On Greedy and Submodular Matrices

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Abstract. We characterize non-negative *greedy matrices*, *i.e.*, $(0,1)$-matrices A such that the problem $\max\{c^T x \mid Ax \leq b,\ x \geq 0\}$ can be solved greedily. We identify so-called *submodular matrices* as a special subclass of greedy matrices. Finally, we extend the notion of greediness to $\{-1,0,1\}$-matrices. We present numerous applications of these concepts.

Keywords: Submodularity, linear programming, max flow

1 Introduction

Discrete optimization problems can often be formulated as linear programs of type

$$\max\{c^T x \mid a \leq Ax \leq b,\ x \geq 0\} \tag{1}$$

with constraint vectors $a, b \in \mathbb{R}^m$, a cost vector $c \in \mathbb{R}^n_+$, and a matrix A with coefficients in $\{-1,0,1\}$. Having ordered the columns so that $c_1 \geq \ldots \geq c_n \geq 0$ holds, one of the most natural approaches to solve (1) is the *greedy algorithm*, which starts with $x = 0$ (if feasible) and subsequently increases in each step the variable x_j with the lowest possible index j until one of the constraints gets tight. If this procedure eventually comes to an end, the resulting final $\overline{x} \in \mathbb{R}^n_+$ is called the *greedy solution* of (1). To ensure that the initial solution $x = 0$ is always feasible, we assume that $a \leq 0 \leq b$. We say that A is *greedy* if the greedy algorithm applied to (1)

(G1) increases $x_1, \ldots, x_n$ each at most once without ever stepping back
(G2) the resulting solution $\overline{x}$ is optimal

for any choice of $a \leq 0 \leq b$ and $c_1 \geq \ldots \geq c_n \geq 0$.

In this paper, we seek to determine greedy $(-1,0,1)$-matrices. Of particular interest is the case of the all one vector $c = 1$ in the LP (1). We call a matrix $A \in \{-1,0,1\}^{m\times n}$ 1-*greedy* if

$$\max\ \{1^T x \mid a \leq Ax \leq b,\ x \geq 0\} \tag{2}$$

can be solved greedily for any $a \leq 0 \leq b$.

In order to identify characterizing or, at least, sufficient conditions for a matrix to be greedy, we first restrict our considerations to binary matrices (*i.e.*, $A \in \{0,1\}^{m\times n}$) in Sections 2 and 3, before we turn to the more general case with possibly negative matrix entries in Section 4.

Let us take a closer look at binary matrices and note that the linear programs (1) and (2), as well as the description of the greedy-algorithm become considerably easier: In the case $A \in \{0,1\}^{m\times n}$, we may assume $a = 0$ (recall that we required $a \leq 0$). Furthermore, we observe that property $(G1)$ is trivially satisfied whenever A has only $(0,1)$-entries.

It follows that the greedy algorithm for binary matrices can be described as follows: Start with $x = 0$ and then raise x_1 until one of the constraints becomes tight, then raise x_2, etc.[4]

1.1 Our contribution and related results

Our contribution goes in two directions. We answer an open question of [3] by characterizing greedy binary matrices in Section 2. Furthermore, we provide the "missing link" between the stream of research on greedy matrices (see, *e.g.*, [3], [4], [5]) and submodular optimization (such as [6], [7],[8]) by introducing the concept of a *submodular matrix*, which turns out to be a special kind of greedy matrix (Section 3). Max flow in (s,t)-planar graphs (with supermodular weights) can easily be seen to fit in our model, as well as Frank's very general model of greedily solvable linear programs [6]. Frank's model itself covers various discrete optimization structures such as polymatroids, supermodular systems, or cut packings. In contrast to previous models, our condition relies only on the ordering of the columns of A and does not necessarily need a lattice structure on the columns. In particular, we do not require the matrix to be "consecutive" in any sense.

In Section 4, we open our model to ternary matrices and introduce the concept of *ordered compatibility*, which ensures that the greedy algorithm never steps backward (property $(G1)$). It will turn out that the max-flow problem in general graphs, as well as Gröflin and Hoffman's ternary lattice polyhedra [10] fit into this model.

As a consequence of ordered compatibility, we show that the greedy algorithm solves the max flow problem optimally as long as the paths are ordered in an appropriate way (for example, *via* a simple "left/right"-relation, or by non-increasing path-lengths).

To give some intuition on our greedy algorithm in both the binary and ternary model, let us consider the max flow problem (with and without weights on the paths).

1.2 (Weighted) max flow

Let $G = (V,E)$ be a (directed or undirected) graph with source and sink node $s,t \in V$, and let $\mathscr{P} \subseteq 2^E$ denote the collection of all simple (s,t)-paths in G (if G is directed, $\mathscr{P}$ consists of all directed paths). If $A \in \{0,1\}^{|E|\times|\mathscr{P}|}$ is the edge-path incidence matrix (*i.e.*, A has entries $a_{eP} = 1$ iff $e \in P$), and $b \in \mathbb{R}_+^{|E|}$ encodes certain edge capacities, then (2) reduces to the classical max flow problem on G, and (1) reduces to a max flow problem on G with certain weights $c(P)$ on the paths $P \in \mathscr{P}$. Several efficient max flow algorithms exist for the unweighted case in general graphs (see, *e.g.*, [12]). For the special case of (s,t)-planar graphs, already Ford and Fulkerson [9] have shown

[4] The *greedy solution* $\bar{x}$ constructed this way is the lexicographically maximal feasible solution.

that the simple greedy strategy of iteratively sending as much flow as possible along the uppermost path in the residual graph works well also for path weights c that are in a sense supermodular. Borradaile and Klein [1] proved that an extension of Ford and Fulkerson's uppermost path algorithm yields the optimum flow (in time $\mathscr{O}(n\log n)$) also on planar graphs that are not necessary (s,t)-planar if no path weights are given (see also [13]). They make use of a lattice structure on the paths induced by the so-called "left/right"-relation (defined below).

For directed graphs, we obtain more structure when we formulate the max flow problem as an LP on a ternary matrix (*i.e.*, with coefficients in $\{-1,0,+1\}$). In this case, we let $\mathscr{P}$ consist of all (directed or undirected) simple (s,t)-paths and consider the corresponding edge-path incidence matrix $A \in \{-1,0,1\}^{|E|\times|\mathscr{P}|}$ with coefficients $a_{eP} = 1$ resp. -1 if P traverses e in forward resp. backward direction, and $a_{eP} = 0$ otherwise. It turns out that the well-known *successive shortest path algorithm* [12] corresponds to our greedy algorithm described above if the columns of A are ordered by non-increasing path-lengths (see Section 4).

2 Binary greedy matrices

We first restrict ourselves to binary matrices and consider linear programs of type

$$\max\ \{c^T x \mid Ax \le b,\ x \ge 0\} \tag{3}$$

with $A \in \{0,1\}^{m\times n}$, $c_1 \ge \ldots \ge c_n \ge 0$ and $b \ge 0$.

We are interested in binary greedy matrices, *i.e.*, $\{0,1\}$-matrices A that guarantee (3) to be greedily solvable for any $c_1 \ge \ldots \ge c_n \ge 0$ and $b \ge 0$ by starting with $x = 0$ and raising the variable x_j in iteration j until one of the constraints becomes tight (for all $j = 1,\ldots,n$).

As mentioned in the Introduction, the problem of characterizing greedy matrices can be reduced to characterizing 1-greedy matrices. Let A_j denote the j-th column of matrix A.

Proposition 1. *A is greedy* $\iff$ *each initial segment* $[A_1,\ldots,A_j]$ *is* 1*-greedy.*

Proof. Write $c \in \mathbb{R}^n$ with $c_1 \ge \ldots \ge c_n \ge 0$ as a conic combination of vectors $(1^T,0^T)$.

So we aim at characterizing 1-greedy matrices in the following. (In [3], another characterization of 1-greedy matrices is derived, which we present below.)

To start with, it is not difficult to obtain sufficient conditions for 1-greediness. For example, it suffices to exclude

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

as submatrices (*cf.* [4]). We will refer to these two 2×3 matrices as the 2×3 *non-greedy matrices*. If A contains a non-greedy 2×3 submatrix A_{IJ}, then we will always assume that $I = \{i_1,i_2\}$ and $J = \{j_0,j_1,j_2\}$ with $j_0 < j_1 < j_2$. (As usual, A_{IJ} denotes the submatrix arising from A by deleting all rows with indices not in $I \subseteq \{1,\ldots,m\}$, and all columns with indices not in $J \subseteq \{1,\ldots,n\}$.)

The mere existence of a non-greedy 2×3 submatrix A_{IJ} is not necessary harmful: For example, if

$$A_j < A_{j_0} + A_{j_1} + A_{j_2} \quad \text{for some } j < j_0 \tag{4}$$

holds, then the greedy algorithm will tighten one of the constraints $i \in \operatorname{supp}(A_j)$ as soon as it raises x_j (or even earlier). (Here, and in the following, the "$\leq$"-relation between two vectors denotes the componentwise "$\leq$"-relation.) As a consequence, even before it reaches x_{j_0} at least one of the variables x_{j_0}, x_{j_1} or x_{j_2} is bound to zero and the greedy algorithm will thus proceed as if A_{IJ} were not there (*cf.* the proof of Theorem 1 below for a rigorous argument).

We therefore call a non-greedy (2×3)-submatrix A_{IJ} *uncritical* if (4) holds and *critical* otherwise. The following result tells us when a critical A_{IJ} destroys the 1-greediness of A and when it does not.

Theorem 1. *A is 1-greedy iff for every critical A_{IJ} there exists $j > j_0$ such that*

$$A_{j_0} + A_j \leq \max\{A_{j_0}, A_{j_1} + A_{j_2}\} \tag{5}$$

holds. (The maximum is taken componentwise.)

Proof. "$\Rightarrow$": Assume A is 1-greedy and A_{IJ} is a critical submatrix. Consider (3) with $b := \max\{A_{j_0}, A_{j_1} + A_{j_2}\}$. The greedy solution $\bar{x}$ has $\bar{x}_1 = \ldots = \bar{x}_{j_0-1} = 0$ (as A_{IJ} is critical) and, obviously, $\bar{x}_{j_0} = 1$. Thus, the greedy solution can only maximize $1^T x$ if it also raises some variable x_j with $j > j_0$ and $A_{j_0} + A_j \leq \max\{A_{j_0}, A_{j_1} + A_{j_2}\}$.

"$\Leftarrow$": Assume that $A \in \{0,1\}^{m\times n}$ satisfies the condition and let $b \geq 0$. We are to show that the greedy solution $\bar{x}$ of (2) is optimal. If $\bar{x} = 0$, then $x = 0$ is the unique feasible solution and hence trivially optimal. Otherwise, let $k \leq n$ be the last index with $\bar{x}_k > 0$. For $j \in \{1,\ldots,n\}$, let $T_j \subseteq \{1,\ldots,m\}$ denote the set of constraints that became tight when raising the jth component to $\bar{x}_j > 0$. Let $T = T_k$ and $T_<$ be the (disjoint) union of all T_j with $j \in \operatorname{supp}(\bar{x})$ and $j < k$. Furthermore, let $U := \{1,\ldots,m\} \setminus (T \cup T_<)$. We concentrate on those A_j, $j > k$ that have $\operatorname{supp}(A_j) \subseteq U \cup T$.

We first show that among all such A_j, there exists a unique one with $\operatorname{supp}(A_j) \cap T$ inclusion-wise minimal. If not, we could choose two such columns, say A_{j_1} and A_{j_2}, with both $\operatorname{supp}(A_{j_1}) \cap T$ and $\operatorname{supp}(A_{j_2}) \cap T$ inclusion-wise minimal. Then with $j_0 = k$, there is a (critical!) submatrix A_{IJ}. Let $j > j_0$ as in the condition of Theorem 1, *i.e.*, such that property (5) holds. In particular, for $i \in T$ (implying $a_{ij_0} = 1$) we find that

$$a_{ij_1} = 0 \quad \Longrightarrow \quad a_{ij} = 0.$$

Together with $a_{ij} = 0$ for $i \in \{i_1, i_2\}$, we thus conclude

$$\operatorname{supp}(A_j) \cap T \subset \operatorname{supp}(A_{j_1}) \cap T,$$

which contradicts the choice of A_{j_1}. Hence this case cannnot occur and we know that among all A_j with $j > k$ and $\operatorname{supp}(A_j) \subseteq U \cup T$, there exists a unique one, say A_{j^*}

with $\text{supp}(A_{j^*}) \cap T$ inclusion-wise minimal. We show by induction on k that the greedy solution is optimal:

Choose any $i \in \text{supp}(A_{j^*}) \cap T$ and decrease b_i by $\varepsilon = \bar{x}_k > 0$. The greedy solution for this modified LP would differ from $\bar{x}$ only in the kth component, which is now set to zero. (Note that raising $\bar{x}_j$ for $j > k$ is impossible for any j: If $\text{supp}(A_j) \cap T_< \neq \emptyset$, this is clear anyway, and if $\text{supp}(A_j) \subseteq U \cup T$, then $i \in \text{supp}(A_{j^*}) \cap T \subseteq \text{supp}(A_j) \cap T$ by our assumption, which prevents us from raising $\bar{x}_j$.) By induction, the new greedy solution for this modified LP is optimal. But then also $\bar{x}$ must have been optimal (w.r.t. the right hand side b), since increasing a single b_i by ε can never increase the objective value by more than ε.

A similar condition was established in[3]:

Theorem 2 ([3]). *$A \in \{0,1\}^{m\times n}$ is 1-greedy iff for every critical A_{IJ} there exists $j > j_0$ such that*

$$a_{ij} = 0 \ \text{ if } i \in I, \text{ and } a_{ij} \leq a_{ij_1} + a_{ij_2} \text{ otherwise.} \tag{6}$$

◇

Condition (6) follows easily from (5). So our condition appears to be stronger. The converse implication (6) ⇒ (5) is less obvious. We have a slight preference for (5), due to its formal similarity with the submodularity concept introduced below.

As a straightforward corollary we observe:

Theorem 3. *The matrix $A \in \{0,1\}^{m\times n}$ is greedy iff for all critical A_{IJ} there exists j with $j_0 < j \leq j_2$ such that*

$$A_{j_0} + A_j \leq \max\{A_{j_0}, A_{j_1} + A_{j_2}\}.$$

Proof. Theorem 1 and Proposition 1.

3 Submodular matrices

A particularly simple class of greedy matrices which we encounter in many applications is provided by the class of so-called submodular matrices as defined below.

Definition 1 (Submodular pair/matrix.). *Relative to a given $A \in \{0,1\}^{m\times n}$, a pair (j,k) of column indices is* submodular *if there exist column indices $j \wedge k < j,k < j \vee k$ such that*

$$A_{j\wedge k} + A_{j\vee k} \leq A_j + A_k \tag{7}$$

holds. The matrix A is submodular *if for any critical submatrix A_{IJ} the pair (j_1, j_2) is submodular.*

Remarks: (1) In practice, the indices $j \wedge k$ and $j \vee k$ are usually unique for each submodular pair (j,k). We do not require any uniqueness here, but assume that indices $j \wedge k$ and $j \vee k$ are somehow fixed for any submodular pair (j,k).

(2) To show that a given matrix A is submodular, it suffices to verify that for each (not necessarily critical) non-greedy 2×3 submatrix A_{IJ} at least one of the three pairs $(j_0, j_1), (j_0, j_2)$ and (j_1, j_2) is submodular: Indeed, if either (j_0, j_1) or (j_0, j_2) is submodular, then A cannot be critical.

Relative to a given submodular $A \in \{0,1\}^{m \times n}$, we call $c \in \mathbb{R}^n$ *supermodular* if

$$c_{j \wedge k} + c_{j \vee k} \geq c_j + c_k$$

holds for any submodular pair (j,k). For example, the constant vector $c = 1$ is always supermodular. So the following Theorem says in particular that submodular matrices are greedy:

Theorem 4. *If $A \in \{0,1\}^{m \times n}$ is submodular and $c \in \mathbb{R}^n_+$ is monotone decreasing (i.e., $c_1 \geq \ldots \geq c_n$) and supermodular, then*

$$\max\{c^T x \mid Ax \leq b,\ x \geq 0\}$$

can be solved greedily.

Proof. Let $\bar{x}$ denote the greedy solution and let x^* be the (unique) lexicographically maximal optimal solution. Assume that $\bar{x} \neq x^*$ and let j_0 be the smallest index with $\bar{x}_{j_0} \neq x^*_{j_0}$. Then $\bar{x}_{j_0} > x^*_{j_0}$ must hold (as $\bar{x}$ is the lexicographically maximal feasible solution). As c is monotone decreasing, increasing $x^*_{j_0}$ to $\bar{x}_{j_0}$ must be compensated by decreasing x^* on at least two further indices $j_1, j_2 > j_0$ (in order to stay feasible) corresponding to some non-greedy (2×3)-submatrix A_{IJ} of A. We claim that A_{IJ} is critical. Indeed, assume to the contrary that there exists $j < j_0$ with $\text{supp}(A_j) \subseteq \text{supp}(A_{j_0}) \cup \text{supp}(A_{j_1}) \cup \text{supp}(A_{j_2})$. Then the greedy algorithm would have tightened some constraint $i \in \text{supp}(A_{j_0}) \cup \text{supp}(A_{j_1}) \cup \text{supp}(A_{j_2})$ when raising x_j or even before, so that certainly there cannot be any feasible solution x which coincides with $\bar{x}$ in components $1, \ldots, j_0 - 1$ and is strictly positive in components j_0, j_1 and j_2. But $x = \frac{1}{2}(\bar{x} + x^*)$ has these properties, a contradiction. Hence submodularity of A implies that (j_1, j_2) is submodular.

But then x^* could be increased on $j_1 \wedge j_2$ and $j_1 \vee j_2$, and decreased on j_1 and j_2, giving rise to another feasible solution, which is lexicographically larger and has an objective value larger than or equal to that of x^*, contradicting the choice of x^*, and completing the proof.

3.1 Example: max flow in (s,t)-planar graphs

Let $G = (V, E)$ with $s, t \in V$ be a (directed or undirected) graph given in a planar embedding with s, t on the outer boundary (*i.e.*, G is a so-called (s,t)-planar graph). Let $\mathscr{P} = \{P_1, \ldots, P_m\}$ denote the collection of all (s,t)-paths in G, ordered from the leftmost to the rightmost path (the "leftmost" path is uniquely constructed by starting at s and always traversing the leftmost (directed) edge), and consider the edge-path incidence matrix $A \in \{0,1\}^{|E| \times |\mathscr{P}|}$. We claim that A is submodular. Indeed, as mentioned in the above Remark, it suffices to show that for any non-greedy (2×3)-submatrix A_{IJ}

at least one of the three pairs $(j_0, j_1), (j_0, j_2)$ and (j_1, j_2) is submodular. Thus, assume that A_{IJ} is such a non-greedy submatrix with $I = \{e_1, e_2\}$. Assume that, say, the path P_{j_1} contains e_1 (but not e_2) and that P_{j_2} contains e_2 (but not e_1). Any two (s,t)-paths form a submodular pair unless one is "to the left" of the other. Thus if none of the three pairs is submodular, then P_{j_0} is left of P_{j_1} and P_{j_1} is left of P_{j_2}. But then, due to planarity, P_{j_1} being in between P_{j_0} and P_{j_2} must also pass through e_2, a contradiction.

3.2 Example: Frank's model [6]

A very far-reaching generalization of Edmonds' polymatroids as well as several other classes of greedily solvable linear programs is provided by Frank's model [6]:

Interpret the $\{0,1\}$-matrix A as the incidence matrix of a (multi-) set family $\mathscr{F} \subseteq 2^E$, *i.e.*, $A \in \{0,1\}^{|E| \times |\mathscr{F}|}$ has entries $a_{eF} = 1$ if $e \in F$ and $a_{eF} = 0$ otherwise. Frank assumes the set family $\mathscr{F}$ to be endowed with some partial order $(\mathscr{F}, \preceq)$. A pair $\{S,T\} \subseteq \mathscr{F}$ is called *intersecting* if there exists some $C \in \mathscr{F}$ with $C \prec S,T$. Two binary operations "$\wedge$" and "$\vee$" are defined on all comparable and intersecting pairs and assume to satisfy

(P1) if $S \preceq T$ then $S \wedge T = S$ and $S \vee T = T$;
(P2) if S,T intersecting, then $S \wedge T \prec S,T \prec S \vee T$.

A function $c \in \mathbb{R}_+^{\mathscr{F}}$ is called *intersecting supermodular* if

$$c(S) + c(T) \leq c(S \wedge T) + c(S \vee T)$$

holds for every intersecting pair $S,T \in \mathscr{F}$ with $c(S), c(T) > 0$. Moreover, c is called *decreasing* if

$$S \preceq T \quad \Longrightarrow \quad c(S) \geq c(T) \quad \forall S,T \in \mathscr{F}.$$

Frank proved that $\max\{c^T x \mid Ax \leq b,\ x \geq 0\}$ can be solved greedily for any intersecting supermodular decreasing function $c \in \mathbb{R}_+^{\mathscr{F}}$ and every $b \in \mathbb{R}_+^E$ if the set system $(\mathscr{F}, \preceq)$ satisfies for all $S,T,U \in \mathscr{F}$:

(P3) if $S \preceq T \preceq U$, then $S \cap U \subseteq T$;
(P4) if S,T are intersecting, then $(S \wedge T) \cup (S \vee T) \subseteq S \cup T$;
(P5) if $S \cap T \neq \emptyset$, then S,T are either intersecting or comparable.

Frank's result follows from Theorem 4. Indeed, order the columns of A according to a linear extension (also known as "topological sorting") of $(\mathscr{F}, \preceq)$ such that $c_1 \geq \ldots \geq c_{|\mathscr{F}|}$ (which is possible as c is decreasing on $(\mathscr{F}, \preceq)$). Now it suffices to prove that A is a submodular matrix:

Let A_{IJ} be a non-greedy submatrix with $I = \{e_1, e_2\}$ and $J = \{F_0, F_1, F_2\}$. Then $F_0 \cap F_1 \neq \emptyset \neq F_0 \cap F_2$. Thus, by property $(P5)$, the pairs $\{F_0, F_1\}$ and $\{F_0, F_2\}$ are either intersecting or comparable. If one of the pairs is intersecting, it is submodular by $(P2)$ and we are done. Else both pairs are comparable, *i.e.*, $F_0 \prec F_1, F_2$, and hence $F_1 \wedge F_2$ exists. Hence, A is submodular unless F_1 and F_2 are comparable. But then $F_0 \prec F_1 \prec F_2$ in contradiction to property $(P3)$.

4 Ternary matrices

Some combinatorial optimization problems allow (or even ask for) an LP-formulation with ternary constraint matrix. Recall from the Introduction that the greedy algorithm for

$$\max\{1^T x \mid a \le Ax \le b,\ x \ge 0\} \tag{8}$$

with $A \in \{-1,0,1\}^{m\times n}$ and $a \le 0 \le b$ starts at $x = 0$ and increases the variable of lowest possible index in each iteration until one of the constraints becomes tight. A ternary matrix is 1-*greedy* if the greedy algorithm never steps backward (property $(G1)$) and the resulting greedy solution $\bar{x}$ is optimal (property $(G2)$).

We first need some notation. As usual, we split any $v \in \mathbb{R}^n$ into its positive and negative part $v^+ \in \mathbb{R}^n$ resp. $v^- \in \mathbb{R}^n$, where

$$v_i^+ := \max\{v_i, 0\} \quad \text{and} \quad v_i^- := |\min\{v_i, 0\}|.$$

Thus, $v = v^+ - v^-$ holds for all $v \in \mathbb{R}^n$. We write $v \preceq w$ if $v^+ \le w^+$ and $v^- \le w^-$. Two vectors v and w are said to be *compatible* if

$$(\text{supp } v^+ \cap \text{supp } w^-) \cup (\text{supp } v^- \cap \text{supp } w^+) = \emptyset.$$

Definition 2 (Compatible solution). *A feasible solution x of (8) is* compatible *if the columns A_j, $j \in \text{supp}(x)$, are pairwise compatible. The linear program (8) is* compatible *if it has a compatible optimal solution.*

Definition 3 (Compatible matrix). *The matrix $A \in \{-1,0,1\}^{m\times n}$ is* compatible *if for any two non-compatible columns $j < k$ there exist two column indices $j \wedge k < j \vee k$ such that $A_{j\wedge k}$ and $A_{j\vee k}$ are compatible and*

$$A_{j\wedge k} + A_{j\vee k} \preceq A_j + A_k$$

holds (implying that $A_{j\wedge k}, A_{j\vee k} \preceq A_j, A_k$).

Proposition 2. *If A is compatible then so is the linear program (8).*

Proof. Let x^* be optimal for (8). If x^* is incompatible, say $\varepsilon := \min\{x_j^*, x_k^*\} > 0$ for some incompatible pair of columns A_j and A_k, then increasing $x_{j\wedge k}^*$ and $x_{j\vee k}^*$ by ε, and decreasing x_j^* and x_k^* by ε does not create any new incompatibilities so that, after a number of such modifications, a compatible optimum x^* is reached.

Definition 4 (Ordered compatible.). *We say that $A \in \{-1,0,1\}^{m\times n}$ is* ordered compatible *if, in addition, the column index $j \wedge k$ satisfies $j \wedge k < k$.*

Remark: As we did in the $(0,1)$-case, we assume throughout that some suitable indices $j \wedge k < j \vee k$ are fixed. The above ordered compatibility condition is weaker than requiring *submodularity* in the sense that

$$j \wedge k < j, k < j \vee k$$

should hold for each non-compatible pair (j,k).

4.1 Example: edge-path incidence matrices in general graphs

Incidence matrices of (s,t)-paths (appropriately ordered) are ordered compatible: Indeed, let $D = (V,E)$ be a digraph with source s and sink t. Assume *w.l.o.g.* that s and t have both degree 1. For each vertex i choose a cyclic ordering π_i of the edges incident to i. The π_i's induce a ordering on the set $\mathscr{P}$ of (s,t)-paths in a natural way:

For example, if D is planar, we may chose each π_i to be the clockwise ordering of the edges around i, which induces the canonical "left to right" ordering on $\mathscr{P}$, starting with the leftmost path and ending with the rightmost path from s to t.

For (s,t)-planar graphs, the corresponding path incidence matrix is even submodular, which explains why flow is never reduced during the augmentation and non-directed paths may be disregarded completely. For other graphs only ordered compatibility can be deduced (see Figure 1 for the planar case).

Proposition 3. *Any (s,t)-path incidence matrix with the path order induced by cyclic orderings on the edges around each vertex is ordered compatible.*

Proof. As above, we assume that s and t have both degree 1. Let $P_1,\ldots,P_r$ be the ordering of the $s-t$ paths induced by cyclic orders π_i on the edges incident with vertex i. Consider two paths P_j and P_k and let P denote the maximal initial subpath contained in both P_j and P_k. Let e denote the last edge in P and let e_j, e_k denote the edges succeeding e on P_j *resp.* P_k. Let i denote the vertex in which P_j and P_k split. Then $j < k$ if and only if $\pi_i = (\ldots,e,\ldots,e_j,\ldots,e_k,\ldots)$. (Note that existence of e is guaranteed by our assumption that s has degree 1.)

Now assume that $P_j^+ \cap P_k^- \neq \emptyset$. Consider $F = P_j + P_k$ (as sum of two vectors in $\mathbb{R}^n$). After removing directed cycles from F (in case there are any), the resulting 2-flow decomposes into $P_{j\wedge k}$ and $P_{j\vee k}$, both following P until the last edge e and then splitting into e_j *resp.* e_k. So $P_{j\wedge k}$ (following e_j) has a smaller index than P_k (following e_k).

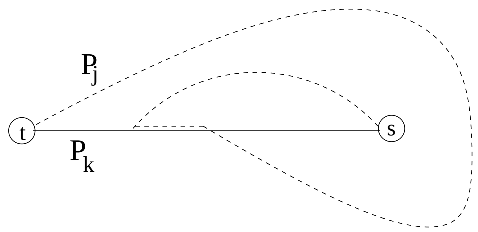

Fig. 1. Two non-compatible (s,t)-paths in a planar graph.

An alternative compatible ordering of $\mathscr{P}$ can be obtained by ordering the paths according to non-increasing length. The straightforward proof is left to the reader.

4.2 Example: Lattice polyhedra [10]

The matrices in lattice polyhedra theory as defined by Gröflin and Hoffman [10] are not only ordered compatible but satisfy the stronger submodularity condition. (These matrices are also called *submodular* in [11]). These associated polyhedra are of type

$$\{x \in \mathbb{R}^E \mid e \leq x \leq d,\ Ax \leq r\}$$

and based on some ternary matrix $A \in \{-1,0,1\}^{L\times E}$ whose row set L forms a lattice $(L \preceq, \wedge, \vee)$ relative to which r is submodular, $e,d \in \mathbb{R}^E_+$, and each column f of A is supermodular on L, and satisfies the consecutivity conditions

$$|f(j) - f(k)| \leq 1 \quad \forall j,k \in L \text{ with } j \preceq k$$
$$|f(j) - f(k) + f(l)| \leq 1 \quad \forall j,k,l \in L \text{ with } j \preceq k \preceq l.$$

(The consecutivity conditions ensure that on any chain in L, a column f takes either non-negative or non-positive values, and whenever $j \prec k \prec l$ and $f(j) = f(l) = 1$ or $= -1$, then $f(k) = 1$ or $f(k) = -1$, respectively.) Gröflin and Hoffman proved that lattice polyhedra are totally dual integral. However, no combinatorial algorithm is known for lattice polyhedra in general (not even in the case of binary matrices).

4.3 Ordered compatibility and greediness

In the following we show that ordered compatible matrices fulfill the first requirement in the definition of 1-greediness:

Proposition 4. *Let A be ordered compatible. Then the greedy algorithm applied to (8) never steps back.*

Proof. When processing x_j for the first time, the greedy algorithm raises x_j until some constraint gets tight. We say that x_j is *blocked* by this constraint. We claim that x_j remains blocked (by either constraint i or some other constraint) from that point on. Assume to the contrary that x_j is unblocked by x_k, $k > j$ (*i.e.*, while the greedy algorithm increases x_k). Just before increasing x_k, variable x_j was blocked by some constraint, say, $a_i x \leq b_i$. Increasing x_k can only unblock x_j if $i \in \text{supp}(A^+_j) \cap \text{supp}(A^-_k)$, so that A_j and A_k are incompatible and $A_{j\wedge k}$ exists. Since $j \wedge k < k$, also variable $x_{j\wedge k}$ is blocked by some constraint i' (at the same point in time, just before increasing x_k). But $A_{j\wedge k} \preceq A_k$, hence i' must also block x_k, a contradiction.

In particular, the greedy algorithm, when applied to 8 with an ordered compatible A, simply raises the variables $x_1, x_2, \ldots, x_n$ in this order until they get blocked, just like in the $(0,1)$-case. (Note that, in contrast to the $(0,1)$-case, however, $\bar{x}$ is in general not lexicographically maximal.) This simple observation immediately implies

Corollary 1. *Path incidence matrices (with path orders induced by cyclic orders π_i around each vertex i) are* 1*-greedy.*

Proof. The greedy algorithm raises $x_1,\dots,x_n$ in this order and the resulting $\bar{x}$ is a max flow (otherwise there were an augmenting path, *i.e.*, a variable x_j that could still be raised).

For planar graphs, the number of augmentations can be shown to be $O(m)$ ([13, 1]). The case of bounded genus is not yet analyzed. For general graphs, it would be interesting to study the running time of the path augmentation method when the ordering of the path is induced by cyclical orderings π_i around each vertex i. Is it polynomial, at least for appropriate choices of π_i? Note that the corresponding greedy algorithm coincides with the well-known ”shortest augmenting path method” if the paths are ordered according to non-decreasing lengths.

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