which $\bar{\sigma}(j_1) \neq \sigma(j_2)$ are zero. This divides the number of possible terms by n' . By using limited expansions as in Equation (D.6), we get that $g_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}, \nu, \bar{\nu}} = O(h^{k\alpha-p})$. Therefore, we have $T_{6,1,1} = O(\frac{nn'h^p}{nn'} h^{k\alpha-p}) = O(h^{k\alpha}) = o(1)$.

Concerning $T_{6,2}$, its variance matrix is given by

$$\begin{aligned} Var[T_{6,2}] &= \frac{\tilde{r}_{n,n'}^2}{|\mathcal{J}_{k,n'}|^2 \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma, \bar{\sigma} \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} \frac{\mathbb{E}[N_\sigma] - f_{\mathbf{Z}', \sigma}}{f_{\mathbf{Z}', \sigma}^2} \frac{\mathbb{E}[N_{\bar{\sigma}}] - f_{\mathbf{Z}', \bar{\sigma}}}{f_{\mathbf{Z}', \bar{\sigma}}^2} \bar{\psi}_\sigma \bar{\psi}_{\bar{\sigma}}^T \bar{g}_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}}, \\ \bar{g}_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}} &:= \mathbb{E} \left[K_{\sigma, \varsigma} K_{\bar{\sigma}, \bar{\varsigma}} (g_\varsigma - \theta_\sigma) (g_{\bar{\varsigma}} - \theta_{\bar{\sigma}}) \right] - \mathbb{E} \left[K_{\sigma, \varsigma} (g_\varsigma - \theta_\sigma) \right] \mathbb{E} \left[K_{\bar{\sigma}, \bar{\varsigma}} (g_{\bar{\varsigma}} - \theta_{\bar{\sigma}}) \right]. \end{aligned}$$

Note that $\bar{g}_{\sigma, \bar{\sigma}, \varsigma, \bar{\varsigma}} = 0$ when $\varsigma \cap \bar{\varsigma} = \emptyset$. This divides the number of terms in the sum above by n , and imposes that $\sigma \cap \bar{\sigma} \neq \emptyset$, which divides the number of terms in the sum above by another n' . Finally, limited expansions gives a bound of $h^{k\alpha-p}$. Summing up all these elements, we obtain $Var[T_{6,2}] = O(\frac{\tilde{r}_{n,n'}^2}{nn'} h^{k\alpha-p}) = O(h^{k\alpha}) = o(1)$. Similarly, we get $\mathbb{E}[T_{6,2}] = o(1)$ by a Taylor expansion.

Appendix D.4. Convergence of T_7 to 0

We recall Equation (D.8):

$$T_7 = \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}| \cdot |\mathfrak{J}_{k,n}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \sum_{\varsigma \in \mathfrak{J}_{k,n}} K_{\sigma,\varsigma} (g_\varsigma - \theta_\sigma) T_{7,\sigma} \bar{\psi}_\sigma,$$

$$T_{7,\sigma} := \frac{1}{f_{\mathbf{Z}',\sigma}} (1 + \alpha_{7,\sigma})^{-3} \left(\frac{N_\sigma - f_{\mathbf{Z}',\sigma}}{f_{\mathbf{Z}',\sigma}} \right)^2, \text{ with } |\alpha_{7,\sigma}| \leq \left| \frac{N_\sigma - f_{\mathbf{Z}',\sigma}}{f_{\mathbf{Z}',\sigma}} \right|.$$

By Lemma 3 applied to $\mathbf{z}_1 = \mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}_{n'} = \mathbf{z}'_{\sigma(n')}$, for $\sigma \in \mathfrak{J}_{k,n'}$, we get

$$\mathbb{P} \left(\sup_{\sigma \in \mathfrak{J}_{k,n'}} |N_\sigma - f_{\mathbf{Z}',\sigma}| \leq \frac{C_{K,\alpha}}{\alpha} h^\alpha + t \right) \geq 1 - 2 \exp \left(- \frac{[n/k]t^2}{h^{-kp}C_1 + h^{-kp}C_2t} \right),$$

for any $t > 0$. Therefore, $\sup_{\sigma \in \mathfrak{J}_{k,n'}} |T_{7,\sigma}| = O_{\mathbb{P}}(h^{2\alpha})$ by choosing $t = h^{\alpha/k}$. Then,

$$|T_7| \leq \sup_{\sigma \in \mathfrak{J}_{k,n'}} |T_{7,\sigma}| \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}| \cdot |\mathfrak{J}_{k,n}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \sum_{\varsigma \in \mathfrak{J}_{k,n}} |K_{\sigma,\varsigma}| \cdot |g_\varsigma - \theta_\sigma| \cdot |\bar{\psi}_\sigma|.$$

The expectation of the double sum is $O(h^\alpha)$, by α -order limited expansions. By Markov's inequality, we deduce

$$T_7 = O_{\mathbb{P}} \left(\tilde{r}_{n,n'} \sup_{\sigma \in \mathfrak{J}_{k,n'}} |T_{7,\sigma}| h^\alpha \right) = O_{\mathbb{P}}(\tilde{r}_{n,n'} h^{3\alpha}) = O_{\mathbb{P}} \left((nn' h^{p+3\alpha})^{1/2} \right),$$

therefore $T_7 = o_{\mathbb{P}}(1)$.

Appendix D.5. Convergence of T_3 to 0

We have

$$T_3 := \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \alpha_{3,\sigma} \cdot \left(\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) - \theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \right)^2 \psi(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}),$$

with $\forall \sigma \in \mathfrak{J}_{k,n'}, |\alpha_{3,\sigma}| \leq C_{\Lambda''}/2$. Therefore

$$\begin{aligned} T_3 &\lesssim \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}|} \sum_{\sigma \in \mathfrak{J}_{k,n'}} \left(\hat{\theta}(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) - \theta(\mathbf{z}'_{\sigma(1)}, \dots, \mathbf{z}'_{\sigma(k)}) \right)^2 \\ &\lesssim \frac{\tilde{r}_{n,n'}}{|\mathfrak{J}_{k,n'}|} \left(\frac{1}{|\mathfrak{J}_{k,n}|} \sum_{\varsigma \in \mathfrak{J}_{k,n}} \frac{K_{\sigma,\varsigma}}{f_{\mathbf{Z}',\sigma}} (g_\varsigma - \theta_\sigma) + K_{\sigma,\varsigma} (g_\varsigma - \theta_\sigma) \left(\frac{1}{N_\sigma} - \frac{1}{f_{\mathbf{Z}',\sigma}} \right) \right)^2 = T_8 + T_9 + T_{10}, \end{aligned}$$

where

$$\begin{aligned}
T_8 &:= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} \frac{K_{\sigma,\varsigma} K_{\sigma,\bar{\varsigma}}}{f_{\mathbf{Z}',\sigma}^2} (g_\varsigma - \theta_\sigma) (g_{\bar{\varsigma}} - \theta_\sigma), \\
T_9 &:= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} \frac{K_{\sigma,\varsigma} K_{\sigma,\bar{\varsigma}}}{f_{\mathbf{Z}',\sigma}^2} (g_\varsigma - \theta_\sigma) (g_{\bar{\varsigma}} - \theta_\sigma) \left(\frac{1}{N_\sigma} - \frac{1}{f_{\mathbf{Z}',\sigma}} \right), \\
T_{10} &:= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} K_{\sigma,\varsigma} K_{\sigma,\bar{\varsigma}} (g_\varsigma - \theta_\sigma) (g_{\bar{\varsigma}} - \theta_\sigma) \left(\frac{1}{N_\sigma} - \frac{1}{f_{\mathbf{Z}',\sigma}} \right)^2.
\end{aligned}$$

We show that $T_8 = o(1)$. The two other terms can be treated in a similar way.

$$\begin{aligned}
\mathbb{E}[|T_8|] &= \mathbb{E} \left[\frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} \frac{|K_{\sigma,\varsigma} K_{\sigma,\bar{\varsigma}}|}{f_{\mathbf{Z}',\sigma}^2} |g_\varsigma - \theta_\sigma| \cdot |g_{\bar{\varsigma}} - \theta_\sigma| \right] \\
&= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma, \bar{\varsigma} \in \mathcal{J}_{k,n}} \int \frac{\prod_{i=1}^k |K_h(\mathbf{z}_{\varsigma(i)} - \mathbf{z}'_{\sigma(i)}) K_h(\mathbf{z}_{\bar{\varsigma}(i)} - \mathbf{z}'_{\sigma(i)})|}{f_{\mathbf{Z}',\sigma}^2} \\
&\quad \cdot \left| g(\mathbf{x}_{\varsigma(1:k)}) - \theta_\sigma \right| \left| g(\mathbf{x}_{\bar{\varsigma}(1:k)}) - \theta_\sigma \right| \prod_{i \in \varsigma(1:k) \cup \bar{\varsigma}(1:k)} f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i) d\mu(\mathbf{x}_i) d\mathbf{z}_i.
\end{aligned}$$

Note that terms for which $\varsigma \neq \bar{\varsigma} \in \mathcal{J}_{k,n'}$ are zero, because the \mathbf{z}'_i are distinct and because of our Assumption 12(i). Therefore, we get

$$\begin{aligned}
\mathbb{E}[|T_8|] &= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|^2} \sum_{\sigma \in \mathcal{J}_{k,n'}} \sum_{\varsigma \in \mathcal{J}_{k,n}} \int \frac{\prod_{i=1}^k K_h(\mathbf{z}_{\varsigma(i)} - \mathbf{z}'_{\sigma(i)})^2}{f_{\mathbf{Z}',\sigma}^2} \left(g(\mathbf{x}_{\varsigma(1:k)}) - \theta_\sigma \right)^2 \prod_{i \in \varsigma(1:k)} f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i) d\mu(\mathbf{x}_i) d\mathbf{z}_i \\
&= \frac{\tilde{r}_{n,n'}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \int \frac{\prod_{i=1}^k K_h(\mathbf{z}_i - \mathbf{z}'_{\sigma(i)})^2}{f_{\mathbf{Z}',\sigma}^2} \left(g(\mathbf{x}_{\varsigma(1:k)}) - \theta_\sigma \right)^2 \prod_{i=1}^k f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}_i) d\mu(\mathbf{x}_i) d\mathbf{z}_i \\
&= \frac{\tilde{r}_{n,n'} h^{-kp}}{|\mathcal{J}_{k,n'}| \cdot |\mathcal{J}_{k,n}|} \sum_{\sigma \in \mathcal{J}_{k,n'}} \int \frac{\prod_{i=1}^k K(\mathbf{t}_i)^2}{f_{\mathbf{Z}',\sigma}^2} \left(g(\mathbf{x}_{\varsigma(1:k)}) - \theta_\sigma \right)^2 \prod_{i=1}^k f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}_i, \mathbf{z}'_{\sigma(i)} + h\mathbf{t}_i) d\mu(\mathbf{x}_i) d\mathbf{z}_i \\
&= O\left(\frac{\tilde{r}_{n,n'} h^{-kp}}{|\mathcal{J}_{k,n}|} \right) = O\left(\left(\frac{n \times n' \times h^{(1-k)p}}{|\mathcal{J}_{k,n}|^2} \right)^{1/2} \right) = o(1). \quad \square
\end{aligned}$$