A Universal Update Principle for Consistent Choice*

Berend Roorda[†] Reinoud Joosten[†]

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Abstract

We propose a framework for complete preference orderings that reconciles two existing principles for conditional choice, without compromise: plan consistency, taking replacement values as non-consequentialist updates, and sequential consistency, identifying one of these updates as the consequentialist one, by a fixed point rule. The static axioms of the framework characterize their existence and uniqueness, and exclude forms of Dutch book opportunities. The normative claim relies on the global interpretation of preference orderings.

We review some implications for preferences based on multiple priors, address an issue in updating capacities, and suggest a weakening of the comonotonicity axiom.

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[†]Faculty of Behavioural, Management and Social Sciences, Department of Industrial Engineering and Business Information Systems, University of Twente, the Netherlands. Email corresponding author: b.roorda@utwente.nl.

The framework supports weak decomposability, as one of the possible directions to weaken the Sure Thing Principle, and betweenness, as the only one under law invariance. We indicate how betweenness accommodates the inverted S-shaped probability weighting common in Prospect Theory, and conclude that the commonly adopted narrow boundaries of rational choice need thorough revision.

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1 Introduction

Models are reference points in thinking, where paradoxes invite to think further. Famous paradoxes in decision theory, in particular those of Allais, Ellsberg, and Rabin, have been very successful in this respect, for decades, and they still are. It is not our aim to add new ideas to this matter, if that were possible, but to combine some findings into a general normative framework: that updating should and should not be consequentialist, that acts have one and several values, that the difference between risk and uncertainty is essential and not so relevant, that forward and backward induction are distinct yet should coincide.

We restrict the attention to the standard class of complete continuous monotone preference orderings on acts (or lotteries) with real outcomes, still the heart of the paradoxes. These can be identified with the unique normalized value function V they induce. We call this the *regular* class.

One of the explanatory factors behind the ongoing debate on consistency of choice, is the difference between the local and global interpretation of a preference ordering, hence of value. The common denominator is that it is the value a decision maker (DM) attributes to obtaining an act: she is indifferent between taking a long position in the act f or receiving the sure amount $V(f)$ (which means paying it when the amount is negative). Under the local interpretation, values are also attributed to an act *as such*, as perhaps suggested by the notation $V(f)$ for the value of an act f. Choices between offering acts, or exchanging one for the other, or between sub-acts, that are taken not in line with the value they *have*, then must reflect a change of taste or belief: V has changed.

Under the global interpretation, however, such differences arise naturally, as consequence of the same taste and belief. For example, the symmetric lottery e with outcome ± 1 dollar with equal probability, has negative value to a risk-averse DM, but it is the same risk aversion that makes her also not be willing to take the other side of the bet. Not V has changed, but the criterion $-V(-e)$ applies - a standard idea in bid-ask price modelling, see e.g. Madan and Cherny (2010). Likewise, as argued in Machina (1989), its value as sub-lottery in a larger bet is its replacement value, which need not be $V(e)$, but depends on the entire bet, *according to* V.

In mathematical terms, under the global interpretation, different types of values of an act f derive from V *as a function*, while under the local interpretation they all derive from $V(f)$ as an outcome, and hence coincide.

This distinction is crucial in the debate on rationality. It makes a big difference whether it is conducted (i) given the local interpretation, (ii) given the global interpretation, (iii) with the aim to compare both, or (iv) to question regular preference orderings as starting point. Confusion easily arises, such as invoking Dutch book arguments based on the local interpretation against adopting the global one. We aim to contribute to category (ii), focusing on the internal consistency of the global interpretation, but meanwhile we briefly reflect on the higher levels (iii) and (iv) as well.

So we adopt the global interpretation, including the variety of values for one and the same act, as different consequences of V for different decisions. In particular, we accept *ex ante* replacement values as *ex post* non-consequentialist updates. We call this principle *free induction*, to emphasize that it aligns backward and forward induction by interpretation, rather than by imposing an axiom.

However, and this has received less attention in the literature, there is a general principle to identify one of those replacement updates as *the* consequentialist update of a sub-act: *values should be in the range of their updates.* This is the notion of sequential consistency[1], axiom S1, which has been developed in a long-standing research line on risk measures, see e.g. Roorda and Schumacher (2016). Three static axioms, S2-4, characterize the existence and uniqueness of updates. A fifth axiom is sometimes imposed, requiring that the willingness to take a long position should not exceed the willingness to keep it. It nicely commutes with the other axioms, and gives Dutch book opportunities no chance.

The consequentialist update is produced by the *fixed point update (fpu)* rule. This is the same as Pires' rule (Pires, 2002), but we found it independently, as a consequence of sequential consistency. Thus we combine a simple consequentialist update rule with dynamic consistency, by free induction, without compromise. It is this combination that we propose as universal updating principle for consistent choice.

That updating remains so incredibly simple relies also on the more subtle nature of the global interpretation: it turns the conflicts between 'multiple selves' under the local interpretation into different answers of the DM to different questions about the same act - we will discern more than ten versions. We see it as an advantage

[1] This is quite different from the notion in Sarin and Wakker (1998), where it is defined as the meta principle of model closedness. We explain later on how we meet this principle.

that at this fundamental level, there is no reference yet to probability, taste, utility, convexity, mixtures, a horse-roulette structure, infinite divisibility of states, or state independence.

Despite the fact that all aforementioned aspects of the global interpretation have been well adressed in the literature, we believe that their synthesis in five axioms gives rise to a rigorous reconsideration of the boundary between rational and irrational decision making. We argue that many forms of so-called *bounded rationality*, as observed in the paradoxes, become *rationality bounded* by the local interpretation, when the global one is adopted. These differences are further illustrated by applying the axioms to the well-known, partly overlapping classes of Multiple Priors (MP) and Rank Dependent Utility (RDU). We address an issue in updating capacities, and suggest a weakening of comonotonicity axiom to make it comply with fixed point updating.

The five axioms capture, in our opinion, the contours of consistent dynamic choice. We could not find a meaningful representation theorem for this entire class. Instead, we describe the subclass in which sub-acts have the same replacement value in encompassing acts of equal value. This axiom of complementary replacement, is nearly the same as *weak decomposability* in Grant et al. (2000). So our framework supports this notion as one of the many ways to relax the Sure Thing Principle consistently. We also show that betweenness (Dekel, 1986; Chew, 1983) is the only extra flexibility when our axioms are combined with law invariance. We note that betweenness allows for practically the same inverted S-shaped probability weighting for binary lotteries as estimated in Tversky and Kahneman (1992).

This paper is organized as follows. Scope and notation are described below. The three subsequent sections introduce respectively the axioms S1-3, S4, and S5. In Section 5 we discuss arbitrage opportunities and the unit of choice underlying our setup. Section 6 addresses the contrast between bounded rationality and rationality bounded, by a series of quotes. Application to the classes MP and RDU is described in Sections 7 and 8. Section 9 is devoted to the weakly decomposable subclass. Betweenness and its link to Prospect Theory are addressed in Section 10, and conclusions follow in Section 11. The appendix contains the proofs and a technical lemma.

1.1 Scope and notation

We consider acts of the form $f : \Omega \to X$, with Ω a finite outcome space, and X a finite interval $[x_*, x^*] \subset \mathbb{R}$ of monetary outcomes. The set of all acts is denoted by $\mathcal{A}(\Omega, X)$, or simply $\mathcal{A}$. If an act f has $f(\omega) = c \in X$ on Ω, it is called a constant (act), and then we use the symbol c also for f. The interval $[\min f, \max f]$ is denoted as range(f), so range$(c) = \{c\}$ for $c \in X$. An act is also called a lottery when an externally given probability measure on Ω is specified. Our scope is the class of regular preference orderings, satisfying the usual basic axioms.

Definition 1.1 $\mathcal{P}$ is the class of preference orderings $\preceq$ on $\mathcal{A}$ that are complete, transitive, monotone (strictly on constants), and continuous.

The equivalence $f \sim c$ if and only if $V(f) = c$ defines a one-to-one correspondence between $\mathcal{P}$ and the class of value functions $V : \mathcal{A} \to X$ that are continuous, monotone, and normalized, i.e., have $V(c) = c$. This V is called the (normalized) *value function* of $\preceq$, and we write $V \in \mathcal{P}$.

Updates are defined with respect to a state space S, identified with a partition of Ω. First we consider one given state space S, later on we take $S = \Omega$ and consider updating for all events $E \subset S$. The sub-act of an act $f \in \mathcal{A}$ in $s \in S$ is denoted as f_s, and $\mathcal{A}_s$ denotes the set of all sub-acts in state s. A consequentialist (state) update of $\preceq$ in s is a preference ordering on $\mathcal{A}_s$, denoted as $\preceq_s$. For the vectors $(\mathcal{A}_s)_{s \in S}$ and $(\preceq_s)_{s \in S}$, we use the notation $\mathcal{A}_1$ and $\preceq_1$, but $(f_s)_{s \in S}$ is simply identified with f. V_s and V_1 are defined analogously. The (vector of) preference ordering(s) $\preceq_1$ is referred to as a (vector) update of $\preceq$. The definition of regularity extends to updates in the obvious way. We write $f_s h$ for the result of pasting sub-act f_s in state s into an act h; $f_E h$ is defined similarly, for events $E \subset S$.

2 The axioms for consequentialist updating

The following axiom imposes a relationship between a preference ordering and its consequentialist updates with respect to a given state space S. It is the cornerstone of our framework. The axioms below apply to $f, g \in \mathcal{A}$ and $c, d \in X$.

S1 *(Sequential Consistency)* If $f_s \sim_s c$ on S, then $f \sim c$.

It is equivalent, in $\mathcal{P}$, to the condition

$$c \preceq_1 f \preceq_1 d \quad \Rightarrow \quad c \preceq f \preceq d. \tag{2.1}$$

In other words, *values should be in the range of their consequentialist updates*, a substantial weakening of the common notion of (state-)monotonicity,

$$f \preceq_1 g \quad \Rightarrow \quad f \preceq g. \tag{2.2}$$

Definition 2.1 $\mathcal{S}$ is the subclass of preferences in $\mathcal{P}$ with unique regular sequentially consistent updates.

The class $\mathcal{S}$ is characterized by the following static axioms.

S2 *(Equal Level Principle)* If $f_s c \sim c \in \text{range}(f_s)$ on S, then $f \sim c$.

S3 *(c-Sensitivity)* If $f_s c \sim c$, then $f_s d \succ d$ for $d < c$ and $f_s d \prec d$ for $d > c$, for all $s \in S$ and $c, d \in \text{range}\, f_s$.

Axiom S2 is the weakening of the Sure Thing Principle (STP) that characterizes existence of consistent updates, under the sensitivity condition in axiom S3 that guarantees their uniqueness.

As shown in the theorem below, consequentialist updating in $\mathcal{S}$ amounts to the following mechanism, which we call *fixed point updating (fpu)*:

$$f_s \sim_s c \quad :\Leftrightarrow \quad f_s c \sim c \text{ with } c \in \text{range}(f_s). \tag{2.3}$$

We call $\preceq_s$ a *fixed point update* of $\preceq$ (in state s) if it satisfies the forward implication in (2.3); it satisfies (2.3) if and only if it is the unique one.

Theorem 1 *A preference ordering $\preceq$ in $\mathcal{P}$ has unique fixed point updates $\preceq_s$ on S if and only if $\preceq$ satisfies axiom S3, and then $\preceq_s$ is given by (2.3), and is regular. The (vector) update $(\preceq_s)_{s\in S} =: \preceq_1$ is then sequentially consistent (axiom S1) if and only if $\preceq$ also satisfies axiom S2, otherwise $\preceq$ has no regular sequentially consistent update.*

So $\mathcal{S}$ is the class of regular preferences that satisfy axiom S2 and S3. In this sense, these axioms are equivalent to axiom S1.

The notation and preceding results generalize from states $s \in S$ to events E in a partition of S in the obvious way. In particular, under the analogues of axiom S2 and S3, the consistent update $\preceq_E$ is then determined by (2.3), with s replaced by E. This satisfies a compatibility property, called commutativity in Gilboa and Schmeidler (1989), which requires that $\preceq_s$ can also be obtained as the update of $\preceq_E$ with $s \in E$. Compatibility is also addressed in (Roorda and Schumacher, 2013, Prop. 4.6) and (Roorda and Schumacher, 2016, Prop. 6.7).

3 Replacement values as embedded updates

The replacement value of a sub-act is the constant for which it can be replaced without changing the value of the encompassing act. To guarantee uniqueness, we impose the following sensitivity condition.

S4 *(Strict state sensitivity)* $(d+\delta)_s f \succ d_s f \quad (d, d+\delta \in X, \delta > 0, f \in \mathcal{A}, s \in S)$.

The axiom guarantees, for all regular preference orderings, that the replacement operator

$$R_s^f : \mathcal{A}_s \to X, \quad R_s^f(g_s) := r \text{ such that } g_s f \sim r_s f,$$

is a well defined, regular value function in each state $s \in S$, for all acts f. Clearly, also replacement values on events $E \subset S$ are then well-defined. [2]

Following the argument in Machina (1989), namely that these replacement values induced by V should also govern conditional choice, *as a consequence of* V, embedded updates are defined by

$$g_E \preceq_E^f h_E \text{ if and only if } g_E f \preceq h_E f, \tag{3.1}$$

cf. Machina and Schmeidler (1992). It is called the f-Bayesian update rule in Gilboa and Schmeidler (1993), in the context of ambiguous beliefs.

So $\preceq_E^f$ denotes the *ex post* preference ordering in state E when bygone states had outcome specified by f (note that only the restriction of f to the complement of E is relevant in this notation), and is represented by

$$V_s^f = R_s^f. \tag{3.2}$$

This rule is as simple as it is important. It aligns *ex ante* and *ex post* conditional choice, without the rigor of the Sure Thing Principle that imposes independence of f at both sides of this equality, but still compatible with the global interpretation that updates should be derived from V as a function. We call it *free induction*, because it aligns forward and backward induction by interpretation rather than restriction of V.

[2] We avoid to use the term certainty equivalent for replacement values, since it is used in the literature for specific notions of replacement with different meaning under the global representation. In Eichberger et al. (2007), the term conditional certainty equivalent consistency is used for what we call the fixed point update rule.

Definition 3.1 (Free Induction) The principle of *free induction* requires that *ex ante* replacement values are interpreted as anticipated *ex post* conditional values, and vice versa.

The principle justification of the distinction between choosing consequentially, according to V_s, and non-consequentially, according to V_s^f, is that the *nature* of choice is different. V does determine the latter choice, but not via V_s. Axiom S2 requires equality only when bygone states are neutral, i.e.,

$$V_s^c(g_s) = V_s(g_s) \text{ for } c = V_s(g_s). \tag{3.3}$$

This is perfectly in line with Machina's argument that a difference between V_s^f and V_s is due to 'borne risk' in f, which is indeed absent when $f = c$. In fact, axiom S2 also implies that for any f with all sub-acts of value equal to $V_s(g_s)$, it holds that $V_s^f(g_s) = V_s(g_s)$. This is how the 'ribs' V_s^f are attached to the 'backbone' V_s. In other cases, however, there is borne risk when s obtains, and a choice for g_s is no longer the same as a 'fresh' choice to obtain g_s.

Axiom S3 is equivalent to the following condition, which we call the *sign property* of embedded updates in each state:

$$\text{if } V_s^d(g_s) - d = 0 \text{ then } V_s^{d'}(g_s) - d' < (>)\ 0 \text{ for } d' > (<)\ d. \tag{3.4}$$

The interpretation is most clear when we assume that axiom S2 also holds for the binary state spaces $\{s, \bar{s}\}$, with $\bar{s}$ the complement of s in S. Then the sign property ensures that embedded updates $r = V_s^f(g_s)$ remain at the same side of $c = V(g_s f)$ as the consequentialist ones, $d = V_s(g_s)$, i.e., $r - c$ and $d - c$ have the same sign. To proceed the analogy, there are ribs to left and right of the backbone, corresponding to negative embeddings (with $c < d, r$) and positive ones (with $c > d, r$). So bygone exposure never turns a sub-act that is 'good' on its own, with $d > c$, into a 'bad' one, with $r < c$, but only affects its degree of goodness. In brief, the inconsequentialist nature of embedded updates is quantitative, not qualitative. This guarantees that values are also in the range of their embedded updates.

Summarizing, we combine two straighforward update rules, (2.3) and (3.2), in one universal update principle, without compromise. It results in the central update V_s, which is by definition consequentialist, and the side-updates V_s^f, which by definition guarantee plan consistency. We believe they reinforce eachother. The simplicity of the fixed point update that defines V_s can only be appreciated in combination with V_s^f. Conversely, the non-consequentialist nature of V_s^f is better understood with V_s as anchor point.

4 Twin consistency

We identify taking a short position in an act with taking the long position in the opposite outcomes. In this way, a preference ordering $\preceq$ induces a counterpart $\preceq^*$ by the well-known reflection principle

$$f \preceq^* g \;\Leftrightarrow\; -g \preceq -f, \tag{4.1}$$

as used e.g. in *conic finance* for modelling market bid- and ask prices (Madan and Cherny, 2010). This is an ordering on the space of acts $-\mathcal{A}$ with outcomes in $-X$. Of particular interest is the case with symmetric outcome range $X = [-x^*, x^*]$, since then also $\preceq^*$ is a preference ordering on $\mathcal{A}$. As $\preceq^*$ shares many properties with $\preceq$, we call them twin preferences. Neither the axioms S1-4 nor the regularity conditions can tell the difference. Correspondingly, we call $V^*(f) = -V(-f)$ the twin value of f induced by $\preceq$. We impose the following axiom.

S5 *(Twin consistency)* If $f \sim c$, then $-f \preceq -c$, i.e., $f \succeq^* c \quad (f \in \mathcal{A} \cap -\mathcal{A})$.

It states that the willingness to obtain something, cannot be less than the willingness to keep it. It would be absurd to buy something for 100 that one finds, at the same time, only worth 80 to keep. In other words, value cannot have negative 'thickness', and the reason to allow it to be strictly positive, is obvious from the example of the symmetric lottery in the introduction. Notice that $V^*(c) = c = V(c)$ for $c \in X \cap -X$, so constants have 'thin' value. Reflection commutes with updating: $(V^*)_s = (V_s)^*$, so we can safely write V_s^*. Similarly, R_s^{*f} is the same as R_s^{f*}.

5 Arbitrage opportunities and unit of choice

A preference ordering is not consistent when it leaves room for arbitrage opportunities, i.e., ways to extract a sure payment from a DM trading according to that preference ordering. Absence of arbitrage has quite strong implications in a trading context, combined with the local interpretation, implying that the DM is willing to buy and sell at the same price, in any amount, and any combination.

The global intrepretation already deprives this argument from its sharpness, since the DM 'buys low, sells high', as a consequence of axiom S5. Furthermore, asset aggregation falls outside our scope. Nevertheless, it could be ruled out rigorously, by imposing

$$V \leq L \leq V^*, \tag{5.1}$$

with L a linear operator that is arbitrage-free when values are thin. This kind of condition, called *consistent risk aversion* in Roorda and Schumacher (2016), would be required when the DM accepts multiple trades at bid price V and ask price V^*.

This raises the more general issue of asset aggregation, and the unit of choice. If the DM has to compare series of decisions, the decision space consists of dynamic strategies, and her objects of choice are in fact the resulting aggregated acts. A tacit assumption in our starting point is that acts resemble the appropriate level of decision, or, at least, the level of interest. Behind this assumption lies the premise that such an appropriate level exists, that it can be meaningful to analyse decisions in a certain degree of isolation. Concretely, that the DM indeed compares (taking long positions in) acts f and g in a set $\mathcal{A}$, resembling the units of object of choice.

Factors like asset aggregation, but also initial wealth, dependence on choice menus, and random choice are essential aspects in real decision making, but we do not believe that leaving them out is the *cause* of the controversies on rationality. The paradoxes invite to a thought experiment within a 'narrow frame', a small room so to speak, with the reward to be found inside the room. Note, however, that within that narrow frame, of just comparing acts with well-defined outcome space, the global interpretation already recognizes that the whole is more than the sum of the parts: $V(f)$ is not recursive in $V_1(f)$. We believe that this better prepares for going out to the large world than the reductionist view of the local interpretation, which tries to understand it as a recursive concatenation of small worlds. The global interpretation has greater expectations.

6 Bounded rationality versus rationality bounded

The five axioms complete our normative framework. As we already emphasized, they are not new, but we believe that the internal consistency of their combination gives rise to rethinking the boundaries of rational choice. A few quotes illustrate how it puts well-established findings under the local interpretation in a different perspective.

To start with, the problematic implication of completeness that a DM should "... always be ready to take one side or the other of any bet" (Binmore, 2017), does not apply under the global interpretation, as illustrated by the elementary lottery in the introduction.

The title of Kahneman et al. (1991), *Anomalies: The endowment effect, loss aversion, and status quo bias*, is in line with their observation that in descriptive models “the important notion of a stable preference order must be abandoned in favor of a preference order that depends on the current reference level.” From our perspective, it is the global interpretation in action, rather than instability, as again illustrated by the elementary lottery.

Recursiveness, backward induction, and state monotonicity (2.2) are closely related concepts that are quite compelling under the local interpretation. For instance, state monotonicity is deemed a “basic rationality axiom” as it appears in the standard Anscomb-Aumann setting, in the extensive exposition on rationality (Gilboa, 2015). However, it is problematic for consequentialist updates under the global interpretation: instead of $V_s(g_s) \geq V_s(h_s)$ implying $g_sf \geq h_sf$, the premise $V_s^f(g_s) \geq V_s^f(h_s)$ does, by free induction. The distinction does not rely on ambiguity. Plan consistency is still guaranteed, not by ‘folding back’ consequentialist decisions, but inconsequentialist ones.

Recursiveness is also one of the cornerstones of the framework in Epstein and Zin (1989): “all utility functions considered in this paper are based on a recursive structure and so are intertemporally consistent”. Under the global interpretation, however, value has thickness, and there is no reason to impose that the thickness of conditional value $V_1^*(g) - V_1(g)$, or other features beside $V_1(g)$, are irrelevant for the contribution of the sub-acts in g to the value $V(g)$ of the whole. We therefore also reject the well-known rectangularity condition on multiple priors in Epstein and Schneider (2003), cf. the next section. More specifically, in conic finance, recursiveness is still commonly imposed in bid and ask prices separately (Madan, 2016), so that the induced dynamic preference reversals (see also below) are avoided. However, as we have indicated in Roorda and Schumacher (2013, Ex. 3.9), this may lead to a market in which round trip costs can always be avoided.

Segal (1990) proposes to maintain recursiveness in valuation, but to reject the axiom of reduction of compound lotteries. From our perspective, it is the other way around: valuation need not be recursive, but there is no problem in taking the law of lotteries, compound or not, as the object of choice.

Relaxing recursiveness leads to so-called preference reversals, a clear sign of inconsistency under the local interpretation. It has been interpreted as evidence of intransitivity in Fishburn (1985). Alternatively, Tversky and Thaler (1990) note that "Unlike the measurement of physical attributes such as mass or length, however, different methods of eliciting preference often give rise to systematically different ordering", and conclude that preferences are not "procedure invariant". From our perspective, we have to disentangle two aspects that they identify: (i) unstable units of choice, and (ii) unstable choice given its unit. The second point has already been addressed, from our perspective: the different procedures relate to different questions. This also applies to the first point: that change of unit of choice affects preferences holds true, *a fortiori*, under the global interpretation - it already recognizes that comparing the whole involves more than comparing the parts when units for the whole are fixed, as argued in the previous section. An extra complication here is the experimental environment, which easily frames the participants' perception of unit of choice, and even may appeal to their cooperation to follow suggestions at this point. This also complicates the validity of the Becker-deGroot-Marschak procedure used to elicit (selling) prices.

We conclude with two quotes on model closedness, i.e., the requirement that updates should be in the same class as initial preferences. Sarin and Wakker (1998) signal a restrictive consequence of model closedness (which they call sequential consistency), when combined with recursiveness: "Rank-dependent and betweenness models can only be used in a restrictive manner, where in at most one stage deviation from expected utility is allowed." We show that there is no such complication for betweenness under the global interpretation, whereas the complication for rank-dependent models partly remains.

Hanany and Klibanoff (2007, 2009) adopt non-consequentialist updating as a way to achieve dynamic consistency, in line with Machina (1989) and the global interpretation. They reject, however, (3.1) as a general update rule, because it lacks the closedness property. "Unfortunately, the Machina-Schmeidler update rule fails closure for any set of preferences that includes non-probabilistically sophisticated members, as long as the preferences satisfy essentially the Savage [1954] axioms without the Sure-Thing Principle (Savage's P2)" We recognize the desirability of model closedness, but, taking the global interpretation one step further, would only impose it for the 'central' fixed point update V_s, since the side updates V_s^f represent a different type of value than V. Ribs are less similar to V than the backbone, in our framework. This distinction resolves the issue in Bayesian updating for the class of max-min expected utility (MEU), addressed in (Hanany and Klibanoff, 2007, Prop. 3). We conjecture that it also partly reconciles (3.1) with model closedness in the smooth ambiguity model, addressed in their 2009 paper, but a thorough analysis is beyond the scope of this paper.

In view of all these issues, inherent in the local interpretation, doubts have been expressed that a universal update principle can exist outside expected utility models (Wakker, 2010; Machina and Viscusi, 2013). We believe it does.

7 Application to multiple priors

We review the implications of the axioms for several well-known classes of preference orderings. To start with, we consider the Multiple Priors (MP) model, also known as Maxmin Expected Utility (MEU), introduced in Gilboa and Schmeidler (1989) in the Anscombe-Aumann setting. Adapted to our (simpler) setting, these are the ones represented by value functions of the form

$$V(f) = u^{-1} \min_{Q \in \mathcal{Q}} E^Q u \cdot f, \tag{7.1}$$

with $\mathcal{Q}$ a set of probability measures on Ω, called priors, E^Q the corresponding expectation operator, and $u : X \to \mathbb{R}$ a strictly increasing continuous utility function applied per outcome. This is precisely the class characterized in Casadesus-Masanell et al. (2000), restricted to finite Ω.

Its simplest form has $\mathcal{Q}$ a singleton $\{Q\}$, and $u(x) = x$. Then $V = E^Q$, axiom S3 and S4 each require that $Q(s) > 0$ on S, and then axiom S2 is always satisfied. Axiom S5 holds with equality, values are 'thin'. Updating is Bayesian. Characterizations of axioms S1-4 for the class of expected utility, $V(f) = u^{-1}E^Q(u \cdot f)$, are directly obtained from the following general result, which is easily verified.

Lemma 7.1 *The regularity conditions and axioms S1-4 are invariant under a strictly increasing continuous utility transformation of X, and fixed point updating commutes with such a transformation.*

Hence, in the analysis of axioms S1-4 for the class MP, we can restrict the attention to $u(x) = x$, and extend V to acts with outcomes in $\mathbb{R}$, in the obvious way. Then V is a so-called *coherent measure*, and we refer to Roorda and Schumacher (2007) for an extensive analysis of sequential consistency for this class.[3]

Axiom S5, however, does impose a restriction on the utility function. It requires that the twin of V in (7.1), given by

$$V^*(f) = \bar{u}^{-1} \max_{Q \in \mathcal{Q}} E^Q \bar{u} \cdot f, \text{ with } \bar{u}(x) := -u(-x),$$

dominates V. It is sufficient for axiom S5 that $\bar{u}u^{-1}$ is convex, and this is also the necessary condition if $\mathcal{Q}$ is a singleton. In general, the implications of axiom S5 for u depend on the set of priors.

[3]In brief, the rectangularity condition in Epstein and Schneider (2003) weakens to a junctedness condition for the priors: vectors of conditional priors from S need not be combinable with *all* priors on S, but only with *at least one*. Extensions to so-called variational preferences can be found in Roorda and Schumacher (2016).

8 Application to Rank Dependent Utility

The Rank Dependent Utility (RDU) model, also called Choquet Expected Utility (CEU), proposed in Quiggin (1982) and Schmeidler (1989), represents preference orderings in terms of capacities. For finite Ω, a capacity ν is a mapping from subsets of Ω to the interval $[0, 1]$, characterized by the properties $\nu(\Omega) = 1$, $\nu(\emptyset) = 0$, and $\nu(A) \leq \nu(B)$ when $A \subseteq B$. For an act $f = (x_1, \dots, x_N)$, now with indices rearranged so that $x_1 \geq x_2 \geq \dots \geq x_N$, define

$$\nu \cdot f \;:=\; \pi_1 x_1 + \pi_2 x_2 + \cdots + \pi_N x_N, \text{ with } \pi_j \;:=\; \nu(\cup_j^N) - \nu(\cup_{j+1}^N), \quad (8.1)$$

where $\cup_j^k$ is the event corresponding to $(x_j, \dots, x_k)$. The conjugate $\bar{\nu}$ is defined by $\bar{\nu}(A) = 1 - \nu(\bar{A})$, with $\bar{A}$ the complement of A in Ω. RDU consists of preference orderings representable by value functions V of the form $V(f) = \nu \cdot (u \cdot f)$, with u a strictly monotone continuous utility function. In view of the previous lemma, we can again assume $u(x) = x$ and take $X = \mathbb{R}$ in the analysis of axioms S1-4. It can be shown that axiom S3 for a capacity ν amounts to

$$\nu(A \cap s) + \bar{\nu}(\bar{A} \cap s) > 0 \text{ on } S \quad (A \subseteq \Omega). \quad (8.2)$$

For binary acts f of the form 1_A, the fpu V_s must therefore coincide with the conditional capacity ν_s defined by

$$\nu_s(A \cap s) = \frac{\nu(A \cap s)}{\nu(A \cap s) + \bar{\nu}(\bar{A} \cap s)} \quad (A \subseteq \Omega). \quad (8.3)$$

So if the update of ν in s is a capacity, then it is ν_s. As shown in Horie (2013), however, this is generally not the case, even when ν is convex (and hence V belongs to the class MP). From our perspective, taking axiom S1 as starting point, the class of capacities has to be adjusted, to meet the requirement of closedness under sequential consistent updating.

8.1 A refinement of comonotonicity

To further analyse the issue, consider the outcome c for $V_s(f_s)$ prescribed by the fixed point update rule,

$$\nu \cdot f_s c = c.$$

The reason that V_s is generally not a capacity, is that a pair of comonotone sub-acts f_s, f'_s, need not have comonotone neutral embeddings $f_s c$, $f'_s c'$, since the rank of c in $f_s c$ need not be the same as the rank of c' in $f'_s c'$. This suggest to weaken the notion of comonotonicity, by imposing the property for pairs $(f, V(f)), (f', V(f'))$.[4] We then say that f, f' are *c-comonotone* (with respect to V).

The corresponding weakening of the comonotonic STP (Chew and Wakker, 1996) for preference orderings is

cC *(c-Comonotonic STP)* For all $E \subset S$, $f_E h \preceq g_E h \Leftrightarrow f_E h' \preceq g_E h'$ whenever the four acts are pairwise c-comonotonic.

The extension of their representation result in terms of outcome-dependent capacities in Chew and Wakker (1996) is straightforward, see Lemma A.1 in the appendix. The corresponding extension of RDU is to apply two capacities inward, from both sides, until they 'meet' the balance point c.[5] Formally, a pair of capaci-

[4]Closedness can also be achieved by replacing c by a fixed embedding x_* or x^* for all acts, cf. (Gilboa and Schmeidler, 1993, Thm. 3.2), the latter one being the Dempster-Shafer rule for updating ambiguous beliefs, which amounts to maximum likelihood updating on the intersection with the class MP. These updates generally do not satisfy axiom S1, but they turn out to play a natural role in the definition of c-equivalents, Definition 9.1.

[5]Interestingly, the special case with one capacity a scalar multiple of the other, appears in the representation results of Horie (2017), generalizing Gul's disappointment aversion theory (Gul, 1991). One of the axioms, 'relative comonotonicity', is nearly the same as c-comonotonicity.

ties $(\hat{\nu}, \check{\nu})$ defines $V(f)$ as the value c such that, for some $m < N$,

$$\hat{\nu}(\cup_1^1)(x_1 - x_2) + \cdots + \hat{\nu}(\cup_1^m)(x_m - c) = \check{\nu}(\cup_{m+1}^N)(c - x_{m+1}) + \cdots + \check{\nu}(\cup_N^N)(x_{N-1} - x_N). \quad (8.4)$$

Recall that we work with utility $u(x) = x$ here. Equivalently,

$$\Sigma_{j=1}^m(\hat{\nu}(\cup_1^j) - \hat{\nu}(\cup_1^{j-1}))(x_j - c) + \Sigma_{j=m+1}^N(\check{\nu}(\cup_j^N) - \check{\nu}(\cup_{j+1}^N))(x_j - c) = 0. \quad (8.5)$$

The necessary and sufficient condition for uniqueness of c, for all acts, is

$$\hat{\nu}(A) + \check{\nu}(\bar{A}) > 0 \quad (A \subset \Omega).$$

The fpu defines V_s by the same rule (8.4), now with both summations restricted to terms with $j \in s$. Axiom S3 now requires that for all $s \in S$,

$$\hat{\nu}(A \cap s) + \check{\nu}(\bar{A} \cap s) > 0 \text{ on } S \quad (A \subset \Omega),$$

so that c is indeed uniquely determined by this recipe. So, this exension of comonotonic independence is closed under fixed point updating.

Axiom S5 just requires that $\hat{\nu} \leq \check{\nu}$, since V^* is represented by the pair $(\check{\nu}, \hat{\nu})$ when V is represented by $(\hat{\nu}, \check{\nu})$.

Axiom S2, however, still puts a severe restriction on the capacities, similar to the standard case. As observed in Sarin and Wakker (1998), the induced outcome weights for an outcome in s cannot depend on its rank with respect to outcomes outside s, i.e., $\hat{\nu}$ and $\check{\nu}$ both have the property that

$$\nu(A \cup A') - \nu(A') = \nu(A) \text{ for } A \subseteq s, A' \subseteq \bar{s}, A' \cap s' \neq s' \text{ on } S.$$

Consequently, $\hat{\nu}$ and $\check{\nu}$ are additive on S, and hence resemble only rank dependency within sub-acts separately.

In brief, the refinement to c-comonotonicity makes it compatible with fixed point updating (axiom S3), but this still clashes with the principle that values should be in the range of their updates (axiom S1, S2). This leaves the accommodation of suitable forms of rank dependency in our framework a topic of future research.

At the technical level, our findings suggest to replace the standard Choquet integral by a centered version, with respect to a two-sided capacity $\nu = (\hat{\nu}, \check{\nu})$,

$$\int f d\nu = c \text{ for } c \text{ such that } \int_c^\infty \hat{\nu}(f \geq x)dx = \int_{-\infty}^c \check{\nu}(f \leq y)dy.$$

This also provides an alternative approach to defining an intrinsic reference point for two-sided Choquet integrals, as proposed in Werner and Zank (2019) in the context of Prospect Theory. In the centered Choquet integral, it is the point c where both integrals 'meet', in dependence on the act under consideration.

9 The subclass with c-equivalents

The axioms so far leave much room for the nature of inconsequentialism of replacement values of sub-acts. For instance, they can be made dependent of both the risk in a sub-lottery, as well as the risk towards it - an idea pursued in Roorda and Schumacher (2016); Roorda and Joosten (2015).

We proceed, however, with describing a simpler subclass, in which the replacement value of a sub-act depends only on the value of the act of which it is part:

$$\text{if } g_s f \sim g_s f' \text{ then } g_s f \sim r_s f \Leftrightarrow g_s f' \sim r_s f' \quad (r \in X, s \in S) \tag{9.1}$$

We do not view it as a compelling principle, but use it to illustrate the distinction between the more than ten types of value arising from the global interpretation. Moreover, it explains how our framework accomodates concept of weak decomposability in Grant et al. (2000).

We define the subclass in terms of the following axiom, equivalent to (9.1).

CR *(Complementary Replacement)* If $f \sim r_s f \sim f_s \bar{r}$ then $f \sim r_s \bar{r}$ $\quad (r, \bar{r} \in X, s \in S)$.

Like (9.1), it says that the replacement value in state s in an act only depends on the replacement value of the bygone part of the act. Notice that this is substantially stronger than axiom S2. The following notion captures the essential feature of this axiom.

Definition 9.1 (c-equivalent) For $V \in \mathcal{P}$ satisfying axioms S2-4 and CR, the c-equivalent of a sub-act is its replacement value in any act of value c. Formally, for a given sub-act $g \in \mathcal{A}_s$, and $c \in X$,

$$r_{c,s}(g) := \begin{cases} r \text{ such that } r_s f \sim g_s f \text{ for (any) } f \text{ s.t. } g_s f \sim c & c \in [V(g_s x_*), V(g_s x^*)] \\ r \text{ such that } r_s x_* \sim g_s x_* & c \prec g_s x_* \\ r \text{ such that } r_s x^* \sim g_s x^* & c \succ g_s x^* \end{cases}.$$

Only in the first case we call the replacement value r *proper*, and we call the interval for c in this case the proper domain of the c-equivalent profile of g. The vector of c-equivalents of an act $f \in \mathcal{A}$ is denoted as $r_{c,1}(f)$, and is called proper if all its entries are proper replacement values.

Obviously, $r_{c,1}(f)$ is always a proper replacement vector for $c = V(f)$, and hence

$$V(f) \overset{(>)}{=} c \text{ if and only if } V(r_{c,1}(f)) \overset{(>)}{=} c. \tag{9.2}$$

In words, serial and parallel replacement of sub-acts coincide under axiom CR.

9.1 Relation with weak decomposability

Axiom CR is largely the same as the axiom of *weak decomposability*, as defined in Grant et al. (2000), although our interpretation in terms of replacement values is somewhat different. Under suitable continuity assumptions, their axiom reads

$$f_E g \sim f, g_E f \sim f \Rightarrow f \sim g \quad (E \subset S).$$

This is equivalent to imposing axiom CR, not only for the given state space S, but also for all binary state spaces $\{E, \bar{E}\}$ with $E \subset S$. They explain how this captures the essential motivation of Savage for the STP, describe its equivalence with the notion of *dynamic programming solvability* in Gul and Lantto (1990), and provide necessary and sufficient conditions for weak decomposability in terms of an additive fixed point representation. Our characterization below is very close to these results. In addition, they show how it leads to the betweenness axiom (Dekel, 1986; Chew, 1983), when combined with probabilistic sophistication, as discussed later on.

To enhance the comparison, we now strengthen the consistency axioms to requirements pertaining to conditioning on any event in the final outcome space Ω.

Definition 9.2 $\mathcal{C}$ is the class of preference orderings that satisfy axioms S3, S4 and CR (hence also S2) for all state spaces S that partition Ω, and $\mathcal{C}_n$ is the subclass with $\sharp\, \Omega = n$.

From a normative perspective, this can be defended as a consistency requirement under all conceivable conditional thinking, regardless the actual information structure that may be externally given. After all, this is how the Allais and Ellsberg preferences become paradoxes.

9.2 Fixed Point Representations

We describe an additive representation for this class $\mathcal{C}$ that is largely the same as in Grant et al. (2000), which relies, in turn, on the results in Segal (1992). We rephrase their results for our simpler setting, characterize axiom S3, and address axiom S5.

Formally, we now take $S = \Omega$, and impose the axioms for all partitions of S. Note that axiom S4 now amounts to strict monotonicity of V. We also write $f = (x_1, \ldots, x_n)$, and $S = \{1, \ldots, n\}$, so both x_j and f_s can be used to indicate the outcome of f in state $s = j$. The interior of $\mathcal{A}$ is denoted as $\mathcal{A}'$, that of X as X'.

The additive representation takes the form of a fixed point condition,

$$V(f) \overset{(>)}{=} c \Leftrightarrow B_c(f) := u_{c,1}(x_1) + \cdots + u_{c,n}(x_n) \overset{(>)}{=} 0, \tag{9.3}$$

where $u_{c,s}(x)$ is the utility of outcome x in state s in an act of value c, defined zero for $x = c$. We call B_c the induced balance function at level c.

Definition 9.3 (Fixed Point Representation) An FPR is a collection $U = \{u_{c,s}\}_{c \in X, s \in S}$ of utility functions

$$u_{c,s} : X \to \mathbb{R}, \text{ strictly increasing, continuous, and with } u_{c,s}(c) = 0, \tag{9.4}$$

that satisfies the following two conditions:

1. the *basic sign property*: for each $c \in X$ and $f \in \mathcal{A}$,

$$B_c(f) = 0 \;\Rightarrow\; B_d(f) < 0 \text{ for } d > c \text{ and } B_d(f) > 0 \text{ for } d < c \tag{9.5}$$

2. the *continuity property*: for each $f \in \mathcal{A}$,

$$\{c \in X \mid B_c(f) \geq 0\} \text{ and } \{c \in X \mid B_c(f) \leq 0\} \text{ are closed sets.} \tag{9.6}$$

That $V(f)$ is well defined by (9.3) for all acts f follows from the fact that the two properties guarantee respectively uniqueness and existence of c such that $B_c(f) = 0$. It is clear that the represented V is regular; conversely, (9.3) only defines a regular V if U has the two properties. Sufficient for these properties is that for each $x \in X$, $c \mapsto u_{c,s}(x)$ is continuous and strictly decreasing, cf. Grant et al. (2000, footnote 17).

The essential domain of $u_{c,s}$ in U is denoted as

$$D_{c,s} = \{x \in X \mid u_{c,s}(x) + \Sigma_{s' \neq s} u_{c,s'}(y) = 0 \text{ for some } y \in X\}, \tag{9.7}$$

and the essential domain of U is the corresponding collection. By standard results, FPRs are unique on their essential domain, modulo a positive scalar for each collection $\{u_{c,s}\}_{s\in S}$. Utility values outside the effective domain are irrelevant for V; only their monotonicity is required to respect the strict inequalities in (9.3).

The characterization of the class $\mathcal{C}$ involves the extension of the sign property (9.5) to partial sums

$$B_{c,E}(f) := \Sigma_{i\in E} u_{c,i}(x_i), \ E \subset S.$$

The *partial sign property* requires that

$$B_{c,E}(f) = 0 \ \Rightarrow \ B_{d,E}(f) < 0 \text{ for } d > c \text{ and } B_{d,E}(f) > 0 \text{ for } d < c. \tag{9.8}$$

Theorem 2 (Fixed Point Valuation) *V belongs to class $\mathcal{C}_n$ if, and only if when $n \neq 3$, it has a regular fixed point representation with the partial sign property (9.8).*

The analysis in Chateauneuf and Wakker (1993) makes clear why an exception has to be made for the case $n = 3$. For $n > 3$, axiom CR entails the so-called Thomsen condition, which can be expressed as a condition on c-equivalents,

$$\text{If } (x, y) \sim^c (x', y') \text{ and } (x', y'') \sim^c (x'', y) \text{ then } (x, y'') \sim^c (x'', y'), \tag{9.9}$$

where $\sim^c$ denotes equality of c-equivalents on the corresponding pair of states. For $n = 3$, however, axiom CR is void, but the Thomsen condition is still required for the existence of an FPR. So, the exception for $n = 3$ can be cancelled if the Thomsen condition for $n = 3$ is added to axiom CR.

We conclude with a remark on axiom S5. The twin V_U^* is represented by

$$U^* = \{u^*_{c,s}\}_{s\in S, c\in X} \text{ with } u^*_{c,s}(x) := -u_{-c,s}(-x).$$

A sufficient, but not necessary, condition for axiom S5 is $U^* \geq U$.

9.3 An overview of values for the same act

We summarize the results by addressing the interpretation of the several types of value induced by one and the same value function. We take starting point in a value function V on three-outcome acts, denoted as $f = (x, y, z)$, specifying the outcomes in the states $S = \{s, s', \tilde{s}\}$, with FPR

$$V(x, y, z) = c \quad \Leftrightarrow \quad u_c(x) + v_c(y) + w_c(z) = 0. \tag{9.10}$$

For the updates, we consider conditioning on $E = \{s, s'\}$. The table below contains four values assigned to the act f, and two updated values, of f_E.

		definition	criterion
1	c	$V(f)$	(9.10)
2	π	$V(f - \pi) = 0$	$u_0(x - \pi) + v_0(y - \pi) + w_0(z - \pi) = 0$
3	c^*	$-V(-f)$	$u_{-c^*}(-x) + v_{-c^*}(-y) + w_{-c^*}(-z) = 0$
4	π^*	$V(\pi - f) = 0$	$u_0(\pi^* - x) + v_0(\pi^* - y) + w_0(\pi^* - z) = 0$
5	d	$V_E(f_E)$	$u_d(x) + v_d(y) = 0$
6	r	$r_{c,E}(f_E)$	$u_c(x) + v_c(y) = u_c(r) + v_c(r)$

The first two values are known as the Willingness to Pay (WTP), where c is the 'choice' variant, and π the 'actual' price that the DM is willing to pay. They are also known as bid prices, where, depending on the context, the choice variant or actual payment is considered. The second couple, c^* and π^*, are the corresponding Willingness to Accept (WTA) figures. Axiom S5 takes care that bid-ask spreads $c^* - c$ and $\pi^* - \pi$ are nonnegative.

The updated values, d and r, correspond to conditional WTP values, both in choice variant. They come in two versions: the consequentialist d can be called the *willingness to conditionally pay*: it governs the (*ex ante* and *ex post*) 'fresh' choice between taking long positions f_E *in case* E obtains, while no position is taken when it does not. To emphasize this latter aspect, we can write $d = V(x, y, -)$, which, by the fixed point rule, is the solution of $V(x, y, d) = d$.[6] So, it is the unconditional WTP (in choice variant) when the amount d the DM is willing to pay is returned in case E does not obtain. In contrast, the (*ex ante* and *ex post*) conditional choice in E, when bygone outcome was z, is governed by r. This is the solution to $V(x, y, z) = V(r, r, z)$.

We can add two 'actual' WTP conditional values to the table as items 7 and 8, and the four WTA versions of 5-8 as items 9-12, to arrive at eight versions of conditional value of f_E, all derived from V.[7]

From this perspective, the Allais and Ellsberg paradoxes can be interpreted as evidence that these differences are important in common decision making. In particular, $d \neq r$, which is already justified when probabilities are known, but can become even more compelling when they are not. Grant et al. (2000, Example E) provides a particularly simple example of an FPR that accomodates the Ellsberg paradox, by loss-aversion with c-dependent reference point. Rabin's paradox can be accommodated similarly, by making risk- and loss aversion depending on c. Notice that balance weight functions $u_{c,s}$ can have both a kink in the *status quo* reference point 0, as well as in the internal reference point c. This observation confirms the extensive analysis of the paradox in Bleichrodt et al. (2017).

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[6]It may be noted here that V_E could be called a 'natural ordering' on 'half-lotteries' $(x, y, -)$, in view of Segal's argument for discriminating between compound and mixture independence, namely that such an ordering does not exist (Segal, 1990, p. 369).

[7]In fact, also V_c can be considered, defined by $V_c(x, y, z) = r$ for $u_c(x) + v_c(y) + w_c(z) = u_c(r) + v_c(r) + w_c(r)$. This can be interpreted as the induced value of f, seen as part of an act that had value c in the past. In this way, model closedness holds also for embedded updating. In view of the quote from Kahneman et al. (1991) in Section 6, it is worth noticing that V_c entirely derives from V for every 'current reference level' c.

10 Law invariance and betweenness

As already indicated, there is a strong link with betweenness, when the axioms are combined with law invariance. A straighforward way to incorporate law invariance in the framework of acts is as follows. Let $\mathcal{D}$ denote the set of all probability distributions Q on X, and let $\phi : \mathcal{D} \to \mathbb{R}$ be a continuous valuation (in the topology of weak convergence). Define, for $K \in \mathbb{N}$, the value function $V_K^\phi(f) = \phi(Q^f)$, with Q^f the law of an act $f : \Omega_K \to X$ when a uniform distribution is assumed on Ω_K, consisting of K elements. This collection of value functions determines ϕ completely, and the axioms induce the following class.

Definition 10.1 $\mathcal{B}$ is the class of continuous valuations $\phi : \mathcal{D} \to X$ for which V_K^ϕ is regular and satisfies axioms S1-4, for all partitions S of Ω_K, for all $K \in \mathbb{N}$.

Regularity (Definition 1.1) amounts to normalization ($\phi(\delta_x) = x$), continuity, and first order stochastic dominance, so that ϕ maps to X. Axiom S2 then amounts to betweenness,

$$\phi(Q) = c = \phi(Q') \Rightarrow \phi(\alpha Q + (1-\alpha)Q') = c \quad (\alpha \in [0,1]), \tag{10.1}$$

and axiom S3 is then equivalent to the sensitivity condition

$$\phi(\alpha Q + (1-\alpha)\delta_c) = c \text{ for some } \alpha \in (0,1) \Rightarrow \phi(Q) = c.$$

Axiom S1 prescribes that the update $\phi_A(Q)$ is determined by the fixed point rule, equivalent to Bayesian updating,

$$\phi_A(Q) = c \quad :\Leftrightarrow \quad \phi(1_A Q + 1_{\bar{A}} c) = c.$$

Axiom S4 is the condition of *strict* first order stochastic dominance. Axiom CR is now redundant, since it is equivalent to betweenness under law invariance.

From the results in Dekel (1986); Chew (1989), the betweenness representation for class $\mathcal{B}$ follows: there must exist a famililiy of balance weight functions $U = \{u_c\}_{c\in X}$ such that

$$\phi(Q) \overset{(>)}{=} c \Leftrightarrow E^Q u_c(x) \overset{(>)}{=} 0,$$

with u_c balance weight functions, satisfying (9.4). Moreover, there exists such a representation U with $c \mapsto u_c(x)$ continuous and strictly decreasing.[8]

10.1 A note on probability weighting in Prospect Theory

There are a few observations in order here, related to linearity in probabilities. The representation U already derives from the restriction of ϕ to binary lotteries. In fact, this can be further restricted to those with one extreme outcome, i.e., the set

$$\{(x, p; y, 1-p) \,|\, p \in (0,1), x \geq y, \text{ and } x = x^* \text{ or } y = x_*\}.$$

We call the restriction of ϕ to this set its *signature*. So ϕ can be elicited on binary lotteries in the signature set, and its extension to general lotteries, and all its updates, then follow from the axioms S1-4.[9]

[8] This representation result can also be derived via the previous theorem, by applying it first for uniform distributions on Ω_K, then for all discrete Q with probabilitie masses multiples of $1/K$, and finally using continuity of ϕ. Monotonicity in c can be derived from the observation that u_c and u_d cannot intersect for both a gain $x > c$ as well as a loss $y < c$, since then $(x, p; y, 1-p)$ would have value c and value d for $p/(1-p) = -u_c(y)/u_c(x)$; hence there must be a scalar so that $u_c > u_d$.

[9] The signature set is called class (2) in the proof of Prop. 1 in Dekel (1986). The extension to all binary lotteries follows from a well-known 'cross condition', that links a lottery $L = (x, p; y, 1-p)$ to the values of those of $(x^*, q; x_*, 1-q)$, $(x^*, p'; y, 1-p')$ and $(x, q'; x_*, 1-q')$, by the requirement that if all three have identical value, then L must have the same value, for p given by $p/(1-p) \times q/(1-q) = p'/(1-p') \times q'/(1-q')$. The extension to non-binary lotteries then follows from their decomposition into lotteries of equal value.

Now, if we return to the binary lotteries $L = (x, p; y, 1-p)$, with $x > y$, the test that L has at least value c, with $c \in [y, x]$, can be expressed as

$$\frac{p}{1-p} \geq \frac{\bar{u}_c(y)}{u_c(x)},$$

where $\bar{u}_c$ denotes $-u_c$. This means that the c-test for L can be rewritten as

$$b\left(\frac{p}{1-p}\right)^a \geq b\,\frac{\bar{u}_c(y)^a}{u_c(x)^a}$$

This is, in fact, the only transformation that leaves the ratio at the right-hand side separable in y and x. In terms of probability weighting, this yields the c-test

$$w(p)\; u_c(x)^a + (1 - w(p))\; bu_c(y)^a \overset{(>)}{=} 0 \text{ with } w(p) = \frac{bp^a}{(1-p)^a + bp^a}. \tag{10.2}$$

This corresponds to the 'linear in log odds' probability transformation (Gonzalez and Wu, 1999). In this sense, betweenness is compatible with probability weighting, for binary lotteries. The classic weighting functions in (Cumulative) Prospect Theory (PT) are of different functional form, but numerically nearly the same (Prelec, 1998). By choosing appropriate parameters a and b for each level c, the PT model in Tversky and Kahneman (1992), restricted to binary lotteries, can be approximated quite closely in the betweenness class.[10]

So the PT model is practically indistinguishable from the betweenness class for the set comprised of all binary lotteries with gains only, losses only, or mixes with value (close to) zero. It is remarkable that the original empirical basis of PT, consisting of such binary lotteries only, also allows for linearity in probabilities, when

[10]They use probability weighting $w^+(p) = \frac{p^\gamma}{(p^\gamma+(1-p)^\gamma)^{1/\gamma}}$ for gains and $w^-(p) = \frac{p^\delta}{(p^\delta+(1-p)^\delta)^{1/\delta}}$ for losses, with $\gamma = 0.61, \delta = 0.69$. This is nearly the same as $w(p)$ in (10.2) with (i) $a = 0.543, b = 0.734$ for gains, and (ii) $a = 0.641, b = 0.834$ for losses. For mixes $L = (x, p; y, 1-p)$ with $x > 0 > y$, the ratio $w^+(p)/(w^+(p) + w^-(1-p))$ corresponds to (iii) $a = 0.651, b = 0,947$. This can be combined in one betweenness model by e.g. taking (iii) for $c = 0$, and a continuous transition towards (ii) and (iii) for c sufficiently positive / negative.

utility is allowed to be dependending on the value c of lotteries. This indicates that the gap between descriptive and normative models may be not as large as generally believed.

11 Conclusions

We have described an axiomatic framework in which different types of value for the same act arise as the consistent application of one preference ordering. Updating is straightforward, both in consequentialist form (by the fixed point update rule) and non-consequentalist form (by free induction). The update principle we propose just combines them, in full harmony. Existence and uniqueness of updates is characterized by just one substantial static restriction (axiom S2), and two static sensitivity conditions (axiom S3 and S4). We have shown how several forms of bounded rationality, under the local interpretation, regain their consistency in this framework. We have indicated some tension with the class of Rank Dependent Utility, and suggest to weaken the comonotonicity axiom in a specific way.

A representation theorem for the subclass with c-equivalents explains how it supports the consistency of weak decomposability (for acts) and betweenness (for lotteries), thus also a form of probability weighting for binary lotteries that is nearly the same as the inverted S-shape commonly used in Prospect Theory.

We conclude that neither the probability weighting in Prospect Theory, as estimated from binary lotteries, nor the three-outcome paradoxes of Allais and Ellsberg, indicate a clear gap between normative and descriptive models, when the global interpretation of preference orderings is adopted.

Current research concentrates on the centipede game, see e.g. Binmore (1997). We currently apply our framework to overcome issues related to updating, and the support of capacities, as addressed in Eichberger et al. (2017), so as to concentrate all attention to the challenge it still poses to the modelling of rationality, knowledge and beliefs in the last couple of stages of the game - a strong invitation to think further.

A Appendix

Proof of Theorem 1

We first prove that (2.3) defines a unique update $\preceq_s$ under axiom S3, for each $s \in S$. Let V denote the (normalized) value function of $\preceq$. Consider, for given $f_s \in \mathcal{A}_s$, the mapping $\rho : c \mapsto V(f_s c)$ on the domain range$(f_s) =: [l, r]$. Since V is continuous and monotone, ρ is continuous, $\rho(l) \geq l$ and $\rho(r) \leq r$. So ρ has a fixed point c' on this domain, i.e., there exists c' satisfying the right-hand side (rhs) of (2.3). Axiom S3 guarantees that such c' is unique, and hence that $\preceq_s$ is uniquely determined by (2.3). This means that $\preceq_1$ is indeed unambiguously defined by (2.3).

This proves the if-part of the first claim of the theorem. The only if-part is obvious from the formulation of S3.

Regularity of $\preceq_s$, under axiom S3, follows straightforwardly from regularity of $\preceq$. In particular, $\preceq_s$ is continuous, because for a series $f_s^k \to f_s$ in $\mathcal{A}_s$, with c_k the unique solution of the rhs of (2.3) for f_s^k, any converging subseries $(c_k)_{k \in \mathcal{I} \subset \mathbb{N}} \to c'$ yields $V(f_s c') = c' \in$ range(f_s), by continuity of V; so c' must be the unique solution of the rhs in (2.3), and hence the full series $(c_k)_{k \in \mathbb{N}}$ is converging to c'.

To see that $\preceq_1$ defined by (2.3) is sequentially consistent if $\preceq$ satisfies axiom S2, consider $f \in \mathcal{A}$ with $f \sim_1 c$. Then (2.3) implies that for all $s \in S$, $f_s c \sim c$ with $c \in$ range(f_s), and by axiom S2, $f \sim c$, so that axiom S1 follows.

It remains to show, under axiom S3, that if $\preceq$ has a regular sequentially consistent update $\preceq_1$, then $\preceq$ must satisfy axiom S2. Let an act $f \in \mathcal{A}$ be given with $f_s c \sim c$ and $c \in$ range(f_s) for all $s \in S$. We have to prove that $f \sim c$. Consider an $s \in S$.

As $\preceq_1$ is regular, there exists $c' \in \text{range}(f_s)$ such that $f_s \sim_s c'$, and hence $f_s c' \sim_1 c'$. But then $f_s c' \sim c'$ by axiom S1, while also $f_s c \sim c$ by assumption, and axiom S3 implies that $c' = c$. Since $s \in S$ was arbitrary, $f_s \sim_s c$ for all $s \in S$, and, again by axiom S1, indeed $f \sim c$.

Lemma on the c-Comonotonic STP

We extend some of the results in Chew and Wakker (1996) to obtain a characterization of the c-comonotonic STP in our relatively simple setting with finite Ω. The essential idea is to refine comonotonic cones Π by pairs (Π, m), in which m marks the rank of c, i.e.,

$$(\Pi, m) = \{(x_1, \ldots, x_N) \,|\, x_1 \geq x_m \geq c \geq x_{m+1} \geq x_n\}.$$

We call $W : \mathbb{R} \times 2^S \to \mathbb{R}$ an outcome-dependent capacity if (i) $W(x, \emptyset) = 0$, (ii) $x \mapsto W(x, A)$ is continuous and (iii) $W(x, B) - W(x, A) - W(y, B) + W(y, A) > 0$ for all $x > y$ and $A \subsetneq B$.[11] Define $W^c(x, A) := W(x, A) - W(c, A)$, and, following their notation, $V(x, B, A) := W(x, B) - W(x, A)$.

A pair $(\hat{W}, \check{W})$ of outcome-dependent capacities represents $\preceq$ if $f = (x_1, \ldots, x_N) \sim c$ for the unique c such that there exists m with

$$\Sigma_{j=1}^{m} \hat{W}^c(x_j, \cup_1^j) - \hat{W}^c(x_j, \cup_1^{j-1}) \; + \; \Sigma_{j=m+1}^{N} \check{W}^c(x_j, \cup_j^N) - \check{W}^c(x_j, \cup_{j+1}^N) = 0. \quad \text{(A.1)}$$

More compactly,

$$f \sim c \text{ when } \Sigma_{j=1}^{n} V^c(x_j, \cup_1^j, \cup_1^{j-1}) = 0, \qquad \text{(A.2)}$$

with V^c defined by, in obvious notation,

$$V^c(x, B, A) := \begin{cases} \hat{V}(x, B, A) - \hat{V}(c, B, A) & (x \geq c) \\ \check{V}(x, \bar{A}, \bar{B}) - \check{V}(c, \bar{A}, \bar{B}) & (x \leq c). \end{cases}$$

[11] Taking $A = \emptyset$ implies strict monotonicity of $x \mapsto W(x, B)$. The capacities induced by W are $\nu_{x,y}$, defined by $\nu_{x,y}(A) := (W(x, A) - W(y, A)/(W(x, \Omega) - W(y, \Omega))$.

Lemma A.1 *A regular preference ordering satisfies c-comonotonicity if it is representable as above. For $n \geq 4$, this is also a necessary condition.*

Proof As in Chew and Wakker (1996, Thm. 1) for simple acts, with the following adjustments. Replace V^{Π} by $V^{\Pi,m}$, the additive representation on (Π, m) of the form $\Sigma_{j=1}^{N} V_j^{\Pi,m}$. Sufficiency follows as in their Lemma 1. Apply the uniqueness argument also to the intersection of comoncones $(\Pi, m) \cap (\Pi, m+1)$, i.e., with $x_m = c$, to deduce that $V_j^{\Pi,m} = V_j^{\Pi,m'}$ in case $j \geq m, m'$ or $j \leq m, m'$ (the argument requires that the intersection is at least of dimension 3, hence the condition $n \geq 4$). So the union of comoncones (Π, m) over m have a common additive representation as in (A.2), for each Π, say $V^{c,\Pi}$ corresponding to pairs $\hat{V}^{\Pi}, \check{V}^{\Pi}$. They satisfy their refinement condition (A4, p17). Since we need not impose that they agree on constants (their condition (A5)), but rather normalize $V^{c,\Pi}$ to zero in c for any $c \in X$, we can simply glue together $\hat{V}^{\Pi}, \check{V}^{\Pi}$ for all fully refined Π's , to arrive at (A.2) as the analog of (2) in their paper. □

Proof of Theorem 2

Sufficiency: we already observed that an FPR defines a regular V. Axiom S2 is an implication of axiom CR, which is addressed below.

To verify axiom S3, consider an act of the form $f_E c$. Then $V(f_E c) = c$ iff $B_c(f_E c) = 0$, hence iff $B_{c,E}(f) = 0$. Then, for $d > c$, $B_d(f_E d) = \Sigma_{s \in E} b_{d,s}(x_s) + \Sigma_{s \in \bar{E}} b_{d,s}(d) < 0$, since the second summation is zero by property (i), and the first is negative by the partial sign property. That V satisfies axiom S4 is obvious.

For axiom CR, consider $f \in \mathcal{A}$, an event $E \subseteq S$, and replacement values $r, \bar{r}$ with $r_s f \sim f \sim f_s \bar{r} \sim c$. This means that $\Sigma_{s \in E} b_{c,s}(x_s) = \Sigma_{s \in E} b_{c,s}(r)$ and $\Sigma_{s \in \bar{E}} b_{c,s}(x_s) = \Sigma_{s \in \bar{E}} b_{c,s}(\bar{r})$. Then $0 = B_c(f) = B_c(r_s \bar{r})$, so $r_s \bar{r} \sim c$. So we have proved that $V \in \mathcal{C}$.

Necessity: Existence of an FPR follows from Chateauneuf and Wakker (1993) and Segal (1992), and the application of these results in Grant et al. (2000). We only address a few adjustments to our setting.

Let be given $V \in \mathcal{C}$ with $n = 4$; the case with $n > 4$ is entirely analogous. We write $f = (x, y, z, b)$ (the b stands for balancing item, namely to level c).

Fix $c \in (x_*, x^*)$, and define $\phi : E \to c - X$ by

$$(x, y, z, c - \phi(x, y, z) \in L_c := \{f \mid V(f) = c\},$$

with domain

$$E := \{(x, y, z) \mid (x, y, z, b) \sim c \text{ for some } b \in X\}.$$

ϕ represents the ordering $\preceq'$ on the first three outcomes (x, y, z) defined by

$$(x, y, z) \preceq' (x', y', z') \text{ when } (x, y, z, b) \sim c \Rightarrow (x', y', z', b) \succeq c,$$

cf.(Grant et al., 2000, Lemma 5).

Axiom CR implies that ϕ has the separability property on E,

$$\phi(x, y, z) = \phi(x', y', z) \Leftrightarrow \phi(x, y, z') = \phi(x', y', z')$$

since both equalities hold true precisely when (x, y) and (x', y') have the same (proper) c-equivalent. It follows from (Chateauneuf and Wakker, 1993, Thm 2.2), which is similar to (Segal, 1992, Thm1), that on the interior of E, $\preceq'$ is additively representable, and hence ϕ takes the form

$$\phi(x, y, z) = \gamma^{-1}(u(x) + v(y) + w(z)). \tag{A.3}$$

with u, v, w satisfying (9.4), and γ strictly increasing, with $\gamma(0) = 0$. The critical condition in their theorem is the connectedness of not only (i) the interior of $\operatorname{int} E$, but also (ii) its subsets with x , y, or z kept fixed, and (iii) those with $\phi(x, y, z)$ kept at fixed level (hence b fixed). This directly follows from the stronger property that the subset of $\operatorname{int} L_c$ with all but two outcomes fixed is path-connected.

From (A.3) it follows that on int E, $V(x,y,z,b) = c$ if and only if $u_c(x) + v_c(y) + w_c(z) + \tilde{u}_c(b) = 0$, with $\tilde{u}_c(b) = -\gamma(c-b)$, also satisfying (9.4). This proves the equalities in (9.3), hence also the inequalities, for $(x,y,z) \in \text{int}\, E$, hence for all (x,y,z,b) on the c-level curve L_c of V with outcomes in (x_*, x^*). The completion to L_c is straightforward, by continuity of V. It remains to show that utility remains finite at the boundary of L_c. If not, there would exist a pair of states s, s' in which utility is unbounded from resp. above and below, say $u_{c,s}(x^*) = \infty$ and $u_{c,s'}(x_*) = -\infty$. Then $f := (x^*, x_*, z, b) \sim c$ for all $z, b \in (x_*, x^*)$ with finite utilities $w_c(z)$, $\tilde{u}_c(b)$, since f can be approximated, in the interior of L_c, by the series (x_k, y_k, z, b) with $x_k = x^* - 1/k$, and y_k such that $u_c(x) + v_c(y) + w_c(z) + \tilde{u}_c(b) = 0$, and V is continuous. This would imply that V is not strictly sensitive in f, contradicting our assumption on V.

Finally, the partial sign property (9.8) follows directly from the premise that V satisfies axiom S3.

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