RADIAL VIBRATIONS OF ECCENTRIC RINGS

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The radial extensional and inextensional modes of vibration of an eccentric circular ring with a rectangular cross section have been studied by perturbation calculations and by measurements on five rings with various eccentricities. In agreement with the theory, only the \( k = 1 \) extensional mode was markedly non-degenerate. Formulae for calculating the frequencies are given.

1. INTRODUCTION

A particular feature of any cylindrically symmetric vibrating system is that generally its eigenfrequencies are (at least) two-fold degenerate as was pointed out by Perrin [1]. This result is an immediate consequence of the rotational symmetry of the system around its axis. A perturbation of this symmetry leads as a rule to splitting of the degenerate eigenfrequency. This effect has been discussed extensively by Charnley and Perrin [2]. They found reasonable agreement between theory and experiment in many cases. The eccentric bell, however, appeared to be an exception [3]. Despite cylindrical symmetry being strongly disturbed by the eccentricity, only very little splitting of the eigenfrequencies was observed. To study this effect in more detail the linear radial extensional and radial inextensional modes of an eccentric homogeneous thin ring of rectangular cross section, for which it is possible to formulate a manageable eigenvalue equation—in contrast with the bell—are considered in this paper.

Let the ring have inner and outer radii \( R_1 \) and \( R_2 \) respectively, and let the centres of the inner and outer circles be a distance \( \Delta \) apart (see Figure 1). The main geometrical properties of the ring are given by the parameters

\[
\delta = \frac{\Delta}{(R_2 - R_1)}, \quad \mu = \frac{(R_2 - R_1)}{(R_2 + R_1)}, \quad \bar{R} = \frac{1}{2}(R_2 + R_1),
\]

where \( \delta \) and \( \mu \) are measures of the eccentricity and thickness respectively. In the present approximation the eigenvalues will be shown to have the form

\[
\omega_k^2 = \omega_{ck}^2 \{1 + \alpha_k \delta^2 + \cdots\}, \quad k \text{ integer},
\]

where the quantities \( \omega_{ck}^2 \) are the "classical" thin ring radial frequencies

\[
Y\rho^{-1}\bar{R}^{-2}(1 + k^2), \quad Y\rho^{-1}\bar{R}^{-2}\mu^2 k^2(k^2 - 1)^2/(k^2 + 1)
\]

for the extensional and inextensional modes respectively [4, 5], in which \( Y \) and \( \rho \) denote Young’s modulus and the density of the material of the ring. The purpose of this paper is to formulate a model with which the coefficients \( \alpha_k \) in equation (1.2) can be calculated and to compare the resulting expressions with experimental data. Note that one expects \textit{a priori} that the two-fold degenerate eigenfrequencies \( \omega_{ck}^2 \) of the concentric ring, given

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in equations (1.3a, b), split as functions of \( \delta \) so that one finds in general two different values of \( \alpha_k \) in each case.

The general outline of the paper is as follows. From the general expression for the Lagrangian of a cylinder, a simplified Lagrangian will be obtained by introducing approximations which are analogous to those leading to the expressions (1.3) for the concentric ring. The choice of a co-ordinate system suitable for the present eccentric system is crucial here (section 2). On the basis of this Lagrangian, the equation of motion is found and a few general properties of the eigenvalue equation for the normal modes are derived (section 3). These properties together with ordinary perturbation methods are used to formulate general results concerning (non)-splitting for the extensional (section 4) and the inextensional modes (section 5) and the coefficients \( \alpha_k \) are calculated. The experimental procedure is described in section 6. In section 7 the theoretical results are summarized and compared with experimental data.

2. THE EIGENVALUE EQUATION

The Lagrangian, i.e., the difference of kinetic and potential energy, for linear vibrations of the cylinder will be formulated in terms of an orthogonal co-ordinate system appropriate for the eccentric cylinder being considered. This Lagrangian, which is expressed in terms of an integral over the volume of the cylinder, can be reduced to a manageable form in a few steps. First, as in the classical thin ring model, the potential energy is greatly simplified by the assumption that the stresses in the radial and axial directions are zero. The only remaining term in the potential energy then corresponds to tangential stress. Next, with the assumptions of radial motion only and of small height of the cylinder the Lagrangian is expressed in terms of the motion of the points in a plane orthogonal to the axis of the cylinder. Further, the assumptions concerning the stress provide a set of differential equations which allow one to reduce the form of the Lagrangian to \( L = \int_0^{2\pi} d\theta L \), where the density \( L \) now is expressed in terms of the motion of a circle with radius \( r \) in this plane and where \( \theta \) is the corresponding polar angle. Due to the eccentricity, \( L \) depends explicitly on \( \theta \). The Lagrangian is given its final form by retaining only the lowest order terms in \( \mu \) (see equation (1.1)), which completes the thin ring approximation. On the basis of this Lagrangian the equation of motion is derived and accordingly the eigenvalue equation for the natural frequencies is formulated.

2.1. THE GENERAL EXPRESSION FOR THE POTENTIAL ENERGY

To describe the motion of the cylinder a co-ordinate system \( \{r, \theta, z'\} \) is introduced which in terms of Cartesian co-ordinates \( \{x, y, z\} \) is given by (see Figure 1)

\[
\begin{align*}
  r^2 &= y^2 + [x - \delta (r - \bar{R})]^2, \quad R_1 \leq r \leq R_2, \\
  \theta &= \arctan \frac{y}{[x - \delta (r - \bar{R})]}, \quad 0 \leq \theta < 2\pi, \quad z' = z. \quad (2.1)
\end{align*}
\]

If \( \delta = 0 \) this system reduces to cylindrical co-ordinates. If \( \delta \neq 0 \), however, it is easily seen that surfaces of constant \( r \) are cylinders with their axes through \( x = \delta (r - \bar{R}) \), \( y = 0 \), the distance between the axes of the cylinders with \( r = R_1 \) and \( r = R_2 \) being equal to \( \Delta \). The cylinder is supposed to fill the space enclosed by the surfaces \( r = R_1, r = R_2, \) and \( z = \pm \frac{1}{2} h_0 \).

This co-ordinate system is obviously not orthogonal; for fixed \( z \) lines of constant \( \theta \) are straight and consequently not orthogonal to lines of constant \( r \) (see Figure 1). On the basis of these co-ordinates however an orthogonal system \( \{\rho, \phi, z\} \) is easily defined as follows

\[
\rho = r, \quad \phi = 2 \arctan \left( (r/r_0)^{-\kappa} \tan \frac{1}{2} \theta \right), \quad R_1 \leq r_0 \leq R_2, \quad (2.2)
\]
where \( r_0 \) is a constant with an arbitrary value between the given boundaries. Lines of constant \( \phi \) are now everywhere orthogonal to lines of constant \( \rho \) (see Figure 1). Further details of this co-ordinate system are given in Appendix A.

It is a matter of standard theory now to express the stress and the potential energy of the cylinder in terms of this orthogonal curvilinear co-ordinate system. The notations adopted by Love [5] (see section 19) are used. The co-ordinates \( \{\rho, \phi, z\} \) depend ultimately on \( x, y, z \). A set of orthonormal basis vectors is then defined as

\[
\mathbf{n}_1 = \nabla \rho / h_1, \quad \mathbf{n}_2 = \nabla \phi / h_2, \quad \mathbf{n}_3 = \nabla z / h_3, \quad (2.3)
\]

where \( \nabla \) denotes the gradient operator \((\partial/\partial x, \partial/\partial y, \partial/\partial z)\) and

\[
h_1 = |\nabla \rho|, \quad h_2 = |\nabla \phi|, \quad h_3 = |\nabla z|. \quad (2.4)
\]

Note that \( h_1 = h_1(\rho, \phi) \) and that \( h_3 \) equals unity. Let \( \mathbf{r}(\mathbf{r}) \) denote the displacement of the point \( \mathbf{r} = (x, y, z) \) and let \( u, v, w \) denote the components of \( \mathbf{r} \) with respect to the basis vectors \( \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \), respectively. Then the traction \( \mathbf{t}_n \) on an infinitesimal surface element in a direction \( \mathbf{n} \) in the cylinder is given by

\[
\mathbf{t}_n = \left[ \lambda' (e_{11} + e_{22} + e_{33}) I + \mu' \begin{pmatrix} 2e_{11} & e_{12} & e_{13} \\ e_{12} & 2e_{22} & e_{23} \\ e_{13} & e_{23} & 2e_{33} \end{pmatrix} \right] \mathbf{n}, \quad (2.5)
\]

where \( I \) denotes the identity operator and the tensor components \( e_{ij} \) are given by (see reference [5], section 20, and make use of \( h_3 = 1 \))

\[
e_{11} = h_1 \mu + h_1 h_2 v(1/h_1), \quad e_{22} = h_2 v + h_1 h_2 u(1/h_2), \quad e_{33} = w, \quad e_{13} = (1/h_1)(h_1 u)_z + h_1 w, \quad e_{23} = h_2 w + (1/h_2)(h_2 v), \quad e_{12} = (h_1 h_2)(h_2 v)_o + (h_2 h_1)(h_1 u)_o. \quad (2.6)
\]

Subscripts \( \rho, \phi, z \) denote partial differentiation. The potential energy now reads [5] (see section 69)

\[
U_{pot} = \int d\mathbf{r} \left( \frac{1}{2}(\lambda' + 2\mu')(e_{11} + e_{22} + e_{33})^2 + 2\mu' \left( \frac{1}{2} e_{12}^2 + \frac{1}{2} e_{13}^2 + \frac{1}{2} e_{23}^2 - e_{11} e_{22} - e_{11} e_{33} - e_{22} e_{33} \right) \right), \quad (2.7)
\]
where the integration is over the volume of the cylinder and \( \lambda' \) and \( \mu' \) are material constants, related to Poisson's ratio \( \sigma \) and Young's modulus \( Y \) by the well-known expressions

\[
\sigma = \frac{\lambda'}{2(\mu' + \lambda')}, \quad Y = \left[ \frac{\mu'fac'}{2+3h'} \right] (2.8)
\]

### 2.2. Restriction to Radial Extensional and Inextensional Modes

With the above results the Lagrangian for the ring can be formulated. The corresponding equation of motion, however, is much too difficult to solve and this Lagrangian has to be simplified. This is done in this section with the aid of the following three connected assumptions: (i) the height of the cylinder is small compared with the average radius; (ii) the displacement component \( w \) in the \( z \)-direction is odd with respect to \( z = 0 \); (iii) the tractions in the directions \( n_1 \) and \( n_3 \) are small compared with the traction in the direction \( n_2 \). In order to satisfy the last assumption it is required that \( T_{n1} = T_{n3} = 0 \) or, in components,

\[
\lambda'(e_{11} + e_{22} + e_{33}) + 2\mu'e_{11} = e_{12} = e_{13} = 0, \quad \lambda'(e_{11} + e_{22} + e_{33}) + 2\mu'e_{33} = e_{13} = e_{23} = 0.
\]

With equations (2.8) these conditions can be expressed conveniently as

\[
e_{13} = e_{23} = 0, \quad e_{33} = e_{11}; \quad e_{12} = 0, \quad e_{11} = -\sigma e_{22}.
\]

(2.10a, b)

With these relations the traction (2.5) becomes

\[
\tau = Y \begin{bmatrix} 0 & e_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} n
\]

(2.11)

and accordingly the potential energy (2.7) takes the form

\[
U_{\text{pot}} = \int d\tau^2 Y e_{22}^2.
\]

(2.12)

To state it in different terms: one considers modes in which the significant contribution to the potential energy originates from pure stretching in the direction of \( n_2 \). This is in the present case the proper analogue of the assumptions leading to the expressions (1.3).

The next step is to reduce the volume integral in equation (2.7) to an integral over an orthogonal cross section of the cylinder: i.e., the plane \( z = 0 \). To this end use is made of assumption (ii): i.e.,

\[
w(\rho, \phi, z) = -w(\rho, \phi, -z).
\]

(2.13)

Further, note that the volume element \( d\tau \) is to be written as \( d\tau = dz \, d\sigma, d\sigma \) being a surface element of the \( (\rho, \phi) \)-plane, and that the integration range for \( z \) is from \(-\frac{1}{2}h_0\) to \(+\frac{1}{2}h_0\). The potential and the kinetic energy then can be written as

\[
U_{\text{pot}} = \frac{1}{2} Y h_0 \int d\sigma \varepsilon_{22}^2 + O(h_0^3),
\]

(2.14)

and (the subscript \( t \) denotes partial differentiation with respect to time)

\[
U_{\text{kin}} = \frac{1}{2} \rho h_0 \int d\sigma (\dot{u}_{0t}^2 + \dot{v}_{0t}^2) + O(h_0^3),
\]

(2.15)

where the subscript \( 0 \) denotes the value in \( z = 0 \) and where for the latter equation (2.13) has been used. Now \( h_0 \) is assumed to be so small that the third order terms may be
omitted, which leaves one with the Lagrangian \( L = U_{\text{kin}} - U_{\text{pot}} \) as

\[
L = \int d\sigma \left\{ \frac{1}{2} \rho (u_{0r}^2 + v_{0\theta}^2) - \frac{1}{2} Ye_{22} \right\}.
\]  

(2.16)

The final step in this section is to reduce the surface integral in this expression to a line integral with integration variable \( \phi \). To this end one uses the relations (2.10b) applied for \( z = 0 \). With equation (2.6), they are written in the form, with the subscript 0 omitted,

\[
\begin{pmatrix}
u
\end{pmatrix}_\rho = \frac{-h_2}{h_1} \begin{pmatrix} 0 & \sigma \end{pmatrix} \begin{pmatrix} u \\
\phi
\end{pmatrix} + \begin{pmatrix} \sigma (h_{2e}/h_2) \\
-(h_2/h_1^2) h_{1\phi} 
\end{pmatrix} \begin{pmatrix} u \\
\phi
\end{pmatrix}.
\]

(2.17)

The subscripts of the column vectors \( (u, v) \) denote partial differentiation of each element of the vector. Now one can proceed as follows. The differential equations (2.17) have the form of an evolution equation \( da(\phi)/d\rho = Ga(\phi) \) where \( G \) is a linear operator. Accordingly the "initial value problem" can be solved: i.e., \( u(\rho, \phi) \) and \( v(\rho, \phi) \) can be expressed in terms of \( u(\rho_0, \phi) \) and \( v(\rho_0, \phi) \) where \( \rho_0 \) equals \( r_0 \) given in equation (2.2). The resulting expressions for \( u \) and \( v \) are then substituted into equation (2.16) and the integration over \( \rho \) is carried out. This leaves one with a Lagrangian of the type \( L = \int d\phi L \), where the density \( L \) depends on the dynamical variables \( u(\rho_0, \phi) \) and \( v(\rho_0, \phi) \) explicitly on \( \phi \). Then one should be able to formulate the equation of motion. This procedure appears to be too difficult, however, because of the very complicated form of the expressions for \( u \) and \( v \), but the thin ring approximation, i.e., retaining only the lowest order terms in \( \mu \) (see equation (1.1)) solves this problem.

2.3. THE THIN RING APPROXIMATION

The problem is simplified if one changes to \( \{r, \theta\} \) co-ordinates. It is a matter of straightforward calculation (Appendix A) to show equations (2.16) and (2.17) then take the forms

\[
L = \frac{2\pi}{R_2} \int_{R_1}^{R_2} d\theta r (1 + \delta \cos \theta) \left\{ \frac{1}{2} \rho (u_{r}^2 + v_{\theta}^2) - \frac{1}{2} Ye^{-2} (v_{\theta} + u)^2 \right\}, \quad \begin{pmatrix} u \\
v
\end{pmatrix}_r = \frac{1}{r} (H_0 + \delta H_1) \begin{pmatrix} u \\
v
\end{pmatrix}.
\]

(2.18, 2.19)

The symbols \( H_0 \) and \( H_1 \) denote operators defined by

\[
H_0 = \left( \begin{array}{cc}
0 & 1 \\
-\partial/\partial \theta & 0
\end{array} \right) - \sigma \left( \begin{array}{cc}
0 & \partial/\partial \theta \\
0 & 0
\end{array} \right),
\]

\[
H_1 = -\left( \begin{array}{cc}
sin \theta & 0 \\
cos \theta & 0
\end{array} \right) \frac{\partial}{\partial \theta} + \left( \begin{array}{cc}
0 & \sin \theta \\
-\sin \theta & \cos \theta
\end{array} \right) - \sigma \cos \theta \left( \begin{array}{cc}
0 & \partial/\partial \theta \\
0 & 0
\end{array} \right),
\]

(2.20)

where the usual "multiplication" rules have to be applied: e.g.,

\[
\left( \begin{array}{cc}
0 & 0 \\
-\partial/\partial \theta & 1
\end{array} \right) \begin{pmatrix} u \\
v
\end{pmatrix} = \begin{pmatrix} 0 \\
-\partial u/\partial \theta + v
\end{pmatrix}.
\]

(2.21)

Because \( H_0 \) and \( H_1 \) do not depend on \( r \) the expression

\[
\begin{pmatrix} u(r, \theta) \\
v(r, \theta)
\end{pmatrix} = e^{H \ln(r)/r_0} \begin{pmatrix} u_0(\theta) \\
v_0(\theta)
\end{pmatrix}, \quad H = H_0 + \delta H_1,
\]

(2.22)
is a solution of equation (2.19) in the sense that \( u(r_0, \theta) = u_0(\theta) \) and \( v(r_0, \theta) = v_0(\theta) \), as is easily verified by differentiation with respect to \( r \). The exponential function in equation (2.22) is defined as (see reference [6], chapter 9, section 1)

\[
e^{H \ln(r/r_0)} = \lim_{n \to \infty} \left[ 1 - (1/n) \ln\left(\frac{r}{r_0}\right)H \right]^{-n}.
\]  

(2.23)

In the concentric case, \( \delta = 0 \), equation (2.19) can be solved directly and it is easily verified that the solution found in this way agrees with the formal solution (2.22).

Before substituting equation (2.22) into equation (2.18) it is convenient to give the latter a more concise form. To this end one can introduce the real linear space of ordered pairs \( \mathbf{a}(\theta) = (u(\theta), v(\theta)) \) where both \( u(\theta) \) and \( v(\theta) \) are periodic with period \( 2\pi \). Further one assumes that for any two pairs \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) the inner product

\[
\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \frac{1}{\pi} \int_0^{2\pi} d\theta (u_1 v_1 + u_2 v_2),
\]  

(2.24)

exists so that one is dealing with a Hilbert space. Obviously any physically relevant pair \( (u, v) \) is an element of this space. If the linear operators \( D \) and \( N \) are defined by

\[
D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} \partial / \partial \theta \\ 0 \end{pmatrix},
\]  

(2.25)

it is easily verified that \( L \) in equation (2.18) takes the form

\[
\pi^{-1} L = \frac{1}{2} \rho \int_{R_1}^{R_2} r \, dr (D \mathbf{a}_n, \mathbf{a}_n) - \frac{1}{2} Y \int_{R_1}^{R_2} \frac{dr}{r} (DN \mathbf{a}, \mathbf{Na}),
\]  

(2.26)

where \( \mathbf{a}(r, \theta, t) \) is considered as an element of the Hilbert space for any value of \( r \) and \( t \). If one substitutes equation (2.22) into equation (2.26) and makes use of the notation \( G^* \) for the Hermitian conjugate of an operator \( G \), which satisfies by definition

\[
(G^* \mathbf{a}, \mathbf{b}) = (\mathbf{a}, G \mathbf{b}),
\]  

(2.27)

then equation (2.26) becomes, apart from the irrelevant factor \( \pi \),

\[
L = \frac{1}{2} \rho \int_{R_1}^{R_2} r \, dr (e^{H \ln(r/r_0)}D e^{H \ln(r/r_0)}a_{0n}, a_{0n}) - \frac{1}{2} Y \int_{R_1}^{R_2} \frac{dr}{r} (e^{H \ln(r/r_0)}N^*DN e^{H \ln(r/r_0)}a_0, a_0).
\]  

(2.28)

In order to write \( L \) as a series expansion with respect to the thickness parameter \( \mu \), \( r_0 \) must be given a particular value. It is appropriate to choose (this choice is discussed below) the geometric mean

\[
r_0 = \sqrt{R_1 R_2}.
\]  

(2.29)

With the expressions

\[
R_1 = r_0 \sqrt{(1 - \mu)/(1 + \mu)}, \quad R_2 = r_0 \sqrt{(1 + \mu)/(1 - \mu)}, \quad r_0^2 = \bar{R}^2 (1 - \mu^2),
\]  

(2.30)

one sees that the Lagrangian (2.28) depends parametrically on \( \bar{R} \) and \( \mu \) and after a simple substitution \( r' = r/r_0 \) it takes the form

\[
L = \frac{1}{2} \rho r_0^2 \int_{f_-}^{f_+} r' \, dr' (e^{H \ln r'}D e^{H \ln r'}a_{0n}, a_{0n}) - \frac{1}{2} Y \int_{f_-}^{f_+} \frac{dr'}{r'} (e^{H \ln r'}N^*DN e^{H \ln r'}a_0, a_0),
\]  

with

\[
f_+ = \sqrt{(1 + \mu)/(1 - \mu)}, \quad f_- = \sqrt{(1 - \mu)/(1 + \mu)}.
\]  

(2.31)
In this expression $L$ depends on $\mu$ through the integration boundaries and the factor $r_0^2$ only. Note that $L$ is an odd function of $\mu$, so that the operators $A$ and $B$ below are even in $\mu$. It is a matter of straightforward differentiation to expand $L$ with respect to $\mu$ and to find

$$L = \frac{1}{2} \rho \tilde{R}^2 \mu \{ (\mathbf{a}_0, \mathbf{a}_0^*) - \omega_0^2 (\mathbf{b}_0, \mathbf{b}_0^*) \}, \quad \omega_0^2 = \frac{Y}{(\rho \tilde{R}^2)}, \quad (2.33)$$

with $A$ and $B$ in lowest order in $\mu$ given by

$$A = 2D + i \mu^2 \left[ (H^2 D + DH^2) + 2 \right], \quad B = 2N^2 D Y + i \mu^2 \left[ 2N^2 D Y + N^2 DH^2 + 2H^2 N^2 D Y + H^2 N^2 D Y \right]. \quad (2.34)$$

$A$ and $B$ being self-adjoint, Hamilton's Action Principle yields finally the equation of motion

$$A \mathbf{a}_{0t} = -\omega_0^2 B \mathbf{a}_0. \quad (2.35)$$

Another choice for $r_0$, say, corresponds with another function $\mathbf{a}_0(\theta) = \mathbf{a}(\theta_0, \theta)$, the transformation from $\mathbf{a}_0$ to $\mathbf{a}_0^*$ being determined by the differential equation (2.19). In other words taking a different value for $r_0$ corresponds with a transformation of the dynamical variable $\mathbf{a}_0$, which does not influence the dynamics resulting from the Lagrangian (2.31). Obviously the operators $A$ and $B$ depend on the choice of $r_0$. The geometric mean (2.29) is chosen because it leads to convenient expressions for $A$ and $B$.

3. GENERAL PROPERTIES OF THE EIGENVALUE EQUATION

Substitution of the normal modes $\mathbf{a}_0(\theta, t) = e^{i \omega t} \mathbf{a}(\theta)$ in the equation of motion (2.35) leads to the eigenvalue equation

$$\omega^2 A \mathbf{a} = \omega_0^2 B \mathbf{a}. \quad (3.1)$$

In this section a few general results are derived concerning this equation and its solutions. These results are based on symmetry properties of the ring.

Obviously reflection with respect to the $\theta = 0$ plane is a symmetry operation for the eccentric cylinder (cf. Figure 1). With this symmetry there corresponds the operator $R$ defined by

$$R \begin{pmatrix} u(\theta) \\ v(\theta) \end{pmatrix} = \begin{pmatrix} u(-\theta) \\ -v(-\theta) \end{pmatrix}. \quad (3.2)$$

This operator $R$ satisfies $R^2 = 1$ and so its eigenvalues are $+1$ or $-1$. The corresponding eigenspaces are called symmetric and antisymmetric respectively and are denoted by $H_S$ and $H_A$. Any element $\mathbf{a} = (u, v)$ can be written uniquely as a sum of a symmetric and an antisymmetric term $\mathbf{a} = \mathbf{a}_S + \mathbf{a}_A$ where

$$\mathbf{a}_S = \frac{1}{2} (\mathbf{a} + R \mathbf{a}), \quad \mathbf{a}_A = \frac{1}{2} (\mathbf{a} - R \mathbf{a}). \quad (3.3)$$

Note that the first and second components of $\mathbf{a}_S$ are even and odd, respectively, and that the inverse holds for the components of $\mathbf{a}_A$, from which property it is easily seen that $\mathbf{a}_S$ and $\mathbf{a}_A$ are orthogonal with respect to the inner product (2.24). Because $R$ represents a symmetry operation it necessarily commutes with the operators $A$ and $B$. Then, according to standard theory [6] the following Theorem applies.

**Theorem 3.1.** The subspaces $H_S$ and $H_A$ are invariant spaces of the operators $A$ and $B$ in equation (3.1): i.e.,

$$AH_S(A) \subset H_S(A), \quad BH_S(A) \subset H_S(A). \quad (3.4)$$
As a consequence of this Theorem the eigenvalue problem (3.1) reduces into two separate problems: one for symmetric, the other one for antisymmetric functions \( a \). Note in particular that this is true for any \( \delta \).

Consider now the behaviour of the eigenvalues and eigenvectors as a function of the parameter \( \delta \). The operators \( A \) and \( B \) in equation (3.1) depend analytically on \( \delta \) and so the same holds for the eigenvalues and eigenvectors \([6]\). Let \( \omega^2(\delta) \) denote a particular eigenvalue and \( a(\delta; \theta) \) the corresponding eigenvector, which is supposed to be normalized. Because \( \omega^2(\delta) \) and \( a(\delta; \theta) \) are analytic functions of \( \delta \), they satisfy the eigenvalue equation also for negative values of the parameter, although by definition \( \delta \) is non-negative (see equation (1.1)). The following Theorem applies, however.

**Theorem 3.2.** Let \( \omega^2(\delta) \) and \( a(\delta; \theta) \) satisfy equation (3.1) and let \( a \) have a unit norm; then

\[
\omega^2(\delta) = \omega^2(-\delta), \quad a(\delta; \theta) = \pm a(-\delta; \theta + \pi).
\]  

**Proof.** From Figure 1 it is immediately clear that negative values of \( \delta \) correspond with a system that is rotated through an angle \( \pi \) which obviously does not influence the eigenvalues. The immediate consequence is not, as one would expect, that \( \omega^2 \) is an even function of \( \delta \) but a more general result can be formulated. To this end note that the combined transformation \( \{ \delta \to -\delta; \theta \to \theta + \pi \} \) leaves the physical system invariant.\(^\dagger\) As a consequence the primed expressions \( \omega^2' \) and \( a' \) defined by

\[
\omega^2'(\delta) = \omega^2(-\delta), \quad a'(\delta; \theta) = a(-\delta; \theta + \pi),
\]

also constitute a solution. If \( \omega^2(\delta) \neq \omega^2'(\delta) \) there are two different branches of eigenvalues and \( a \) and \( a' \) are linearly independent for any \( \delta \) (see Figure 2(a)). Note that for \( \delta = 0 \)

![Figure 2](image-url)

(a) Eigenvalues if \( \omega^2(0) \) is twofold degenerate; the eigenvectors \( a(\delta) \) and \( a'(\delta) \) are linearly independent for any \( \delta \). (b) Eigenvalues if \( \omega^2(0) \) is non-degenerate; the eigenvectors \( a(\delta) \) and \( a'(\delta) \) are linearly dependent for any \( \delta \).

the eigenvalue \( \omega^2(0) = \omega^2'(0) \) is degenerate. In sections 4 and 5, however, it will be shown that the unperturbed eigenvalues one is dealing with are non-degenerate: i.e., with respect to the symmetric and the antisymmetric spaces separately. In that case there is only one eigenvalue \( \omega^2(\delta) \) originating at \( \delta = 0 \) in each case which corresponds with an eigenspace of unit dimension in a neighbourhood of \( \delta = 0 \). Consequently one has

\[
\omega^2'(\delta) = \omega^2(\delta), \quad a'(\delta; \theta) = \gamma a(\delta; \theta),
\]  

(3.7)

where \( \gamma \) equals \( \pm 1 \) because of the assumed normalization. Then, however, one obtains with equation (3.6) the final result (3.5).

\(^\dagger\) The transformation \( \delta \to -\delta, \theta \to \theta + \pi \) leaves the operators \( A \) and \( B \) in equation (2.34) invariant as one may verify.
Finally a general result about splitting will be derived. It has been seen already (Theorem 3.1) that equation (3.1) can be solved in the symmetric and antisymmetric subspaces $H_S$ and $H_A$ separately and that if $\delta = 0$ the sets of eigenvalues are equal in each case because of the cylindrical rotation symmetry. There is, however, no a priori reason to expect this equality if $\delta \neq 0$. To clarify this matter one has to know more about the operators $A$ and $B$, and an important result is formulated in the lemma below. To prepare for this lemma an orthogonal basis in the present Hilbert space is introduced:

$$\psi_{ISk} = (k \cos k\theta, -\sin k\theta), \quad k = 1, 2, \ldots, \psi_{IAk} = (k \sin k\theta, \cos k\theta), \quad k = 0, 1, 2, \ldots, \psi_{ESk} = (\cos k\theta, k \sin k\theta), \quad k = 0, 1, 2, \ldots, \psi_{EAk} = (\sin k\theta, -k \cos k\theta), \quad k = 1, 2, \ldots,$$

(3.8)

the sets $\{\psi_S\}$ and $\{\psi_A\}$ being bases in the spaces $H_S$ and $H_A$, respectively. The functions with subscripts $E$ and $I$ represent the extensional and inextensional modes of the concentric ring, as will be shown in sections 4 and 5. The definition of the functions $\psi$ can be extended without problem to include all integer values of $k$. Obviously the elements of the extended set $\{\psi_k\}$ are mutually orthogonal and this set is complete. It is not a basis, however, because of the existence of linear relations.

**Lemma 3.3.** Let $G$ be a linear operator defined on a domain in $H$ with invariant subspaces $H_S$ and $H_A$; then, for any integer $k$,

$$G\psi_{IS,ak} = \sum_{l=-\infty}^{+\infty} c_{IKl}\psi_{IS,al} + \sum_{l=-\infty}^{+\infty} c_{IKl}\psi_{ES,al},$$

(3.9a)

$$G\psi_{ES,ak} = \sum_{l=-\infty}^{+\infty} c_{IKl}\psi_{IS,al} + \sum_{l=-\infty}^{+\infty} c_{IKl}\psi_{ES,al},$$

(3.9b)

where the coefficients $c_{IKl}$, etc., are the same in the symmetric and antisymmetric cases, respectively. The proof is given in Appendix B.

This result, if applied on the operators $A$ and $B$, suggests that the matrix elements of $A$ and $B$ in each subspace $H_S$ and $H_A$ are identical, so that there is no splitting. Although this is not true (see equation (4.9) and following) this Lemma appears to be very important with respect to (non)-splitting. A first consequence is the Theorem formulated below. Theorem 4.1 in section 4 is a second consequence.

It will be shown (sections 4 and 5) on the basis of equation (3.1) that the frequencies of the extensional and inextensional modes have the form $\omega^2_k = \omega_0^2 \lambda_k(\delta)$ and $\omega^2_k = \omega_0^2 \mu^2 \gamma_k(\delta)$, respectively, where $\lambda$ and $\gamma$ are solutions of an eigenvalue equation of the type

$$\tilde{\lambda}(I + G_1\delta + G_2\delta^2)a = (H_0 + H_1\delta + H_2\delta^2)a,$$

(3.10)

$I$ being the identity and $G_1$ and $H_1$ being self-adjoint linear transformations. In the extensional case equation (3.10) is an eigenvalue problem in the space $H_S$ (or $H_A$) spanned by the set $\{\psi_{ISk}, \psi_{ESk}\}$ (or $\{\psi_{IAk}, \psi_{EAk}\}$). In the inextensional case equation (3.10) is a problem in the inextensional subspace $H_{IS}$ (or $H_{IA}$) spanned by the set $\{\psi_{ISk}\}$ (or $\{\psi_{IAk}\}$). Let $\{\tilde{a}_k\}$ denote an orthonormal set of eigenvectors of the unperturbed problem ($\delta = 0$) and $\{\lambda_k\}$ the corresponding set of eigenvalues, which are assumed to be non-degenerate. Then the solutions of equation (3.10) have the form (from Theorem 3.2)

$$\alpha_k(\tilde{a}_k, (H_2 - \lambda_k G_2)\tilde{a}_k) + \sum_{i \neq k} (\lambda_k - \lambda_i)^{-1}(\tilde{a}_i, (H_1 - \lambda_k G_1)\tilde{a}_k)^2,$$

(3.11)

Lemma 3.3 can now be applied which leads immediately to the following Theorem.
THEOREM 3.4. The coefficient operator \( G_2 \) and \( H_2 \) in equation (3.10) do not contribute to splitting of eigenfrequencies in order \( \delta^2 \).

Proof. Lemma 3.3 shows immediately that the first term in equation (3.11) in the symmetric case equals that in the antisymmetric case. The second term depends on \( G_1 \) and \( H_1 \) only.

Consequently as far as one is interested in splitting in order \( \delta^2 \) only, there is no need to consider the validity of the \( \delta^2 \)-terms in \( A \) and \( B \).

4. THE EXTENSIONAL MODES

In this section perturbation methods are used, together with the results of the foregoing section, to study how the eigenvalues \( \omega^2 \) of equation (3.1) depend on \( \delta \) if \( \mu = 0 \). In that case equation (3.1) reduces to the form (3.10)

\[
\tilde{\lambda}(I + \delta C)a = (N^+ N + \delta N^+ CN)a; \quad C = \begin{pmatrix} \cos \theta & 0 \\ 0 & \cos \theta \end{pmatrix}, \quad \tilde{\lambda} = \omega^2/\omega_0^2. \tag{4.1a, b}
\]

For the interpretation of equations (4.1) it might be interesting to note that correspondingly the kinetic and potential energies are given by, the angle brackets denoting the inner product in the Euclidean space \( \mathbb{R}^2 \),

\[
\frac{1}{2} \int_{-\pi}^{+\pi} d\theta \langle a_\theta, \rho (1 + \delta \cos \theta) a_\theta \rangle, \quad \frac{1}{2} \int_{-\pi}^{+\pi} d\theta \langle N a_\theta, Y(1 + \delta \cos \theta) N a_\theta \rangle, \tag{4.2}
\]

which expressions are the same as those for an inhomogeneous circular ring with density and Hooke’s constant being given by \( \rho(\theta) = \rho(1 + \delta \cos \theta) \) and \( Y(\theta) = Y(1 + \delta \cos \theta) \). Note that the original expressions given in equation (2.33) cannot be interpreted in this way.

4.1. THE CONCENTRIC RING

If \( \delta = 0 \) equation (4.1) reduces further to

\[
\tilde{\lambda} a = N^+ N a. \tag{4.3}
\]

The Hermitian conjugate \( N^\dagger \), being defined by \( (N^\dagger a, b) = (a, N b) \), is given by (see equation (2.25))

\[
N^\dagger = \begin{pmatrix} 1 \\ -d/d\theta \\ 0 \end{pmatrix}, \tag{4.4}
\]

and one verifies easily that

\[
N^\dagger N \psi_{\text{EK}} = \lambda_k \psi_{\text{EK}}, \quad \lambda_k = 1 + k^2; \quad N^\dagger N \psi_{\text{IK}} = 0. \tag{4.5a, b}
\]

Thus it follows that the concentric ring has eigenvalue zero with an eigenspace spanned by the vectors \( \psi_{\text{IK}} \) and eigenvalues \( \lambda_k \) corresponding with eigenfunctions \( \psi_{\text{EK}} \). The latter represent the classical extensional modes (see equation (1.3a)). Furthermore it is easily seen that the vectors \( \psi_{\text{IK}} \) not only satisfy equation (4.5b) but also \( N \psi_{\text{IK}} = 0 \). As a consequence equation (4.1) has an eigenvalue zero for any value of \( \delta \) with eigenspace spanned by the set \( \{ \psi_{\text{IK}} \} \). This is easily understood because the vectors \( \psi_{\text{IK}} \) represent the inextensional flexural modes (see section 5), which should have eigenvalue zero in the limiting case \( \mu = 0 \).

4.2. SPLITTING

It must be emphasized that the eigenvalue problem (4.1) will be solved for symmetric and antisymmetric eigenfunctions separately. Furthermore note that \( \lambda_k \) is non-degenerate.
in each case and that consequently, according to Theorem 3.2, the eigenvalue can be written as

$$\lambda_k(\delta) = \lambda_k(1 + \alpha_k \delta^2 + \cdots).$$

(4.6)

If one now applies the usual perturbation techniques (see Appendix C) and makes use of the particular properties of the present unperturbed problem, in particular that the basis vectors \( \psi_t \) correspond with zero eigenvalue, one obtains for \( \alpha_k \) the equation

$$\lambda_k \alpha_k = \sum_a \lambda_a (\bar{\psi}_{t_a} C \bar{\psi}_{E_k})^2 + \sum_i (\lambda_k - \lambda_i)^{-1} (\bar{\psi}_{E_i} [N^+CN - \lambda_k C] \bar{\psi}_{E_k})^2,$$

(4.7)

where the bar denotes normalization. This equation holds true both in the symmetric and in the antisymmetric case. Note however that the lower boundaries for \( i \) in the summations are zero or unity according to equation (3.8) and that the case \( k = 0 \) applies in the symmetric case only.

In order to evaluate \( \alpha_k \), one now can take a closer look at the operators \( C \) and \( N^+CN \). In Appendix B (equations (B13) and (B19)) it is shown that

$$2C\psi_{E_k} = \lambda_k^{-1} \{ -\psi_{E_k} + (1 + k (k - 1)) \psi_{E_k} \} + \lambda_k^{-1} \{ \psi_{E_{k+1}} + (1 + k (k + 1)) \psi_{E_{k+1}} \},$$

(4.8a)

$$N^+CN\psi_{E_k} = \frac{1}{2} \lambda_k \{ \psi_{E_{k+1}} + \psi_{E_{k+1}} \}.$$

(4.8b)

Combining both equations gives

$$\{ N^+CN - \lambda_k C \} \psi_{E_k} = \frac{1}{2} \lambda_k \{ -\psi_{E_{k-1}} + \lambda_k^{-1} (k - 1) + \psi_{E_{k+1}} + \lambda_k^{-1} (k + 1) \} + \text{inextensional terms},$$

(4.8c)

These relations hold for any integer \( k \) if the definition of the vectors \( \psi_k \) is extended accordingly. Further they hold equally well (Lemma 3.3) in both the symmetric and the antisymmetric case. This suggests that the value of \( \alpha_k \) is the same in each case. This, however, is not true in general because the extended sets \{\( \psi_{E_k}, \psi_{E_k} ; |k| < \infty \), although complete, are not bases. As a matter of fact the following linear relations exist:

$$\psi_{IS - k} = -\psi_{IS k}, \quad \psi_{ES - k} = \psi_{ES k}, \quad \psi_{IA - k} = \psi_{IA k}, \quad \psi_{EA - k} = -\psi_{EA k}, \quad |k| < \infty.$$

(4.9)

These relations are different in each case, which causes in general a difference in the values of \( \alpha_k \). With equation (3.8) it follows easily that \( \psi_{ISO} = \psi_{EO0} = 0 \) and further

$$C\psi_{ES0} = \frac{1}{2} (\psi_{IS1} + \psi_{ES1}), \quad C\psi_{ES1} = \frac{1}{2} (\psi_{ES0} + \frac{1}{2} \psi_{ES2} + \frac{3}{2} \psi_{ES2}),$$

$$C\psi_{EA1} = \frac{1}{2} (-\psi_{IA0} + \frac{1}{2} \psi_{IA2} + \frac{3}{2} \psi_{EA2}).$$

(4.10a)

$$\{ N^+CN - \lambda_0 C \} \psi_{ES0} = \frac{1}{2} \psi_{ES1} + \text{inextensional terms},$$

$$\{ N^+CN - \lambda_1 C \} \psi_{ES1} = \frac{3}{2} \psi_{ES2} + \text{inextensional terms},$$

$$\{ N^+CN - \lambda_1 C \} \psi_{EA1} = \frac{2}{3} \psi_{EA2} + \text{inextensional terms}.\quad (4.10b)$$

From equations (4.10a) in particular it is seen that differences in the values for \( \alpha_k \) may arise if \( k = 1 \). For \( k \geq 2 \), however, the following result is easily proved.

**Theorem 4.1.** The coefficients \( \alpha_{Sk} \) (symmetric case) and \( \alpha_{Ak} \) (antisymmetric case) are equal for any \( k \geq 2 \).

**Proof.** Both \( C \) and \( N^+CN \) produce, if acting on \( \psi_{E_k} \), only \( k - 1 \) and \( k + 1 \) terms (see equation (4.8)). Consequently, because of the orthogonality of the basis vectors, only matrix elements \([C]_{k,k+1}\) and \([N^+CN]_{k,k+1}\) lead to non-zero contributions in equation
If \( k \geq 2 \), however, these matrix elements are determined by equation (4.8) and consequently they have the same value in the symmetric and in the antisymmetric case, which proves the Theorem.

The \( k = 0 \) antisymmetric extensional mode being trivially equal to zero (see equation (3.8)), there is no splitting in this case. And as a consequence of the above Theorem the \( k = 1 \) mode is the only candidate for splitting (in order \( \sigma^2 \)).

4.3. CALCULATIONS

In the calculations, one can first consider the case \( k \geq 2 \). It is easily seen that equation (4.7) reduces to

\[
\lambda_k \alpha_k = \lambda_k \{(\vec{\psi}_{I_{k-1}}, C\vec{\psi}_{EK})^2 + (\vec{\psi}_{I_{k+1}}, C\vec{\psi}_{EK})^2 \}
\]

\[
+ (\lambda_k - \lambda_{k-1})^{-1}(\vec{\psi}_{I_{k-1}}, \{N^+CN - \lambda_k C\}\vec{\psi}_{EK})^2
\]

\[
+ (\lambda_k - \lambda_{k+1})^{-1}(\vec{\psi}_{I_{k+1}}, \{N^+CN - \lambda_k C\}\vec{\psi}_{EK})^2, \quad k \geq 2.
\]

(4.11)

According to the inner product definition (2.24), the norms of the vectors \( \psi \) are given by

\[
\|\psi_{I_k}\| = \|\psi_{EK}\| = (1 + k^2)^{1/2}, \quad k = 1, 2, \ldots
\]

(4.12)

With equation (4.8) the relevant inner products in equation (4.11) are then easily expressed in terms of \( \lambda_k \) as

\[
(\vec{\psi}_{In}, C\vec{\psi}_{EK})^2 = \frac{1}{4}\lambda_n \lambda_k^{-1}, \quad (\vec{\psi}_{En}, [N^+CN - \lambda_k C]\vec{\psi}_{EK})^2 = \frac{1}{4}\lambda_k \lambda_n^{-1}(\lambda_n - (1 + kn))^2, \quad n = k \pm 1,
\]

(4.13)

and with the identities

\[
(\lambda_n - (1 + kn))^2 = \lambda_n - 1, \quad n = k \pm 1, \quad \lambda_n \lambda_k^{-1} = (\lambda_k - \lambda_n)^{-1}(\lambda_n^{-1} - \lambda_k^{-1}),
\]

(4.14)

equation (4.11) takes the form

\[
4\alpha_k = (1 - \lambda_k^{-1})((\lambda_k - \lambda_{k-1})^{-1} + (\lambda_k - \lambda_{k+1})^{-1}).
\]

(4.15)

Inserting the expression for \( \lambda_k \) one finally obtains

\[
\alpha_k = \frac{1}{2}(\frac{k^2}{(k^2 + 1)})(\frac{1}{2k - 1} - \frac{1}{2k + 1}].
\]

(4.16)

Consider now the case \( k = 0 \). The concentric mode \( \psi_{E0} \) in this case is the vector \( (1, 0) \) and is usually called the breather mode. There is no splitting problem in this case because the antisymmetric counterpart \( \psi_{E0} \) is the zero vector. The lower boundaries for \( i \) in both sums in equation (4.7) being unity in this case, this equation reads

\[
\lambda_0 \alpha_0 = \lambda_0 (\vec{\psi}_{I_1}, C\vec{\psi}_{EO})^2 + (\lambda_0 - \lambda_1)^{-1}(\vec{\psi}_{E1}, [N^+CN - \lambda_0 C]\vec{\psi}_{EO})^2.
\]

(4.17)

From the first expressions in equations (4.10a) and (4.10b), respectively, it follows immediately that both squares in this equation are equal, so that with \( \lambda_0 = 1 \) and \( (\lambda_0 - \lambda_1)^{-1} = -1 \), the two terms in equation (4.17) cancel and

\[
\alpha_0 = 0.
\]

(4.18)

Finally consider the case \( k = 1 \). In the symmetric case equation (4.7) reads

\[
\lambda_1 \alpha_{S1} = \lambda_1 (\vec{\psi}_{IS_2}, C\vec{\psi}_{ES1})^2 + (\lambda_1 - \lambda_0)^{-1}(\vec{\psi}_{ES0}, [N^+CN - \lambda_1 C]\vec{\psi}_{ES1})^2
\]

\[
+ (\lambda_1 - \lambda_2)^{-1}(\vec{\psi}_{ES2}, [N^+CN - \lambda_1 C]\vec{\psi}_{ES1})^2.
\]

(4.19)
With equations (4.10a) and (4.12) one obtains

\[ (\psi_{ES2}, C\psi_{ES1})^2 = \frac{1}{160} k_3^2 \lambda_1^{-1}. \]  

(4.20)

With equation (4.10b) one sees that the second term on the right-hand side of equation (4.19) is zero and that

\[ (\psi_{ES2}, [N^* CN - \lambda_1 C]\psi_{ES1})^2 = \frac{1}{2} k_3^2 \lambda_1^{-1}, \]  

(4.21)

so that with these expressions inserted into equation (4.19) there remains

\[ \alpha_{S1} = -\frac{1}{24}. \]  

(4.22)

In the antisymmetric case equation (4.7) reads

\[ \lambda_1 \alpha_{A1} = \lambda_1 (\psi_{IA0}, C\psi_{EA1})^2 + \lambda_1 (\psi_{IA2}, C\psi_{EA1})^2 + (\lambda_1 - \lambda_2)^{-1} (\psi_{IA2}, [N^* CN - \lambda_1 C]\psi_{EA1})^2. \]  

(4.23)

Obviously the second and the third terms on the right-hand side are equal to the first and the third terms respectively in the expression (4.19) for \( \alpha_{S1} \). Because the second term in the latter equals zero one finds

\[ \alpha_{A1} = \alpha_{S1} + (\psi_{IA0}, C\psi_{EA1})^2. \]  

(4.24)

Now note that \( \|\psi_{IA0}\|^2 = 2 \) (see the inner product (2.24)) and evaluate the square with equation (4.10a) to obtain \( \alpha_{A1} - \alpha_{S1} = \frac{1}{4} \) so that

\[ \alpha_{A1} = \frac{5}{24}. \]  

(4.25)

5. THE INEXTENSIONAL MODES

On the basis of equation (3.1) an eigenvalue equation will be derived that describes how the frequencies of the inextensional modes depend on \( \delta \). As in the case of the extensional modes, it will be shown that there is no splitting in order \( \delta^2 \) if \( k = 1 \). Because the \( k = 0 \) and \( k = 1 \) modes represent rigid rotation and translation of the ring respectively there is no splitting in these cases.

It will be recalled that equation (3.1) has the form

\[ \omega^2 (A_0 + \mu^2 A_2) a = \omega^2 (B_0 + \mu^2 B_2) a, \]  

(5.1)

where the coefficient operators \( A_1 \) and \( B_1 \) depend on \( \delta \): i.e.,

\[ A_1 = A_{11} + A_{12} \delta + \cdots, \quad B_1 = B_{11} + B_{12} \delta + \cdots. \]  

(5.2)

The operators \( A_{ij} \) and \( B_{ij} \) are evaluated by substitution of the expressions (2.20) and (2.25) for \( D \) and \( H \) into equation (2.34). Equation (5.1) has been studied in section 4 for the case \( \mu = 0 \), where it has been shown that it has an eigenvalue zero with infinite multiplicity and eigenspace spanned by the set \( \{ \psi_\ell \} \). The first order perturbation expressions with respect to \( \mu^2 \) of this zero eigenvalue are

\[ \omega^2 = \omega^2 \mu^2 \gamma(\delta), \quad a = a_0(\delta) + \mu^2 a_2(\delta), \]  

(5.3)

where \( a_0(\delta) \) is a linear combination of the \( \psi_\ell \). After substitution of equations (5.3) into equation (5.1) one finds, in the usual way, i.e., by taking terms of equal order in \( \mu^2 \) together and with \( B_0 \psi_\ell = 0 \), an eigenvalue equation for \( \gamma \),

\[ \gamma(\delta) P_1 A_0(\delta) P_1 a_0 - P_1 B_2(\delta) P_1 a_0, \]  

(5.4)

where \( P_1 \) denotes the orthogonal projection on the inextensional subspace.
5.1. THE CONCENTRIC RING

To solve the concentric ring eigenvalue problem, one needs \( A_{00} \) and \( B_{20} \), which, from equation (2.34), are

\[
A_{00} = 2I, \quad B_{20} = \frac{1}{3}(2N^tN + N^tNH^2 + 2H_0N^tNH_0 + H_0^{12}N^tN).
\]  

(5.5)

One verifies easily \( N\psi_{Rk} = 0 \) so that, equivalently, \( NP \) is the zero operator and the same holds for \( (NP)^t = P^tN^t \). Consequently equation (5.4) reduces in the concentric case to

\[
\tilde{\gamma}P^t\alpha_0 = \frac{1}{3}P^tH_0^tN^tNH_0P^t\alpha_0.
\]  

(5.6)

With the expressions for \( H_0 \) and \( N \) (see equations (2.20) and (2.25)) the matrix elements of the right-hand side operator in equation (5.6) are evaluated straightforwardly to give

\[
(\psi_{Rk}, \frac{1}{3}H_0^tN^tNH_0\psi_{Rn}) = \gamma_k(\psi_{Rk}, \psi_{Rn}), \quad \gamma_k = \frac{1}{3}k^2(k^2 - 1)/(k^2 + 1),
\]  

(5.7a, b)

and consequently the \( \gamma_k \) are the eigenvalues of equation (5.6), corresponding with a two-dimensional eigenspace spanned by \( \psi_{SK} \) and \( \psi_{AK} \). Thus the classical frequencies \( \omega_k^2 = \omega_0^2 + \gamma_k \) are found.

5.2. SPLITTING

According to Theorem 3.1, equation (5.4) can be solved for the symmetric and the antisymmetric cases separately. In each case \( \gamma_k \) is a non-degenerate eigenvalue and consequently \( \tilde{\gamma}(\delta) \) can be written in the form (Theorem 3.2)

\[
\tilde{\gamma}(\delta) = \gamma_k(1 + \alpha_k\delta^2 + \cdots),
\]  

(5.8)

and, as in the extensional case, one can consider the difference of the values of \( \alpha_k \) in the symmetric and in the antisymmetric case. To this end note that only the \( A_{01} \) and \( B_{21} \) terms in the expressions for \( A_0(\delta) \) and \( B_0(\delta) \) (see equations (2.34) and (2.25)) contribute to splitting in order \( \delta^2 \) (Theorem 3.4). These two operators are given by (see equations (2.34), (2.20) and (2.25))

\[
P^tA_{01}P^t = 2P^tCP^t, \quad P^tB_{21}P^t = \frac{1}{3}P^t[H_0^tN^tNH_1 + H_0^tN^tCNH_0 + H_0^tN^tNH_0]P^t.
\]  

(5.9)

With \( C = \cos \theta \) it is clear that \( P^tA_{01}P^t \), if acting on a vector \( \psi_{Rk} \), produces only \( \psi_{Rk+1} \) and \( \psi_{Rk-1} \) terms. The same holds true for \( P^tB_{21}P^t \), as one sees with the expressions (2.20) for \( H_0 \) and \( H_1 \). An analogous result, however, was the basis of the proof of Theorem 4.1 and consequently this Theorem applies just as well for the coefficients \( \alpha_k \) in equation (5.8). Evaluation of these coefficients is straightforward but complicated. The results are given in section 7.

Finally consider the cases \( k = 0 \) and \( k = 1 \), for which \( \gamma_k = 0 \). Apart from the trivial mode \( \psi_{ISO} = (0, 0) \) the corresponding eigenvectors are

\[
\psi_{IS1} = (\cos \theta, -\sin \theta), \quad \psi_{IA1} = (\sin \theta, \cos \theta), \quad \psi_{IA0} = (0, 1),
\]  

(5.10)

which obviously (remember the definition of \( u \) and \( v \)) represent translations \( (k = 1) \) and a rotation of the ring, respectively. These modes should exist also if \( \delta \neq 0 \) with corresponding eigenvalues \( \tilde{\gamma}(\delta) \) equal to zero. With the relations

\[
H_0\psi_{IS1} = H_0\psi_{IA1} = 0, \quad H^t_0\psi_{IA0} = \psi_{IA1}, \quad N_0\psi_{IA1} = 0,
\]  

(5.11)

which are easily verified, one shows straightforwardly that this is indeed the case \( (H = H_0 + \delta H_1; \text{see equations (2.20 and (2.34))}) \).
6. EXPERIMENTAL PROCEDURE

Radial characteristic frequencies were measured on a set of five rings which, eccentricity apart, had substantially similar dimensions. They were turned from a single sheet of mild steel on a lathe fitted with a magnetic chuck. In preliminary trials one accurately machined concentric ring had shown pronounced splitting of its lower frequencies due to internal strain. The rings, therefore, were made about 2 mm oversize and together with two rods, cut in directions along and across the sheet, were annealed in air at 600°C for 1 h before being turned to their final dimensions.

The rods were both milled to a rectangular cross-section with parallel ends. From their lengths and lowest longitudinal resonances the value of $Y/p$ was determined as $(2.699 \pm 0.001) \times 10^7$ m$^2$ s$^{-2}$.

Measurements of the outside and inside diameters and eccentricity of the centres were made using a G.S.I.P. Universal Measuring Machine Type 214B. Readings were taken to the nearest 0.5 µm. The circularity was checked with a Talyrond 100.

In Table 1 the diameters quoted are each the mean of six diameters at 30° separation. The “errors” denote the lack of circularity: i.e., the maximum deviation from the circle with the mean diameter. Eccentricity, $\Delta$, is the distance between the centres of the inner and outer circles.

<table>
<thead>
<tr>
<th>Ring</th>
<th>Outer diameter (mm)</th>
<th>Inner diameter (mm)</th>
<th>Eccentricity</th>
<th>Height (mm)</th>
<th>$\delta^2$</th>
<th>$\mu^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200.664 ± 0.016</td>
<td>191.208 ± 0.016</td>
<td>0</td>
<td>5.07</td>
<td>0</td>
<td>5.822 × 10$^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>200.088 ± 0.016</td>
<td>189.991 ± 0.016</td>
<td>0.481</td>
<td>5.05</td>
<td>9.096 × 10$^{-3}$</td>
<td>6.700 × 10$^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>200.083 ± 0.025</td>
<td>190.090 ± 0.025</td>
<td>0.976</td>
<td>5.03</td>
<td>3.816 × 10$^{-2}$</td>
<td>6.560 × 10$^{-4}$</td>
</tr>
<tr>
<td>4</td>
<td>200.024 ± 0.020</td>
<td>189.934 ± 0.015</td>
<td>2.019</td>
<td>5.05</td>
<td>0.1602</td>
<td>6.695 × 10$^{-4}$</td>
</tr>
<tr>
<td>5</td>
<td>200.043 ± 0.075</td>
<td>190.085 ± 0.060</td>
<td>3.042</td>
<td>5.05</td>
<td>0.372</td>
<td>6.515 × 10$^{-4}$</td>
</tr>
</tbody>
</table>

To measure its characteristic frequencies the ring was suspended in a vertical plane on the sharp point of a hook formed on the end of a helical spring. In earlier work [2] this support had proved to be the most satisfactory in that it caused minimum extra splitting of otherwise almost degenerate pairs. To avoid any further mechanical loading the ring was excited acoustically with a loudspeaker and its vibration was detected with a Brüel and Kjaer (B & K) capacity transducer Type MM0004. This fed through a B & K measuring amplifier Type 2606 and slave-filter Type 2020 which followed the frequency of the oscillator, Type 1022. Output was to an oscilloscope and B & K level recorder Type 2304. Methods of measuring the frequencies and separation of close doublets have been discussed in detail in an earlier paper [7].

7. RESULTS

7.1. EXTENSIONAL RADIAL FREQUENCIES

Although the primary object of the investigation was to study the splitting of resonances which would be degenerate in a concentric ring, the perturbation calculations in section 4 also predict the changes which should occur due to eccentricity when splitting does not occur.

It will be recalled that for concentric rings the radial radian frequencies are given by

$$\omega_{\pm}^2 = \frac{Y}{\rho} \left( \frac{1}{R^2} \right) (1 + k^2)$$

(7.1a)
and that for eccentric rings these frequencies are modified to

$$\alpha_k^2 = \alpha_k^2[1 + \alpha_k \delta^2], \quad (7.1b)$$

$$\alpha_0 = 0, \quad \alpha_1 = -1/24 \text{ for symmetrical vibration,} \quad (7.2a)$$

$$\alpha_1 = +5/24 \text{ for antisymmetrical vibration,} \quad (7.2b)$$

$$\alpha_k = \frac{1}{4}(k^2/(k^2 + 1))\left[[1/(2k - 1)] - [1/(2k + 1)]\right], \quad k \geq 2.$$  

Owing to the 20 kHz upper frequency limit of the apparatus it was not possible to test this expression for $k > 2$.

Data for $k = 0$ are given in Table 2. All modes were singlets and agreement between predicted and measured frequencies was good.

**Table 2**  
*Extensional modes $k = 0$*

<table>
<thead>
<tr>
<th>Ring</th>
<th>Experimental frequency (Hz)</th>
<th>Calculated frequency (Hz)</th>
<th>Exp./calc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8420.7</td>
<td>8439.9</td>
<td>0.998</td>
</tr>
<tr>
<td>2</td>
<td>8460.6</td>
<td>8478.7</td>
<td>0.998</td>
</tr>
<tr>
<td>3</td>
<td>8458.5</td>
<td>8476.6</td>
<td>0.998</td>
</tr>
<tr>
<td>4</td>
<td>8461.2</td>
<td>8481.3</td>
<td>0.998</td>
</tr>
<tr>
<td>5</td>
<td>8456.0</td>
<td>8477.6</td>
<td>0.997</td>
</tr>
</tbody>
</table>

Data for $k = 1$ are given in Table 3, both experiment and theory showing doublets for the eccentric rings but with no splitting predicted for the concentric case. However, the theory takes no account of the point loading imposed on the ring by its support. That this caused the 0.11% splitting, rather than some imperfection in the ring, was shown firstly by the degree of splitting being independent of the point of support, and secondly by the point of support locating the nodal pattern.

**Table 3**  
*Extensional modes $k = 1$*

<table>
<thead>
<tr>
<th>Ring</th>
<th>Experimental frequencies (Hz)</th>
<th>Calculated frequencies (Hz)</th>
<th>Exp./calc.</th>
<th>Percentage splitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11897.4 11910.5</td>
<td>11935.8 11935.8</td>
<td>0.997 0.998</td>
<td>0.11 0</td>
</tr>
<tr>
<td>2</td>
<td>11948.5 11976.1</td>
<td>12002.0 12002.0</td>
<td>0.997 0.998</td>
<td>0.23 0.113</td>
</tr>
<tr>
<td>3</td>
<td>11942.8 12009.7</td>
<td>12035.3 12035.3</td>
<td>0.997 0.998</td>
<td>0.56 0.475</td>
</tr>
<tr>
<td>4</td>
<td>11922.8 12179.5</td>
<td>12193.0 12193.0</td>
<td>0.997 0.999</td>
<td>2.13 1.98</td>
</tr>
<tr>
<td>5</td>
<td>11828.8 12560.0</td>
<td>12446.6 12446.6</td>
<td>0.994 1.009</td>
<td>6.00 4.53</td>
</tr>
</tbody>
</table>

The eccentric rings were all supported from their narrowest point and the "extra" effect of the support may therefore be taken into account. Subtracting 0.11% from the observed splitting then brings theory and experiment into good agreement except for Ring 5. The substantially greater splitting shown by the latter may have been due to its relatively poor circularity and/or effects of order $\delta^4$. 
For \( k = 2 \) (see Table 4) Ring 5 again shows a much greater splitting than the others. All show some splitting where none was predicted, but substantially less than for \( k = 1 \) and of the same order as for the concentric ring. Again, the predicted and experimental frequencies are in good agreement.

### Table 4

**Extensional modes \( k = 2 \)**

<table>
<thead>
<tr>
<th>Ring</th>
<th>Experimental frequencies (Hz)</th>
<th>Calculated frequency (Hz)</th>
<th>Mean exp./calc.</th>
<th>Percentage splitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18 815.1</td>
<td>18 892.2</td>
<td>0.997</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>18 899</td>
<td>18 961.2</td>
<td>0.997</td>
<td>0.09</td>
</tr>
<tr>
<td>3</td>
<td>18 899</td>
<td>18 964.0</td>
<td>0.997</td>
<td>0.12</td>
</tr>
<tr>
<td>4</td>
<td>18 949</td>
<td>19 005.3</td>
<td>0.997</td>
<td>0.02</td>
</tr>
<tr>
<td>5</td>
<td>18 981</td>
<td>19 050.6</td>
<td>0.998</td>
<td>0.72</td>
</tr>
</tbody>
</table>

### Table 5

**Constants used to calculate the inextensional frequencies**

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \gamma )</th>
<th>( \alpha )</th>
<th>( k )</th>
<th>( \gamma )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.4</td>
<td>-0.2814</td>
<td>8</td>
<td>1302.6</td>
<td>-0.7380</td>
</tr>
<tr>
<td>3</td>
<td>19.2</td>
<td>-0.6107</td>
<td>9</td>
<td>2107.3</td>
<td>-0.7407</td>
</tr>
<tr>
<td>4</td>
<td>70.6</td>
<td>-0.6867</td>
<td>10</td>
<td>3234.7</td>
<td>-0.7426</td>
</tr>
<tr>
<td>5</td>
<td>184.6</td>
<td>-0.7141</td>
<td>11</td>
<td>4760.7</td>
<td>-0.7440</td>
</tr>
<tr>
<td>6</td>
<td>397.3</td>
<td>-0.7268</td>
<td>12</td>
<td>6769.3</td>
<td>-0.7450</td>
</tr>
<tr>
<td>7</td>
<td>752.6</td>
<td>-0.7338</td>
<td>13</td>
<td>9352.7</td>
<td>-0.7457</td>
</tr>
</tbody>
</table>

### Table 6

**Ring 1, inextensional radial frequencies**

<table>
<thead>
<tr>
<th>( k )</th>
<th>Experimental (Hz)</th>
<th>Calculated (Hz)</th>
<th>Exp./calc.</th>
<th>Percentage splitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>312.55</td>
<td>312.75</td>
<td>315.50</td>
<td>0.991</td>
</tr>
<tr>
<td>3</td>
<td>881.99</td>
<td>882.40</td>
<td>892.38</td>
<td>0.989</td>
</tr>
<tr>
<td>4</td>
<td>1686.8</td>
<td>1687.8</td>
<td>1711.9</td>
<td>0.986</td>
</tr>
<tr>
<td>5</td>
<td>2718.6</td>
<td>2720.7</td>
<td>2767.2</td>
<td>0.983</td>
</tr>
<tr>
<td>6</td>
<td>3972.9</td>
<td>3974.9</td>
<td>4059.4</td>
<td>0.979</td>
</tr>
<tr>
<td>7</td>
<td>5442.3</td>
<td>5445.7</td>
<td>5887.2</td>
<td>0.974</td>
</tr>
<tr>
<td>8</td>
<td>7120.8</td>
<td>7125.8</td>
<td>7350.4</td>
<td>0.969</td>
</tr>
<tr>
<td>9</td>
<td>9003.3</td>
<td>9009.0</td>
<td>9349.0</td>
<td>0.963</td>
</tr>
<tr>
<td>10</td>
<td>11 081.0</td>
<td>11 087.4</td>
<td>11 583</td>
<td>0.957</td>
</tr>
<tr>
<td>11</td>
<td>13 350</td>
<td>13 357</td>
<td>14 052</td>
<td>0.950</td>
</tr>
<tr>
<td>12</td>
<td>15 780</td>
<td>15 805</td>
<td>16 756</td>
<td>0.942</td>
</tr>
</tbody>
</table>
### Table 7

**Ring 2, inextensional radial frequencies**

<table>
<thead>
<tr>
<th>$k$</th>
<th>Experimental (Hz)</th>
<th>Calculated (Hz)</th>
<th>Exp./calc.</th>
<th>Percentage splitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>336.69</td>
<td>337.36</td>
<td>339.56</td>
<td>0.992</td>
</tr>
<tr>
<td></td>
<td>949.52</td>
<td>950.17</td>
<td>958.98</td>
<td>0.990</td>
</tr>
<tr>
<td>4</td>
<td>1814.21</td>
<td>1815.49</td>
<td>1838.1</td>
<td>0.987</td>
</tr>
<tr>
<td>5</td>
<td>2923.5</td>
<td>2925.0</td>
<td>2972.3</td>
<td>0.984</td>
</tr>
<tr>
<td>6</td>
<td>4270.0</td>
<td>4270.9</td>
<td>4360.0</td>
<td>0.980</td>
</tr>
<tr>
<td>7</td>
<td>5842.1</td>
<td>5846.7</td>
<td>6000.8</td>
<td>0.974</td>
</tr>
<tr>
<td>8</td>
<td>7639.5</td>
<td>7646.0</td>
<td>7894.4</td>
<td>0.968</td>
</tr>
<tr>
<td>9</td>
<td>9653.6</td>
<td>9659.3</td>
<td>10041</td>
<td>0.962</td>
</tr>
<tr>
<td>10</td>
<td>11873.6</td>
<td>11878.1</td>
<td>12440</td>
<td>0.955</td>
</tr>
<tr>
<td>11</td>
<td>14288</td>
<td>14295</td>
<td>15091</td>
<td>0.947</td>
</tr>
<tr>
<td>12</td>
<td>16887</td>
<td>16901</td>
<td>17995</td>
<td>0.939</td>
</tr>
</tbody>
</table>

### Table 8

**Ring 3, inextensional radial frequencies**

<table>
<thead>
<tr>
<th>$k$</th>
<th>Experimental (Hz)</th>
<th>Calculated (Hz)</th>
<th>Exp./calc</th>
<th>Percentage splitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>332.69</td>
<td>333.40</td>
<td>334.52</td>
<td>0.996</td>
</tr>
<tr>
<td></td>
<td>932.88</td>
<td>933.92</td>
<td>940.14</td>
<td>0.993</td>
</tr>
<tr>
<td>4</td>
<td>1780.5</td>
<td>1782.3</td>
<td>1800.0</td>
<td>0.990</td>
</tr>
<tr>
<td>5</td>
<td>2861.1</td>
<td>2869.0</td>
<td>2909.4</td>
<td>0.985</td>
</tr>
<tr>
<td>7</td>
<td>5729.6</td>
<td>5733.3</td>
<td>5872.1</td>
<td>0.976</td>
</tr>
<tr>
<td>8</td>
<td>7528.4</td>
<td>7533.3</td>
<td>7724.6</td>
<td>0.975</td>
</tr>
<tr>
<td>9</td>
<td>9469.5</td>
<td>9474.5</td>
<td>9824.3</td>
<td>0.964</td>
</tr>
<tr>
<td>10</td>
<td>11646.8</td>
<td>11654.1</td>
<td>12171</td>
<td>0.957</td>
</tr>
<tr>
<td>11</td>
<td>14017</td>
<td>14028</td>
<td>14765</td>
<td>0.950</td>
</tr>
<tr>
<td>12</td>
<td>16573</td>
<td>16588</td>
<td>17606</td>
<td>0.942</td>
</tr>
</tbody>
</table>

### Table 9

**Ring 4, inextensional radial frequencies**

<table>
<thead>
<tr>
<th>$k$</th>
<th>Experimental (Hz)</th>
<th>Calculated (Hz)</th>
<th>Exp./calc.</th>
<th>Percentage splitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>329.36</td>
<td>329.74</td>
<td>332.22</td>
<td>0.992</td>
</tr>
<tr>
<td>3</td>
<td>902.77</td>
<td>903.82</td>
<td>913.33</td>
<td>0.989</td>
</tr>
<tr>
<td>4</td>
<td>1714.34</td>
<td>1716.56</td>
<td>1739.4</td>
<td>0.986</td>
</tr>
<tr>
<td>5</td>
<td>2759.30</td>
<td>2760.00</td>
<td>2806.0</td>
<td>0.983</td>
</tr>
<tr>
<td>7</td>
<td>5506.6</td>
<td>5512.6</td>
<td>5655.5</td>
<td>0.974</td>
</tr>
<tr>
<td>8</td>
<td>7200.8</td>
<td>7209.0</td>
<td>7437.4</td>
<td>0.969</td>
</tr>
<tr>
<td>9</td>
<td>9101.6</td>
<td>9110.6</td>
<td>9457.3</td>
<td>0.963</td>
</tr>
<tr>
<td>10</td>
<td>11187</td>
<td>11209</td>
<td>11715</td>
<td>0.956</td>
</tr>
<tr>
<td>11</td>
<td>13480</td>
<td>13498</td>
<td>14210</td>
<td>0.949</td>
</tr>
<tr>
<td>12</td>
<td>15956</td>
<td>15966</td>
<td>16944</td>
<td>0.942</td>
</tr>
</tbody>
</table>
Radial Vibrations of Eccentric Rings

Table 10

Ring 5, inextensional radial frequencies

<table>
<thead>
<tr>
<th>k</th>
<th>Experimental (Hz)</th>
<th>Calculated (Hz)</th>
<th>Exp./calc.</th>
<th>Percentage splitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>315.12</td>
<td>317.13</td>
<td>0.995</td>
<td>0.336</td>
</tr>
<tr>
<td>3</td>
<td>823.92</td>
<td>824.73</td>
<td>0.989</td>
<td>0.098</td>
</tr>
<tr>
<td>4</td>
<td>1545.9</td>
<td>1567.8</td>
<td>0.987</td>
<td>0.213</td>
</tr>
<tr>
<td>5</td>
<td>2481.0</td>
<td>2518.0</td>
<td>0.985</td>
<td>0.020</td>
</tr>
<tr>
<td>6</td>
<td>3610.5</td>
<td>3681.9</td>
<td>0.981</td>
<td>0.152</td>
</tr>
<tr>
<td>7</td>
<td>4940.3</td>
<td>5058.6</td>
<td>0.977</td>
<td>0.166</td>
</tr>
<tr>
<td>8</td>
<td>6460.4</td>
<td>6647.9</td>
<td>0.973</td>
<td>0.206</td>
</tr>
<tr>
<td>9</td>
<td>8167.7</td>
<td>8449.4</td>
<td>0.968</td>
<td>0.226</td>
</tr>
<tr>
<td>10</td>
<td>10 059.0</td>
<td>10 160.0</td>
<td>0.962</td>
<td>0.214</td>
</tr>
<tr>
<td>11</td>
<td>12 128</td>
<td>12 290</td>
<td>0.957</td>
<td>0.181</td>
</tr>
<tr>
<td>12</td>
<td>14 340</td>
<td>14 590</td>
<td>0.950</td>
<td>0.348</td>
</tr>
</tbody>
</table>

The values of splitting is roughly the same for all k except k = 2, which tends to higher splitting than the rest. While some splitting will be due to the support, much of it is probably due to terms in order $\delta^4$. It is interesting that in their calculations concerning the vibrations of eccentric cylinders Tonin and Bies [8] expect only two pairs of modes to split, and both of these correspond to the $k = 2$ case for a ring. They predict substantial splitting for their (1, 2) modes, with the symmetric frequency increasing from the "concentric" value and the asymmetric frequency decreasing: the (3, 2) modes should have very slight splitting at extreme eccentricity.

As with the extensionals

$$\omega_k^2 = \omega_e^2 (1 + \alpha_k \delta^2),$$

but now

$$\omega_{ek} = (Y/\rho)(\mu^2/R_e^2)\gamma_k, \quad \gamma_k = \frac{1}{2}k^2(k^2 - 1)^2/(k^2 + 1).$$

The coefficients $\alpha_k$ follow from straightforward but lengthy analysis of equation (5.4). These calculations will be published in the near future. The coefficients are

$$\alpha_k = \frac{1}{2} + \frac{1}{2}[(p_+^2/q_+^2) + (p_+^2/q_-^2)]/(k^2 + 1),$$

$$p_+ = 2k^3 + 6k^2 + 3k + 7, \quad q_+ = 2k^3 + 3k^2 + 3k + 1,$$

$$p_- = 2k^3 - 6k^2 + 3k - 7, \quad q_- = 2k^3 - 3k^2 + 3k - 1.$$
8. CONCLUSION

The present theory gives useful expressions for the radial extensional and inextensional vibrations of an eccentric ring and predicts splitting only of \( k = 1 \) extensional modes, in order \( \delta^2 \). The question then arises if in higher order of \( \delta \) splitting is to be expected for other values of \( k \). It may be significant that the \( k = 2 \) modes show more splitting than those of higher \( k \).

To answer this question in detail one must calculate the 4th order perturbation terms and know the operators \( A \) and \( B \) in much more detail. Also, it would be very difficult to test the theory by looking for such a small amount of splitting with a ring, where the effect of the support can never be negligible.

However, if results similar to those above can be obtained for more general eccentric systems it may be possible to test them on eccentric cones and bells where, for some modes at least, the support has negligible effect. Such a generalization of the essentials of the present theory (see Theorems 3.1-3.4 and their proofs) seems to be possible and will be the object of further study.

ACKNOWLEDGMENTS

The authors are indebted to Dr R. Perrin for suggesting this topic, and subsequently for many helpful discussions, and to Mr P. L. Langdon for measuring the dimensions of the rings and rods.

REFERENCES

2. T. CHARNLEY and R. PERRIN 1973 Acustica 28, 139-146. Perturbation studies with a thin circular ring.

APPENDIX A

Consider the polar-like co-ordinates \( \{r, \theta\} \) in the \( x-y \) plane given by the implicit relations

\[
\begin{align*}
    r^2 &= y^2 - (x - \delta (r - \bar{R}))^2, \\
    \tan \theta &= y / (x - \delta (r - \bar{R})).
\end{align*}
\]

The lines of constant \( r \) are circles with their centre at \( x = \delta(r - \bar{R}), y = 0 \). By implicit differentiation it is easily shown that

\[
\begin{align*}
    \nabla r &= (1 + \delta \cos \theta)^{-1} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \\
    \nabla \theta &= r^{-1} (1 + \delta \cos \theta)^{-1} \begin{pmatrix} -\sin \theta \\ \cos \theta + \delta \end{pmatrix}.
\end{align*}
\]

(A2, A3)
Lines of constant \( \theta \) have slope \( \frac{\partial \theta}{\partial x} = -\sin \theta / (\cos \theta + \delta) \) which depends on \( \theta \) only so that these lines are straight (see Figure 1). The surface \( d\sigma \) of an infinitesimal area is given by

\[
d\sigma = r(1 + \delta \cos \theta) \, d\theta \, dr,
\]

which follows from \( d\sigma = a \times r d\theta \), where \( a = dr|\nabla r|^{-1} \) is the orthogonal distance between the lines \( r = r' \) and \( r' = r' + dr \). Obviously, as can be seen from equations (A2) and (A3), this system is not orthogonal. Define, however, the system \( \{\rho, \phi\} \) as follows:

\[
\tan \frac{1}{2}\phi = \left(\frac{r}{r_0}\right)^{-\delta} \tan \frac{1}{2}\theta, \quad \rho = r.
\]

This system is orthogonal as can be seen from the expressions for the gradients below. First calculate \( \nabla \phi \). With equation (A5)

\[
\frac{1}{\cos^{\delta/2}} \frac{1}{2} \nabla \phi = \left(\frac{r}{r_0}\right)^{-\delta} \frac{1}{2} \cos^{\delta/2} \nabla \theta - \delta \left(\frac{r}{r_0}\right)^{-\delta} r^{-1} \tan \frac{1}{2}\theta \nabla r.
\]

Substitution of equations (A2) and (A3), making use of some well-known goniometrical relations and of

\[
\left(\frac{r}{r_0}\right)^{-\delta} \left(\cos^{\delta/2}\phi\right) = \sin \phi / \cos \theta,
\]

the latter being an immediate consequence of equation (A5), then gives

\[
\nabla \phi = r^{-1} \frac{\sin \phi}{\sin \theta} \left(\frac{-\sin \theta}{\cos \theta}\right).
\]

Obviously \( \nabla \phi \) is perpendicular to \( \nabla \rho = \nabla r \) and so the \( \{\rho, \phi\} \) system is orthogonal.

Equation (2.17) from the main text

\[
\left(\begin{array}{c}
u \\ \sigma \\
\end{array}\right)_\rho = \frac{h_2}{h_1} \left(\begin{array}{c}0 \\ 1 \\
\end{array}\right) \left(\begin{array}{c}u \\ \sigma \\
\end{array}\right)_\phi.
\]

will now be written in terms of \( \{r, \theta\} \) co-ordinates. The symbols \( h_{1,2} \) denote

\[
h_1 = |\nabla \rho| = (1 + \delta \cos \theta)^{-1}, \quad h_2 = |\nabla \phi| = \rho^{-1} (\sin \phi / \sin \theta).
\]

Among the differential operators one has the following relations

\[
\partial / \partial \phi = \theta_\phi (\partial / \partial \theta) + r_\phi (\partial / \partial r), \quad \partial / \partial \rho = \theta_\rho (\partial / \partial \theta) + r_\rho (\partial / \partial r),
\]

where \( \theta_\phi \) denotes partial differentiation of \( \theta(\rho, \phi) \), etc. From equation (A5) it follows immediately that

\[
r_\rho = 1, \quad r_\phi = 0; \quad \theta_\rho = \delta r^{-1} \sin \theta, \quad \theta_\phi = \sin \theta / \sin \phi,
\]

where for the latter use has been made of equation (A7). With equation (A11) it follows easily that the left-hand side of equation (A9) equals

\[
\left(\begin{array}{c}u \\ \sigma \\
\end{array}\right)_r + \left(\begin{array}{cc}\theta_\rho & \theta_\phi (h_2 / h_1) \sigma \\
(h_2 / h_1) \theta_\phi & \theta_\rho \\
\end{array}\right) \left(\begin{array}{c}u \\ \sigma \\
\end{array}\right)_\phi.
\]

With equations (A13) and (A10) one obtains

\[
\theta_\phi (h_2 / h_1) = r^{-1}(1 + \delta \cos \theta)
\]

and substitution of equation (A13) gives equation (A14) its final form:

\[
\left(\begin{array}{c}u \\ \sigma \\
\end{array}\right)_r + r^{-1} \left(\begin{array}{cc}\delta \sin \theta & \sigma (1 + \delta \cos \theta) \\
1 + \delta \cos \theta & \delta \sin \theta \\
\end{array}\right) \left(\begin{array}{c}u \\ \sigma \\
\end{array}\right)_\phi.
\]
Now consider the right-hand side of equation (A9). First note that $h_{2,1}/h_2 = (\ln h_2)_T$ and thus one obtains, with equation (A10),

$$h_{2,1}/h_2 = -\rho^{-1} - (\cos \theta / \sin \theta) \theta_p = -\rho^{-1}(1 + \delta \cos \theta), \quad (A17)$$

the second equality following with equation (A13). Differentiation of $h_1$ in equation (A10) with respect to $\phi$ and substitution of the expressions for $h_1, h_2$ and $\theta_p$ in the result gives

$$(h_2^2/h_1)h_{1,\phi} = \delta \rho^{-1} \sin \theta. \quad (A18)$$

With the above results the right-hand side of equation (A9) can now be written as

$$r^{-1}( -e(1 + \delta \cos \theta) \begin{pmatrix} \delta \sin \theta \\ -\delta \sin \theta \end{pmatrix} (u) ) \quad (A19)$$

and thus equation (A9) is equivalent to

$$\begin{pmatrix} u \\ v \end{pmatrix}_r = -\frac{1}{r} \begin{pmatrix} \delta \sin \theta & e(1 + \delta \cos \theta) \\ 1 + \delta \cos \theta & -\delta \sin \theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (A20)$$

Finally the quantity

$$e_{22} = h_2 v_{\phi} + u h_1 (h_2^{-1})_T \quad (A21)$$

is written in terms of $\{r, \theta\}$-co-ordinates. With equations (A11)-(A13) one obtains

$$h_2 v_{\phi} = r^{-1} v_\phi, \quad (A22)$$

and with equations (A10) and (A17)

$$u h_1 h_2 (h_2^{-1})_T = v^{-1} u, \quad (A23)$$

and as a result

$$e_{22} = r^{-1} (v_\phi + u). \quad (A24)$$

APPENDIX B

In order to prove Lemma 3.3 the following complex functions $\tilde{\psi}$ are introduced:

$$\tilde{\psi}_{ik} = e^{ik\theta} \begin{pmatrix} k \\ i \end{pmatrix}, \quad \tilde{\psi}_{ik} = e^{ik\theta} \begin{pmatrix} 1 \\ -ik \end{pmatrix}, \quad |k| = 0, 1, 2, \ldots. \quad (B1)$$

They are related to the real functions $\psi$ in equation (3.9) by

$$\tilde{\psi}_{ik} = \psi_{ik}^l + i\psi_{ik}^r, \quad \tilde{\psi}_{ik} = \psi_{ik}^s + i\psi_{ik}^t, \quad (B2)$$

where the definition of $\psi_k$ is extended to all integer values of $k$. The vectors $\tilde{\psi}$ are elements of the complex extension $\tilde{H}$ of the real space $H$ [9]. The elements of $\tilde{H}$ have the form $a + ib$ with $a$ and $b$ in $H$ and the inner product is given by

$$(a + ib, c + id) = (a, c) + (b, d) + i((a, d) - (b, c)). \quad (B3)$$

One verifies easily that the set $\{\tilde{\psi}_{ik}, \tilde{\psi}_{ik}^r\}$ is an orthogonal basis in $\tilde{H}$. Let $\tilde{G}$ denote the extension of $G$ to $\tilde{H}$: i.e., by definition

$$\tilde{G}\tilde{\psi}_{ik} = G\psi_{ik} + iG\psi_{ik}^r. \quad (B4)$$
with an analogous expression for $\tilde{G}\tilde{\psi}_{E_k}$. The set $\{\tilde{\psi}_{H_k}, \tilde{\psi}_{E_k}\}$ being a basis one also has the unique expression

$$\tilde{G}\tilde{\psi}_{H_k} = \sum_{i=-\infty}^{+\infty} c_{R_{H_k}} \tilde{\psi}_{H_i} + \sum_{i=-\infty}^{+\infty} c_{R_{E_k}} \tilde{\psi}_{E_i}. \quad (B5)$$

Now one can claim that the coefficients $c_{R_{H_k}}$ and $c_{R_{E_k}}$ are real. Equation (3.9a) in Lemma 3.3 then follows immediately after substitution of equation (B2) in the right-hand side and of equation (B4) in the left-hand side of equation (B5). Equation (3.9b) can be proved analogously. To show that the coefficients in equation (B5) are real note that, with equation (B5)

$$(\tilde{\psi}_{H_i}, \tilde{G}\tilde{\psi}_{H_k}) = c_{R_{H_k}} (\tilde{\psi}_{H_i}, \tilde{\psi}_{H_i}), \quad (\tilde{\psi}_{E_i}, \tilde{G}\tilde{\psi}_{H_k}) = c_{R_{E_k}} (\tilde{\psi}_{E_i}, \tilde{\psi}_{E_i}) \quad (B6)$$

but also, with equations (B3) and (B4),

$$(\tilde{\psi}_{H_i}, \tilde{G}\tilde{\psi}_{H_k}) = (\psi_{H_{IS_h}, G}\psi_{H_{IS_k}}) + (\psi_{H_{IA_h}, GI}\psi_{H_{IA_k}}) + i[(\psi_{H_{IS_h}, G}\psi_{H_{IA_k}}) - (\psi_{H_{IA_h}, GI}\psi_{H_{IS_k}})]. \quad (B7)$$

The right-hand side of this last expression is real because of the invariancy of the subspaces $H_S$ and $H_A$ with respect to $G$ (Theorem 3.1) and because of their orthogonality and so, with equation (B6), $c_{R_{H_k}} (\tilde{\psi}_{H_i}, \tilde{\psi}_{H_i})$ is real. The second factor, being a norm, is real and consequently $c_{R_{E_k}}$ is real. Analogous to equation (B7) there is an expression for $(\tilde{\psi}_{E_i}, G\tilde{\psi}_{H_k})$, with which it is shown in the same way that $c_{R_{E_k}}$ is real.

Now consider the action of the operators one is dealing with on the set (B1). Making use of

$$\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \quad (B8)$$

one finds easily ($C$ denotes multiplication with $\cos \theta$),

$$2C\tilde{\psi}_{H_k} = e^{i(k+1)\theta} \begin{pmatrix} k \\ 1 \end{pmatrix} + e^{i(k-1)\theta} \begin{pmatrix} k \\ 1 \end{pmatrix}. \quad (B9)$$

A simple calculation shows that

$$\begin{pmatrix} k \\ 1 \end{pmatrix} = \frac{1 + k(k+1)}{1 + (k+1)^2} \begin{pmatrix} k+1 \\ i \end{pmatrix} - \frac{1}{1 + (k+1)^2} \begin{pmatrix} 1 \\ -ik-i \end{pmatrix}, \quad (B10a)$$

but also

$$\begin{pmatrix} k \\ 1 \end{pmatrix} = \frac{1 + k(k-1)}{1 + (k-1)^2} \begin{pmatrix} k-1 \\ i \end{pmatrix} + \frac{1}{1 + (k-1)^2} \begin{pmatrix} 1 \\ -ik+i \end{pmatrix}. \quad (B10b)$$

Introducing

$$\lambda_k = 1 + k^2 \quad (B11)$$

one finds

$$2C\tilde{\psi}_{H_k} = \lambda_{k+1}^{-1} \{[1 + k(k+1)]\tilde{\psi}_{H_{k+1}} - \tilde{\psi}_{H_{k-1}}\} + \lambda_{k-1}^{-1} \{[1 + k(k-1)]\tilde{\psi}_{H_{k-1}} + \tilde{\psi}_{H_{k+1}}\}. \quad (B12)$$

The corresponding expression for $\tilde{\psi}_{E_k}$ is found by application of the equality

$$-i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{\psi}_{H_k} = \tilde{\psi}_{E_k}, \quad i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{\psi}_{E_k} = \tilde{\psi}_{H_k}, \quad (B13)$$

to equation (B12) and reads

$$2C\tilde{\psi}_{E_k} = \lambda_{k+1}^{-1} \{[1 + k(k+1)]\tilde{\psi}_{E_{k+1}} + \tilde{\psi}_{H_{k+1}}\} + \lambda_{k-1}^{-1} \{[1 + k(k-1)]\tilde{\psi}_{E_{k-1}} - \tilde{\psi}_{H_{k-1}}\}. \quad (B14)$$
Next consider the operator $N^CN$. With the representation

$$N = \begin{pmatrix} 1 & d/d\theta \\ 0 & 0 \end{pmatrix}$$ (B15)

one easily verifies

$$\tilde{N}\tilde{\psi}_{ik} = 0, \quad \tilde{N}\tilde{\psi}_{Ek} = \lambda_k e^{ik\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ (B16)

With the first of the above equations one finds

$$N^CN\tilde{\psi}_{ik} = 0.$$ (B17)

Proceeding with the second one, one obtains

$$2CN\tilde{\psi}_{Ek} = \lambda_k \{e^{i(k+1)\theta} + e^{i(k-1)\theta}\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ (B18)

With

$$N^* = \begin{pmatrix} 1 & 0 \\ -d/d\theta & 0 \end{pmatrix}$$ (B19)

one easily arrives at

$$N^+CN\tilde{\psi}_{Ek} = \frac{1}{2}\lambda_k \{\tilde{\psi}_{E,k+1} + \tilde{\psi}_{E,k-1}\}.$$ (B20)

APPENDIX C

Consider the eigenvalue problem

$$\lambda(I + \delta G_1 + \delta^2 G_2)a = (H_0 + \delta H_1 + \delta^2 H_2)a.$$ (C1)

Let $\{\lambda_i\}$ denote the eigenvalues of the unperturbed problem and $\{\tilde{a}_i\}$ the corresponding complete orthonormal set of eigenvectors. Let the $k$th eigenvalue be non-degenerate and assume a priori that this eigenvalue has the form

$$\lambda^* = \lambda_k(1 + \alpha \delta^2 + \cdots),$$ (C2)

which is true in the present case. Substitute this expression together with

$$a = \tilde{a}_k + \delta a_{k1} + \delta^2 a_{k2} + \cdots$$ (C3)

into equation (C1). Then one finds the hierarchy

$$(\lambda_k - H_0)\tilde{a}_k = 0, \quad (\lambda_k - H_0)a_{k1} = (-\lambda_k G_1 + H_1)\tilde{a}_k,$$ (C4a, C4b)

$$(\lambda_k - H_0)a_{k2} + \lambda_k \alpha a_k = (H_1 - \lambda_k G_1)\tilde{a}_{k1} + (H_2 - \lambda_k G_2)\tilde{a}_k.$$ (C4c)

Choose as usual $a_{k1}$ orthogonal to $\tilde{a}_k$ and write

$$a_{k1} = \sum_{i \neq k} a_{k1i} \tilde{a}_i.$$ (C5)

Substitution of this expression into equation (4.6) and taking the inner product with $\tilde{a}_i$ yields the following expressions for the coefficients $a_{k1i}$:

$$a_{k1i} = (\lambda_k - \lambda_i)^{-1}(\tilde{a}_i, (H_1 - \lambda_k G_1)\tilde{a}_k), \quad i \neq k.$$ (C6)
The coefficient $\alpha_k$ in equation (C2) is found after substitution of equations (C5) and (C6) into equation (C4c) and taking the inner product with $\bar{a}_k$ yields

$$\lambda_k \alpha_k = (\bar{a}_k, (H_2 - \lambda_k G_2) \bar{a}_k) + \sum_{i \neq k} (\lambda_k - \lambda_i)^{-1} |(\bar{a}_i, (H_1 - \lambda_k G_1) \bar{a}_k)|^2.$$  \hspace{1cm} (C7)