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Brief paper

Solvability conditions and design for state synchronization of multi-agent systems[☆]

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ABSTRACT

This paper derives conditions on the agents for the existence of a protocol which achieves synchronization of homogeneous multi-agent systems (MAS) with partial-state coupling, where the communication network is directed and weighted. These solvability conditions are necessary and sufficient for single-input agents and sufficient for multi-input agents. The solvability conditions reveal that the synchronization problem is primarily solvable for two classes of agents. This first class consists of at most weakly unstable agents (i.e. agents have all eigenvalues in the closed left half plane) and the second class consists of at most weakly non-minimum-phase agents (i.e. agents have all zeros in the closed left half plane). Under our solvability condition, we provide in this paper a design, utilizing H_∞ optimal control.

1. Introduction

Many researchers have been working on the synchronization problem of multi-agent systems (MAS) over the last decade. The area has spread from theoretical research to different applications, such as robot networks, sensor networks, power networks, social networks, and so on. The goal of synchronization is to secure asymptotic agreement on a common state (*state synchronization*) or output trajectory (*output synchronization*) among agents of the network through decentralized control protocols. Early work can, for instance, be found in Olfati-Saber and Murray (2004), Ren and Atkins (2007), Tuna (2008), Wu and Chua (1995) and Yang, Roy, Wan, and Saberi (2011) for the state synchronization that requires homogeneous MAS (i.e. agents are identical), and in Bai, Arcak, and Wen (2011), Chopra and Spong (2008a), Kim, Shim, and Seo (2011), Wieland, Sepulchre, and Allgöwer (2011), Xiang and Chen (2007) and Zhao, Hill, and Liu (2010) for the output synchronization of heterogeneous MAS.

For state synchronization, two kinds of communication networks are considered, one pertaining to full-state coupling and the other to partial-state coupling. We consider purely decentralized protocols which are non-introspective and do not share protocol states over the network. For MAS with full-state coupling, there is no restriction on the agent model. That is, synchronization can be always be achieved for stabilizable agents via purely decentralized protocols (e.g., Wang, Cheng, and Hu, 2008; Wieland, Kim, Scheu, and Allgöwer, 2008). For MAS with partial-state coupling, work has focused on conditions to achieve consensus by using static output-feedback protocols, for example (Ma & Zhang, 2010; Tuna, 2009; Xia & Scardovi, 2016a, 2016b). It is shown that only a restricted class of agents satisfy these solvability conditions. When using dynamic output-feedback protocols, synchronization can be achieved for a larger class of agents. It is well-known in the literature that the synchronization problem can be solved via solving a simultaneous stabilization problem. The solvability of these simultaneous stabilization problems requires a specific and stringent requirement on the right half plane (RHP) poles and zeros of the agent model, that is the simultaneous presence of RHP poles and zeros is not allowed. Therefore, in the literature on MAS with partial-state coupling and purely decentralized protocols, three kinds of restrictions can be observed.

Agents are at most weakly unstable In Seo, Shim, and Back (2009), general multi-input multi-output (MIMO) agents with all eigenvalues in the closed-left half plane are considered and a low-gain based protocol is proposed. This work is extended in Wang, Saberi, Stoorvogel, Grip, and Yang (2013) to tolerate time delay in the input for each agent. In Yang, Saberi, Stoorvogel, and Grip (2014), pre-compensators are developed to shape non-identical agents to almost identical agents with all eigenvalues in the closed-left half plane.

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Agents are at most weakly non-minimum-phase More specifically, Grip, Saberi, and Stoorvogel (2015) and Zhang, Saberi, Grip, and Stoorvogel (2015) require the agents to be minimum-phase. Chopra and Spong (2008a) require agents to be weakly minimum-phase, and Stoorvogel, Zhang, Saberi, and Grip (2014) and Stoorvogel, Saberi, and Zhang (2016) require agents to be weakly non-minimum-phase with the restriction that the Jordan blocks associated to zeros on the imaginary axis are at most size 2.

Agents are passive with positive definite storage function This implies, among other conditions, that the agents are neutrally stable and weakly minimum-phase. See Arcak (2007), Bai et al. (2011), Chopra and Spong (2008b), Yao, Guan, and Hill (2009) and Zhao et al. (2010) and references therein.

In this paper our contribution is twofold.

- For agents with single input, we give a necessary and sufficient condition for the solvability of state synchronization of a MAS.
- For agents with multiple inputs, we give a sufficient condition for the solvability of state synchronization of a MAS. The sufficient condition identifies two major classes of agents, for which the synchronization problem can be solved: at most weakly unstable agents (poles in the closed left half plane) and at most weakly non-minimum-phase agents (zeros in the closed left half plane).

Under the sufficient condition we provide a protocol design that utilizes H_∞ optimal control.

1.1. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. We denote by col$\{x_1,\ldots,x_N\}$, a column vector with $x_1,\ldots,x_N$ stacked together. $A\otimes B$ depicts the Kronecker product between A and B. I denotes the identity matrix and 0 denotes the zero matrix with their dimensions clear from the context.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V},\mathcal{E},\mathcal{A})$ where $\mathcal{V}=\{1,\ldots,N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighting matrix, and $a_{ij}>0$ iff $(i,j)\in\mathcal{E}$. Each pair in $\mathcal{E}$ is called an *edge*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1,\ldots,i_k\}$ such that $(i_j,i_{j+1})\in\mathcal{E}$ for $j=1,\ldots,k-1$. A *directed tree* with *root* r is a subset of nodes of the graph $\mathcal{G}$ such that a path exists between r and every other node in this subset. A *directed spanning tree* is a directed tree containing all the nodes of the graph. The set of all root agents for a graph $\mathcal{G}$ is denoted by $\Pi_{\mathcal{G}}$.

For a weighted graph $\mathcal{G}$, a matrix $L=[\ell_{ij}]$ with

$$\ell_{ij}=\begin{cases}\sum_{k=1}^{N}a_{ik}, & i=j,\\ -a_{ij}, & i\neq j,\end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. In the case where $\mathcal{G}$ has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector **1**.

The remainder of the paper is organized as follows. Section 2 describes the synchronization problem for MAS with partial-state coupling to be solved in this paper. Section 3 provides some preliminaries. Section 4 develops the solvability condition for the synchronization problem.

2. Problem formulation

Consider a MAS composed of N identical linear time-invariant continuous-time agents of the form,

$$\begin{aligned}\dot{x}_i &= Ax_i+Bu_i,\\ y_i &= Cx_i,\end{aligned}\qquad (i=1,\ldots,N) \tag{1}$$

where $x_i\in\mathbb{R}^n$, $u_i\in\mathbb{R}^m$, $y_i\in\mathbb{R}^p$ are respectively the state, input, and output vectors of agent i.

The communication network provides each agent with a linear combination of its own output relative to that of other neighboring agents. In particular, each agent $i\in\{1,\ldots,N\}$ has access to the quantity,

$$\zeta_i=\sum_{j=1}^{N}a_{ij}(y_i-y_j)=\sum_{j=1}^{N}\ell_{ij}y_j, \tag{2}$$

where $[a_{ij}]$ is the weighting matrix and $[\ell_{ij}]$ is the Laplacian matrix of the associated graph $\mathcal{G}$ with nodes corresponding to the agents in the network.

Definition 1. *Let $\mathbb{G}^N$ denote the set of directed graphs with N nodes that contain a directed spanning tree. For any given $\alpha\geq\beta>0$, the subset $\mathbb{G}^N_{\alpha,\beta}\subset\mathbb{G}^N$ requires additionally that the corresponding Laplacian matrix has the property that its nonzero eigenvalues have a real part larger than or equal to β and have an amplitude less than α.*

According to Ren and Beard (2005, Lemma 3.3), the Laplacian matrix L associated to any graph $\mathcal{G}\in\mathbb{G}^N_{\alpha,\beta}$ has a simple eigenvalue at the origin, with corresponding right eigenvector **1**. Let $\lambda_1,\ldots,\lambda_N$ denote the eigenvalues of L such that $\lambda_1=0$ and $\mathrm{Re}(\lambda_i)>0, i=2,\ldots,N$.

Since all the agents in the network are identical, we pursue state synchronization among agents. That is, we require

$$\lim_{t\to\infty}(x_i(t)-x_j(t))=0,\qquad \forall i,j\in\{1,\ldots,N\}. \tag{3}$$

We formulate the state synchronization problem for a network with identical agents as follows.

Problem 1. Consider a MAS described by (1) and (2). Let $\mathbf{G}$ be a given set of directed graphs such that $\mathbf{G}\subseteq\mathbb{G}^N$. The *state synchronization* problem with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear time-invariant dynamic protocol of the form,

$$\begin{cases}\dot{\chi}_i &= A_c\chi_i+B_c\zeta_i,\\ u_i &= C_c\chi_i+D_c\zeta_i,\end{cases} \tag{4}$$

for $i=1,\ldots,N$ where $\chi_i\in\mathbb{R}^{n_c}$, such that, for any graph $\mathcal{G}\in\mathbf{G}$ and for all initial conditions for the agents and their protocol, state synchronization among agents is achieved.

3. Preliminaries

As shown in Seo et al. (2009) and Yang et al. (2014), the state synchronization among agents in the network with partial-state coupling can be solved by equivalently solving a robust stabilization problem. The closed-loop of agent (1) and protocol (4) can be written as

$$\begin{cases}\dot{\bar{x}}_i &= \begin{pmatrix}A & BC_c\\ 0 & A_c\end{pmatrix}\bar{x}_i+\begin{pmatrix}BD_c\\ B_c\end{pmatrix}\zeta_i,\\ y_i &= \begin{pmatrix}C & 0\end{pmatrix}\bar{x}_i,\end{cases} \tag{5}$$

for $i=1,\ldots,N$ where

$$\bar{A}=\begin{pmatrix}A & BC_c\\ 0 & A_c\end{pmatrix},\quad \bar{B}=\begin{pmatrix}BD_c\\ B_c\end{pmatrix},\quad \bar{C}=\begin{pmatrix}C & 0\end{pmatrix}\text{ and }\bar{x}_i=\begin{pmatrix}x_i\\ \chi_i\end{pmatrix}.$$

Define $\bar{x} = \mathrm{col}\{\bar{x}_1, \ldots, \bar{x}_N\}$. Then, the overall dynamics of the N agents can be written as

$$\dot{\bar{x}} = (I_N \otimes \bar{A} + L \otimes \bar{B}\bar{C})\bar{x}. \tag{6}$$

Let $L = TJT^{-1}$, where J is the Jordan canonical form of the Laplacian matrix L such that the first diagonal entry, denoted by λ_0, is zero, and the other entries, denoted by λ_i ($i = 2, \ldots, N$), are non-zero. Moreover, the first column of T equals **1**. Then, we will make use of the following well-known lemma.

Lemma 1. *The MAS (6) achieves state synchronization if and only if the following $N-1$ subsystems*

$$\dot{\tilde{\eta}}_i = (\bar{A} + \lambda_i \bar{B}\bar{C})\tilde{\eta}_i, \quad i = 2, \ldots, N, \tag{7}$$

are globally asymptotically. Moreover, the synchronized trajectory is given by

$$\dot{\eta}_1 = \bar{A}\eta_1, \quad \eta_1(0) = (w \otimes I_{n+n_c})\bar{x}(0), \tag{8}$$

where $w = (w_1, \ldots, w_N)$ is the first row of T^{-1}, i.e. the normalized left eigenvector[1] *associated with the zero eigenvalue. This shows that the modes of the synchronized trajectory are determined by the eigenvalues of $\bar{A}$, which is the union of the eigenvalues of A and A_c. The complete dynamics depends on both $\bar{A}$ and a weighted average of the initial conditions of the agents and their protocols.*

Following Wu (2007, Lemma 2.29), one can conclude that $\eta_1(0)$ is only a linear combination of initial conditions of root agents and their protocol. Recall that the set of all root agents for a graph $\mathcal{G}$ is denoted by $\Pi_{\mathcal{G}}$. Then, the synchronized trajectory given by (8) can be written explicitly as

$$\eta_1(t) = e^{\bar{A}t} \sum_{i \in \Pi_{\mathcal{G}}} w_i \bar{x}_i(0), \tag{9}$$

which is the weighted average of the trajectories of root agents and their protocol. In the case of state synchronization via stable protocol, the synchronized trajectory is given by

$$\eta_1(t) = e^{At} \sum_{i \in \Pi_{\mathcal{G}}} w_i x_i(0). \tag{10}$$

In light of the definition of Problem 1 where synchronization is formulated for a set of graphs **G**, we basically obtain a *robust* stabilization problem, i.e. the stabilization of the system

$$\mathcal{S}: \begin{cases} \dot{x} = Ax + \lambda Bu \\ y = Cx \end{cases} \tag{11}$$

via a protocol

$$\mathcal{H}: \begin{cases} \dot{\chi} = A_c\chi + B_c y \\ u = C_c\chi + D_c y \end{cases} \tag{12}$$

for any λ which is a non-zero eigenvalue of a Laplacian matrices associated with a graph in the given set of graphs **G**.

4. Solvability conditions for synchronization

Let $P_0 \geq 0$ be the unique solution, see Willems (1971), of the Riccati equation,

$$A'P_0 + P_0A - P_0BB'P_0 = 0 \tag{13}$$

such that $A - BB'P_0$ has all eigenvalues in the closed left half plane. Moreover, let $Q_0 \geq 0$ be the solution of the linear matrix inequality, see Schumacher (1983),

$$G(Q_0) = \begin{pmatrix} AQ_0 + Q_0A' + BB' & Q_0C' \\ CQ_0 & 0 \end{pmatrix} \geq 0, \tag{14}$$

such that $\operatorname{rank} G(Q_0) = \rho$ where $\rho = \operatorname{normrank} C(sI - A)^{-1}B$ and

$$\operatorname{rank} \begin{pmatrix} sI - A & AQ_0 + Q_0A' + BB' & Q_0C' \\ -C & CQ_0 & 0 \end{pmatrix} = n + \rho$$

for all s in the open right half plane.

4.1. Necessary conditions for single-input agents

Theorem 1. *Consider a MAS described by (1) and (2) with a scalar input with (A, B) stabilizable and (C, A) detectable. Then, the problem of synchronization with partial state coupling as defined in Problem 1 is solvable for $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ for any $\alpha \geq \beta > 0$ only if $P_0Q_0 = 0$.*

Proof. We need to verify whether for any $\alpha \geq \beta > 0$ there exists a controller $\mathcal{H}$ of the form (12) for the system (11) such that the interconnection is stable for all λ with $\operatorname{Re}\lambda \geq \beta$ and $|\lambda| < \alpha$. We will weaken the above requirement by requiring that the interconnection is stable for any $\lambda \in \Lambda_\rho$ where

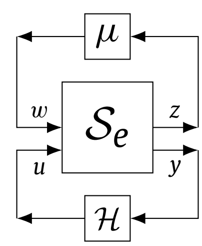
$$\Lambda_\rho := \left\{ \lambda \in \mathbb{C}^+ \;\middle|\; \lambda = \tfrac{\rho}{1+\rho} + \mu \text{ where } \mu \in \mathbb{C} \text{ such that } |1 - 2\rho\mu| \leq 1 \right\} \tag{15}$$

for any $0 < \rho < 1$. Note that any compact set contained in $\mathbb{C}^+$ will be in Λ_ρ for sufficiently small ρ. In particular note that for any $\alpha > \beta > 0$ there exist ρ_1 and ρ_2 such that:

$$\Lambda_{\rho_1} \subset \left\{ \lambda \in \mathbb{C}^+ \mid \operatorname{Re}\lambda \geq \beta, |\lambda| < \alpha \right\} \subset \Lambda_{\rho_2}.$$

Hence the problem is solvable for any α, β if and only if, for any $\rho > 0$, we can find a protocol such that the interconnection is stable for any $\lambda \in \Lambda_\rho$.

The interconnection of $\mathcal{S}$ and $\mathcal{H}$ can be written as



where

$$\mathcal{S}_e: \begin{cases} \dot{x} = Ax + \frac{\rho}{1+\rho}Bu + Bw, \\ y = Cx, \\ z = u. \end{cases} \tag{16}$$

Denote the transfer function of the interconnection of $\mathcal{S}_e$ and $\mathcal{H}$ (with input w and output z) by S_{cl}. We find that the interconnection of $\mathcal{S}$ and $\mathcal{H}$ is stable if the interconnection of $\mathcal{S}_e$ and $\mathcal{H}$ is internally stable and

$$1 - \mu S_{\mathrm{cl}}(s) \neq 0 \tag{17}$$

for all μ such that $|1 - 2\rho\mu| \leq 1$ and for all $s \in \mathbb{C}^+$. We note that

$$(1 - 2\rho\mu)(1 - 2\rho\mu^*) \leq 1 \iff \frac{1}{\mu} + \frac{1}{\mu^*} \geq 2\rho. \tag{18}$$

This implies that the interconnection satisfies (17) if and only if

$$S_{\mathrm{cl}}(s) + S^*_{\mathrm{cl}}(s) < 2\rho$$

for all $s \in \mathbb{C}^+$ which is equivalent to

$$(\rho - S_{\mathrm{cl}}(s) - 1)^* (\rho - S_{\mathrm{cl}}(s) - 1) < (\rho - S_{\mathrm{cl}}(s) + 1)^* (\rho - S_{\mathrm{cl}}(s) + 1)$$

for all $s \in \mathbb{C}^+$ which yields

$$\left\| (\rho - S_{\mathrm{cl}} + 1)^{-1} (\rho - S_{\mathrm{cl}} - 1) \right\|_\infty < 1.$$

[1] The left eigenvector is normalized in the sense of $\sum_{j=1}^N w_j = 1$.

Next, note that $q = -v + (\rho - S_{cl}(s))(v - q)$ yields that the transfer function from v to q is equal to

$$(\rho - S_{cl}(s) + 1)^{-1}(\rho - S_{cl}(s) - 1).$$

Therefore we can rewrite our original question as trying to find a stabilizing controller of the form (12) for the following system:

$$\bar{\mathcal{S}}_e : \begin{cases} \dot{x} = Ax + \frac{\rho}{1+\rho}Bu + \frac{1+\rho}{2}B(v-q), \\ y = Cx, \\ q = -\frac{2}{1+\rho}u + \rho(v-q) - v, \end{cases}$$

such that the closed loop transfer matrix from v to q has an H_∞ norm less than 1. This system $\bar{\mathcal{S}}_e$ can be rewritten as

$$\bar{\mathcal{S}}_e : \begin{cases} \dot{x} = Ax + Bu + Bv, \\ y = Cx, \\ q = -\frac{2}{(\rho+1)^2}u + \frac{\rho-1}{\rho+1}v. \end{cases} \tag{19}$$

In order to check whether we can make the H_∞ norm less than 1 we use the approach presented in Stoorvogel (1996). The state feedback Riccati equation is

$$A'P + PA - \nu PBB'P = 0,$$

such that $A - \nu BB'P$ has all its eigenvalues in the closed left half plane. Here $\nu = \rho(\rho^2 + \rho + 1)$. Note that $\nu \to 0$ as ρ goes to 0. If A has all eigenvalues in the closed left half plane, we find $P = 0$ and $P_0 = 0$. On the other hand, if A has at least one eigenvalue in the open right half plane then we have $P = \frac{1}{\nu}P_0$, where P_0 is given in Theorem 1.

Next, we consider the observer Riccati equation. Note that this is a singular problem and hence we need to consider the quadratic matrix inequality. We find that

$$G_\eta(Q) = \begin{pmatrix} AQ + QA' + \eta BB' & QC' \\ CQ & 0 \end{pmatrix} \geq 0,$$

such that $\operatorname{rank} G_\eta(Q) = \rho$ where $\rho = \operatorname{normrank} C(sI - A)^{-1}B$ and

$$\operatorname{rank}\begin{pmatrix} sI - A & AQ + QA' + \eta BB' & QC' \\ -C & CQ & 0 \end{pmatrix} = n + \rho$$

for all s in the open right half plane where $\eta = \frac{(\rho+1)^2}{4\rho}$. Note that η goes to infinity when ρ goes to 0. If (A, B, C) is at most weakly non-minimum-phase, we find $Q = 0$ and $Q_0 = 0$. Otherwise, if A has an invariant zero in the open right half plane then we have $Q = \eta Q_0$, where Q_0 is given in Theorem 1.

Therefore we note that the state feedback and observer Riccati equations always have a solution for $\rho < 1$. However, there is a third condition $\lambda_{\max}(PQ) < 1$, where $\lambda_{\max}$ denotes the spectral radius. We have

$$PQ = \frac{\eta}{\nu}P_0Q_0.$$

Since η/ν goes to infinity as ρ increases to 1 we note that the only case where we can find a solution for all $\rho < 1$ is the case where $P_0Q_0 = 0$. The final condition from Stoorvogel (1996) is that for any $s_0 \in \mathbb{C}^0$ which is an invariant zero of either $(A, B, C, 0)$ or $(A, B, 0, I)$ there should exist K such that

$$\| -\tfrac{2}{(\rho+1)^2}KC(s_0I - A - BKC)^{-1}B + \tfrac{\rho-1}{\rho+1}I\| < 1.$$

It is easy to check that it is sufficient to prove that we can find K such that

$$\|KC(s_0I - A - BKC)^{-1}B\| \leq 1.$$

If $s_0I - A$ is invertible, we can choose $K = 0$. Otherwise, we choose S and T such that:

$$s_0I - A = S\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}T, \quad B = S\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}T.$$

Stabilizability of (A, B) guarantees that B_2 is surjective while detectability of (C, A) guarantees that C_2 is injective. We denote by B_2^r and C_2^ℓ the right- and left-inverses of B_2 and C_2 respectively. We then choose $K = -B_2^rC_2^\ell$. We obtain:

$$\begin{aligned} &KC(s_0I - A - BKC)^{-1}B \\ &= B_2^r\begin{pmatrix} C_2^\ell C_1 & I \end{pmatrix}\begin{pmatrix} I + B_1B_2^rC_2^\ell C_1 & B_1B_2^r \\ C_2^\ell C_1 & I \end{pmatrix}^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ &= B_2^r\begin{pmatrix} C_2^\ell C_1 & I \end{pmatrix}\begin{pmatrix} I & -B_1B_2^r \\ -C_2^\ell C_1 & I + C_2^\ell C_1B_1B_2^r \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ &= B_2^r\begin{pmatrix} 0 & I \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = I \end{aligned}$$

which completes the verification of the necessary conditions from Stoorvogel (1996). ■

4.2. Sufficient conditions for general agents and design

The sufficient condition for partial-state synchronization is stated as follows:

Theorem 2. *Consider a (possibly multi-input) MAS described by (1) and (2) where (A, B) is stabilizable and (C, A) is detectable. Then, the problem of synchronization with partial-state coupling as defined in Problem 1 is solvable for $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ for any $\alpha > \beta > 0$ if $P_0Q_0 = 0$.*

Proof. We rely on arguments from the proof of Theorem 1 with some modifications to make the argument valid for multi-input systems.

We first choose ρ such that any λ with $\operatorname{Re}\lambda \geq \beta$ and $|\lambda| \leq \alpha$ is contained in Λ_ρ which is defined in (15). Our objective is to design a controller $\mathcal{H}$ of the form (12) which stabilizes (11) for any $\lambda \in \Lambda_\rho$. We again define the system $\mathcal{S}_e$ by (16) and we need to ensure that the interconnection of $\mathcal{S}_e$ and $\mathcal{H}$ is internally stable and

$$\det(I - \mu S_{cl}(s)) \neq 0 \tag{20}$$

for all μ such that $|1 - 2\rho\mu| \leq 1$ and for all $s \in \mathbb{C}^+$. It is sufficient if we achieve:

$$S_{cl}(s) + S_{cl}^*(s) < 2\rho I. \tag{21}$$

After all if (20) is not satisfied then there exists a vector $v \in \mathbb{C}^n$ such that $\mu S_{cl}(s)v = v$ which implies

$$\mu v^*S_{cl}(s)v = v^*v$$

and hence:

$$v^*S_{cl}(s)v + v^*S_{cl}^*(s)v = \left(\frac{1}{\mu} + \frac{1}{\mu^*}\right) > 2\rho v^*v$$

using (18) which yields a contradiction with (21).

Note that in Theorem 1 we developed necessary conditions and used that (20) and (21) are identical for scalar input systems. Here we want sufficient conditions for solvability and we can rely on the fact that for multi-input systems (21) implies (20). Using arguments similar as before, it is sufficient that we find a controller $\mathcal{H}$ which stabilizes the system $\bar{\mathcal{S}}_e$ given by (19) and yields an H_∞ norm from v to q less than 1. We can use the results from Stoorvogel (1996) to design such a controller. ■

The combination of Theorems 1 and 2 results in a necessary and sufficient condition for the single-input case:

Corollary 1. *Consider a MAS described by (1) and (2) with a single input with (A, B) stabilizable and (C, A) detectable. Then, the problem of synchronization with partial state coupling as defined in Problem 1 is solvable for $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ for any $\alpha > \beta > 0$ if only if $P_0Q_0 = 0$.*

The class of MAS that satisfies our sufficient condition contains two cases: At most weakly unstable agents, i.e., the eigenvalues of A are in the closed left half plane and at most weakly non-minimum-phase agents, i.e., the invariant zeros of (A, B, C) are in the closed left half plane. It is easy to verify that the first case is equivalent to the condition $P_0 = 0$ while the second case is equivalent to the condition $Q_0 = 0$. Outside of these two cases, agents could be exponentially unstable and strictly non-minimum-phase. In that case, both $P_0 \neq 0$ and $Q_0 \neq 0$, but the condition $P_0Q_0 = 0$ can still be satisfied and therefore the state synchronization problem could still be solved, even though this will rarely occur.

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