Robust bus scheduling considering transfer synchronizations

Konstantinos Gkiotsalitis
University of Twente

Center for Transport Studies (CTS)
Department of Civil Engineering
P.O. Box 217
7500 AE Enschede
The Netherlands
Email: k.gkiotsalitis@utwente.nl

Oskar A.L. Eikenbroek
University of Twente

Center for Transport Studies (CTS)
Department of Civil Engineering
P.O. Box 217
7500 AE Enschede
The Netherlands
Email: o.a.l.eikenbroek@utwente.nl

Oded Cats
Delft University of Technology

Department of Transport and Planning
P.O. Box 5048
2600 GA Delft
The Netherlands
Email: o.cats@tudelft.nl

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ABSTRACT
In this study, we model the bus scheduling problem considering transfer synchronizations. Our mathematical model accounts for the variability of the travel and dwell times of bus trips, the regularity of individual bus lines, and the regulatory constraints related to the schedule sliding prevention and the layover time limits. To perform a synchronization of multiple bus lines that is robust to travel time and dwell time variabilities, we tackle this problem using the minimax decision rule of 2-player games where one player selects the optimal dispatching times for some specific travel and dwell time noise, whereas the other player selects the worst-case travel and dwell time noise for a given dispatching time solution. In a validation of this approach in two bus lines in Stockholm using 1 month of actual vehicle location and passenger counting data, we demonstrate the potential improvement in terms of service regularity and increased synchronizations. Finally, we examine how the performance of bus schedules in common-case scenarios is affected when their robustness to extreme deviations is increased.

Keywords: timetabling; high-frequency services; robust optimization; transfer coordination; non-linear programming

INTRODUCTION

Bus timetabling is a sub-problem of the tactical planning phase. Other tactical planning stages such as route design, frequency settings and vehicle allocation precede the timetabling problem (Ceder (*1*), Farahani et al. (*2*), Gkiotsalitis and Cats (*3*), Gkiotsalitis and Kumar (*4*)). After the timetabling stage, transport operators can apply dynamic control strategies such as bus holding, stop-skipping or dispatching time changes (Gavriilidou and Cats (*5*), Adamski and Turnau (*6*), Fu and Yang (*7*), Zhao et al. (*8*), Bartholdi and Eisenstein (*9*), Hickman (*10*), Gkiotsalitis and Stathopoulos (*11*), Berrebi et al. (*12*), Eberlein et al. (*13*)) to adjust to the spatio-temporal variations of passenger demand and travel times and improve the service reliability.

In practice, the timetables of bus lines are typically determined at the line level treating each bus line in isolation Gkiotsalitis and Maslekar (*14*). Timetabling approaches that account for the network-wide synchronization do not consider the variability of the bus travel times during the actual operations (Chakroborty et al. (*15*), Wong et al. (*16*)). This, however, yields high inefficiencies which were already outlined in the early 1990s (Bookbinder and Desilets (*17*)).

This study focuses on the determination of the dispatching times of bus trips that favor the synchronization among different bus lines and are *robust* to travel time variations during the daily operations. In this study, we solve the network-wide synchronized scheduling problem to generate timetables with trip dispatching times that favor the synchronization among different bus lines while also improving the regularity of each individual bus line. Note that in line with the theory of robust optimization, we refer as robust timetables the timetables that maintain their operational performance in worst-case scenarios of travel time disturbances Bertsimas and Sim (*18*).

The remainder of this study is structured as follows: in the remaining part of this section we discuss the most relevant research studies in bus scheduling considering transfer synchronizations. In section 2, we formulate our problem and we introduce the objectives and constraints of our main mathematical model. The mathematical model is presented in section 3 and is reformulated to ensure its feasibility after relaxing its soft constraints. In section 4, we introduce our solution method to the minimax problem which is based on the concept of alternating optimization. A detailed demonstration of our approach in a toy network is presented in section 5. This demonstration facilitates the reproduction of our work. In addition, the application of our approach to two bi-directional bus lines in Stockholm is presented in the same section. Finally, section 6 provides the concluding remarks and discusses the future direction.

Related Studies

Coordinating multiple bus lines by synchronizing their timetables has been studied by (Daduna and Voß (*19*), Jansen et al. (*20*), Wong et al. (*16*), Vansteenwegen and Van Oudheusden (*21*), Gkiotsalitis and Maslekar (*22*)). A typical objective of that problem is the reduction of passenger waiting times at transfer stops while maintaining even dispatching headways (Gkiotsalitis et al. (*23*)). Cevallos and Zhao (*24*) and Cevallos and Zhao (*25*) proposed simple pertubations by merely shifting the pre-existing timetables to solve the aforementioned problem and resorted into a Genetic Algorithm (GA) given the computational complexity of the problem.

Zhigang et al. (*26*) coupled the problem of vehicle scheduling with the timetabling problem that considers transfer synchronizations. Despite that, most approaches treat those two problems in isolation because, as it was demonstrated in Zhigang et al. (*26*), those two problems can only be solved at different levels using bi-level programming.

Typically, the timetabling problem that considers network-wide synchronizations is solved with heuristics due to its computational complexity. This results in approximate solutions, such as in the works of Ceder et al. (*27*) and Ceder and Tal (*28*) that introduced a mixed integer linear program and used a heuristic algorithm for its solution. The follow-up works of Eranki (*29*) and Ibarra-Rojas and Rios-Solis (*30*) modified the mathematical model of Ceder and Tal (*28*) by relaxing the synchronization requirements and allowing a time buffer for transfer synchronization. With this time buffer, a synchronization is achieved even if bus trips from different lines do not arrive at the transfer point simultaneously, but with a small delay that lies within the limits of the pre-set time buffer. Both works of Eranki (*29*) and Ibarra-Rojas and Rios-Solis (*30*) resorted into heuristics. Notably, Ibarra-Rojas and Rios-Solis (*30*) developed a multi-start, local search algorithm for converging as close as possible to the global optimum.

Coffey et al. (*31*) treated the synchronization problem as a demand-supply matching problem. In their approach, they optimized the timetables of public transport modes by matching the passenger demand expressed via journey planners with the public transport supply in order to reduce missed connections. Other works that expand the synchronization problem to mixed (rail-bus) operations such as Chien and Schonfeld (*32*), Sun and Hickman (*33*), Sivakumaran et al. (*34*), Verma and Dhingra (*35*) proposed multi-modal synchronization methods based on the so-called "feeder model" that prioritizes the transport modes and forces the bus schedules to adjust to the less flexible rail schedules.

In the above-mentioned works, the variability of travel and dwell times and the resulting effect on the number of boardings/alightings at stops was not considered at the optimization stage. However, this is a very important aspect because the expected and the actual arrival times of buses at stops can differ significantly in real operations resulting in missed connections. For instance, Knoppers and Muller (*36*) explored the waiting times of passengers at transfer stops in the case of rail synchronization and showed that synchronization has no effect in real operations if the arrival times at the transfer stops fluctuate significantly from the expected ones.

The travel time variability was explored in the work of Hall et al. (*37*). Hall et al. (*37*) studied thoroughly the importance of travel time variability at the multi-line synchronization problem. The main focus of Hall et al. (*37*) was on real-time bus holding of buses at transfer stops for improving synchronization via adjusting to the travel time changes and not on network-wide synchronized scheduling. In their work, the bus trips were held at the transfer stops in anticipation of the arrival of passengers from other trips in order to perform the transfer. In addition, the transfer times were minimized under stochastic travel time conditions by modeling the noise of the bus arrivals at the transfer stop with the use of normal distributions.

As in Hall et al. (*37*), our work considers the potential variability in the travel times and dwell times of daily trips and has the following additional features: (i) is concerned with tactical planning, in particular bus timetabling (i.e., offline optimization of the dispatching times of the daily trips); (ii) it has a dual objective and minimizes the regularity of individual bus lines while ensuring the synchronization of trips at the transfer stops; and (iii) considers operational regulatory constraints such as schedule sliding prevention and layover time limits.

PROBLEM FORMULATION

Travel time and dwell time noise due to external traffic or incidents is one of the key reasons behind the unreliability of the bus operations. Most timetables assume that the operational arrival times of bus trips at stops will be close to their expected values, something that is rarely the case in real-world operations. Several studies such as Berrebi et al., Gkiotsalitis and Cats, Gkiotsalitis and Van Berkum (*12*, *38*, *39*) have identified the travel time noise that affects the boardings and alightings of passengers as the main factor of bus bunching and missed passenger transfers. The mechanism behind it is that if one bus is postponed because of traffic, the headway with its preceding bus increases and this will increase the number of boardings (and the associated dwell time); therefore, this domino-effect will make it difficult for the bus which is left behind to rebound later on.

This work makes a number of reasonable assumptions such as: (i) the number of bus trips per line is decided during the frequency settings phase and all of the assigned trips are expected to be performed during the day; (ii) bus trips from the same line are not expected to overtake one another (an assumption that is used in several studies such as Xuan et al. (*40*), Chen et al. (*41*)); and (iii) the actual travel time of a bus trip between two consecutive stops can deviate from its expected value due to external traffic or road works.

Before proceeding to the description of the multi-line synchronization problem, the following notation is introduced.

NOMENCLATURE

Sets

$L = \{1,...,l,...\}$	are the different bus lines in the study area
$N(l) = \{1,...,n,...\}$	is the set of all daily trips of each bus line $l \in L$
$S(l) = \{1...,s,...\}$	is the set of bus stops of each bus line $l \in L$ ordered from the first to the last
B_{lj}	all transfer stops between lines l and j where the arrival times of trips that belong to line l need to be synchronized with the arrival times of trips that belong to line j

Parameters

f_l	is the number of trips for each line $l \in L$ which are needed to fulfill the demand (note: the number of trips is already determined at the frequency settings stage)
T	the planning period (note: the suggested planning period is at most one day of operations)
$h_l^* = \frac{T}{f_l}$	the ideal headway of bus line $l \in L$ that should be maintained at all bus stops for attaining a perfectly regular service (sec)
$t_{l,n,s}$	denotes the expected travel time of bus trip n of line l between stops $s-1$ and s (sec)
δ_l^{min}	is the dispatching time of the first trip of the planning period (sec)
δ_l^{max}	is the latest possible time where all trips of line $l \in L$ must have completed their service for preventing schedule sliding (sec)
$k_{l,n,s}$	is the expected dwell time of bus trip n of bus line l at stop s (sec)
ψ_l	is the required layover time for line l after completing each bus trip (sec)

Decision Variables

$x_{l,n}$	is the dispatching time of the n^{th} trip that belongs to line l (sec)

Uncertainty Parameters

$\xi_{l,n,s} \in [\xi_{l,s}^{min}, \xi_{l,s}^{max}]$	is an uncertain parameter that represents the travel time 'noise' between stops $s-1$ and s for trip n of line l (in sec). The parameter $\xi_{l,n,s}$ can take any value within the range $[\xi_{l,s}^{min}, \xi_{l,s}^{max}]$ where $\xi_{l,s}^{min}$ is the minimum possible travel time and $\xi_{l,s}^{max}$ the maximum possible travel time (i.e., free-flow travel time) between stops $s-1$ and s
$\zeta_{l,n,s} \in [\zeta_{l,s}^{min}, \zeta_{l,s}^{max}]$	is the uncertain parameter that represents the dwell time noise at stop s for trip n of line l (in sec)

Note that we do not make any assumption with respect to the probability distribution of the uncertain parameters $\xi_{l,n,s}$ and $\zeta_{l,n,s}$, but we allow them to take any value withing the uncertainty sets $[\xi_{l,s}^{min}, \xi_{l,s}^{max}]$ and $[\zeta_{l,s}^{min}, \zeta_{l,s}^{max}]$. Following the above notation, the arrival time of any trip n that belongs to a bus line $l \in L$ at stop $s \in S(l) \setminus \{1\}$ is:

$$a_{l,n,s} = x_{l,n} + \sum_{z=2}^{s} \left(t_{l,n,z} + \xi_{l,n,z}\right) + \sum_{z=1}^{s-1} \left(k_{l,n,z} + \zeta_{l,n,z}\right) \tag{1}$$

where $\xi_{l,n,z}$ is the travel time deviation from the expected travel time value $t_{l,n,z}$ for the road section defined by bus stops $z-1$ and z and $\zeta_{l,n,z}$ is the dwell time deviation from the expected dwell time at stop z. In Eq.1 the arrival time of a trip n at stop s is set equal to departure time of the trip, $x_{l,n}$, plus the sum of the expected travel times and the respective travel time deviations between consecutive stops until reaching stop s, $\sum_{z=2}^{s} \left(t_{l,n,z} + \xi_{l,n,z}\right)$, plus the expected dwell time at each bus stop until reaching stop s, $\sum_{z=1}^{s-1} \left(k_{l,n,z} + \zeta_{l,n,z}\right)$. From Eq.1 one can note that the arrival times of buses at stops vary based on the departure times of the trips and the travel time/dwell time noise.

Formulating the objectives of the Network-wide synchronization problem

To increase the regularity of bus services, the actual time headways[1] at bus stops should be as close as possible to their scheduled values. The ideal headway $h_l^* = \frac{T}{f_l}$ of a bus line $l \in L$ is already defined at the frequency settings stage. In addition, the time headway between two consecutive services $n-1, n$ of line l at stop s is:

$$h_{l,n,s} = a_{l,n,s} - a_{l,n-1,s} \text{ where } n \in N(l) \setminus \{1\} \tag{2}$$

The difference between the actual headways and the ideal headways at stops is the sole key performance indicator of regularity-based services and has been in use in London, Singapore, Barcelona and many other densely populated areas where the bus services operate in high frequencies (Randall et al. (*42*)). The main reason of its use in high-frequency services is that it indicates the excessive waiting times of passengers at stops, where the excessive waiting times are the difference between the actual waiting times and the scheduled ones. Note that in high-frequency services, the waiting time of a passenger of trip n at stop s is half the headway between trip n and

[1]the time headway between two consecutive bus trips $n, n+1$ of a line l at bus stop s is the headway between the front bumpers of the respective buses at the time of their arrival at stop s

trip $n-1$, $\frac{h_{l,n,s}}{2}$, because the passenger arrivals at stops are considered as random[2] and are uniformly distributed.

In order to reduce the difference between the actual waiting times of passengers at stops and the ideal ones for a bus line $l \in L$, one should minimize the sum of the squared difference between the actual and the ideal headways:

$$\sum_{s \in S(l)} \left(\sum_{n \in N(l) \setminus \{1\}} \left(\frac{h_{l,n,s}}{2} - \frac{h_l^*}{2} \right)^2 \right) = \frac{1}{4} \sum_{s \in S(l)} \left(\sum_{n \in N(l) \setminus \{1\}} \left(h_{l,n,s} - h_l^* \right)^2 \right) \tag{3}$$

Remark 2: The key performance indicators of service regularity, such as the excess waiting time indicator, consider the squared difference between the actual and the ideal headways because the squared difference penalizes progressively the headway deviations from the ideal case. Consequently, the optimization is steered towards avoiding extreme abnormalities.

Remark 3: If all bus lines $l \in L$ have the same importance, the average regularity level of all bus lines can be expressed as $\sum_{l \in L} \frac{1}{4} \sum_{s \in S(l)} \left(\sum_{n \in N(l) \setminus \{1\}} \left(h_{l,n,s} - h_l^* \right)^2 \right)$. Notwithstanding, if some bus lines have more importance than others, the network-wide regularity can be indicated by the weighted sum of the daily excessive waiting times for all bus lines:

$$\sum_{l \in L} \frac{w_l}{4} \sum_{s \in S(l)} \left(\sum_{n \in N(l) \setminus \{1\}} \left(h_{l,n,s} - h_l^* \right)^2 \right) \tag{4}$$

where w_l are weight factors that give more importance to the regularity of some bus lines in the expense of others. Note that $w_l \geq 0, \forall l \in L$, and $\sum_{l \in L} w_l = 1$.

Plugging Eq.1 and 2 into Eq.4 yields:

$$\begin{aligned} f(x,\xi,\zeta) := \sum_{l \in L} \frac{w_l}{4} \sum_{s \in S(l)} \sum_{n \in N(l) \setminus \{1\}} \Bigg(\Big(x_{l,n} + \sum_{z=2}^{s} \left(t_{l,n,z} + \xi_{l,n,z} \right) + \sum_{z=1}^{s-1} \left(k_{l,n,z} + \zeta_{l,n,z} \right) \Big) - \\ \Big(x_{l,n-1} + \sum_{z=2}^{s} \left(t_{l,n-1,z} + \xi_{l,n-1,z} \right) + \sum_{z=1}^{s-1} \left(k_{l,n-1,z} + \zeta_{l,n-1,z} \right) \Big) - h_l^* \Bigg)^2 \end{aligned} \tag{5}$$

where $f(x,\xi,\zeta)$ is the daily, network-wide excessive waiting time of passengers that indicates the service regularity.

Now let us consider the waiting times of passengers at transfer stops. Reckon that B_{lj} is the set with all transfer stops between lines l and j where the arrival times of trips that belong to line l need to be synchronized with the arrival times of trips that belong to line j. Let also Y_{lj}^{bnm} be a dummy variable where $Y_{lj}^{bnm} = 1$ if trip $n \in N(l)$ needs to synchronize its arrival time with trip $m \in N(j)$ at the transfer stop $b \in B_{lj}$ and $Y_{lj}^{bnm} = 0$ otherwise. Ceder et al. (27) considers a perfect synchronization when trip n arrives at the transfer stop $b \in B_{lj}$ exactly at the same time as trip $m \in N(j)$. In this way, the waiting times of passengers that want to transfer from bus trip $n \in N(l)$

[2] several studies, such as Randall et al. (42), Welding (43), have shown that passengers cannot synchronize their arrivals to the arrivals of the bus trips in high-frequency services.

to bus trip $m \in N(j)$ at bus stop $b \in B_{lj}$ are minimized when $a_{l,n,b} - a_{j,m,b} = 0$. Later studies by Eranki (*29*) and Ibarra-Rojas and Rios-Solis (*30*) proposed a more flexible scheme where the bus trip n is still considered synchronized if it arrives within a time range of $[0, \Delta t]$ seconds after the arrival of trip m at the transfer stop b.

In our study, we also allow a more flexible synchronization by treating the required synchronizations at transfer stops as problem constraints:

$$0 \leq Y_{lj}^{bnm}\left(a_{l,n,b} - a_{j,m,b}\right) \leq \Delta t, \quad \forall l, j \in L, \forall n \in N(l), \forall m \in N(j) \setminus \{1\}, \forall b \in B_{lj} \tag{6}$$

Obviously, when the dummy variable $Y_{lj}^{bnm} = 0$ the inequalities of Eq.6 hold for any value of the arrival times $a_{l,n,b}$ and $a_{j,m,b}$ because in such case there is no requirement to synchronize the arrival time of trip $n \in N(l)$ with the arrival time of trip $m \in N(j)$ at stop $b \in B_{lj}$. Plugging Eq.1 into Eq.6 yields the expanded form:

$$\begin{aligned} 0 \leq Y_{lj}^{bnm}\Bigg(\Big(x_{l,n} + \sum_{z=2}^{b}\left(t_{l,n,z} + \xi_{l,n,z}\right) + \sum_{z=1}^{b-1}\left(k_{l,n,z} + \zeta_{l,n,z}\right)\Big) - \\ \Big(x_{j,m} + \sum_{z=2}^{b}\left(t_{j,m,z} + \xi_{j,m,z}\right) + \sum_{z=1}^{b-1}\left(k_{j,m,z} + \zeta_{j,m,z}\right)\Big)\Bigg) \leq \Delta t, \\ \forall l, j \in L, \forall n \in N(l), \forall m \in N(j) \setminus \{1\}, \forall b \in B_{lj} \end{aligned} \tag{7}$$

Regulatory constraints

This study considers layover constraints. The layover time of a bus that finishes one bus trip is the minimum required time before starting its next trip. Hence, the layover time is equal to the required time for traveling from the last stop of the finished trip to the first stop of the next trip (known as deadheading time) plus the recovery time for the bus driver (in most cases, bus drivers must take a short break after completing a bus trip).

Let us consider a dummy variable $\Phi_{n,n'}^{l}$ where $\Phi_{n,n'}^{l} = 1$ if bus trip $n' \in N(l)$ is operated after the completion of bus trip $n \in N(l)$ by the same bus and $\Phi_{n,n'}^{l} = 0$ otherwise. Then, if the required layover time for bus line $l \in L$ is ψ_l where the layover time consists of the required deadhead time for traveling from the last to the first stop and the resting time of the bus driver, the dispatching time, $x_{l,n'}$, of trip n' should satisfy the inequality:

$$\Phi_{n,n'}^{l}\left(x_{l,n'} - \left(x_{l,n} + \sum_{s \in S(l)\setminus\{1\}}\left(t_{l,n,z} + \xi_{l,n,z}\right) + \sum_{s \in S(l)}\left(k_{l,n,z} + \zeta_{l,n,z}\right)\right)\right) \geq \Phi_{n,n'}^{l}\psi_l \ , \ \forall n, n' \in N(l), \forall l \in L \tag{8}$$

which yields:

$$\Phi_{n,n'}^{l}x_{l,n'} \geq \Phi_{n,n'}^{l}\left(\psi_l + x_{l,n} + \sum_{s \in S(l)\setminus\{1\}}\left(t_{l,n,z} + \xi_{l,n,z}\right) + \sum_{s \in S(l)}\left(k_{l,n,z} + \zeta_{l,n,z}\right)\right) \ , \ \forall n, n' \in N(l), \forall l \in L \tag{9}$$

Note that the inequality of Eq.9 is satisfied when bus trip n' is not operated by the same bus as trip n because in such case $\Phi^l_{n,n'} = 0$ and the inequality holds. If $\Phi^l_{n,n'} = 1$ the dispatching time of trip $n' \in N(l)$ which is denoted as $x_{l,n'}$ should be greater than (i) the arrival time of trip n at the last stop plus the dwell time at that stop, plus (ii) the layover time ψ_l.

Finally, to prevent schedule sliding and maintain the duration of the planned operations, all trips of any bus line $l \in L$ must have been completed before time δ_l^{max}. The schedule sliding constraint ensures that the operations of the examined planning period are not prolonged because this will have adverse effects on future operations and increase the working hours of bus drivers (the latter is not typically allowed because of the respective contractual agreements). Avoiding schedule sliding yields the following inequality constraints:

$$x_{l,n} + \sum_{s\in S(l)\setminus\{1\}} (t_{l,n,z} + \xi_{l,n,z}) + \sum_{s\in S(l)} (k_{l,n,z} + \zeta_{l,n,z}) \leq \delta_l^{max} \quad , \quad \forall n \in N(l), l \in L \tag{10}$$

which ensures that each trip n of line l has arrived at the last stop and has completed all passenger alightings before time δ_l^{max}.

MATHEMATICAL PROGRAM OF THE NETWORK-WIDE SYNCHRONIZATION PROBLEM

The proposed network-wide synchronization problem that explicitly considers uncertain travel and dwell times is formulated as a mathematical optimization problem. The mathematical program can be written in a compact form as:

$$(Q): \quad \min_{x} \max_{\xi,\zeta} \quad f(x,\xi,\zeta) \tag{11}$$

$$\text{s.t.:} \quad x \in \mathscr{F}(\xi,\zeta) = \left\{ \, x \;\middle|\; x \text{ satisfies Eq. } 7, 9, 10 \, \right\} \tag{12}$$

$$x_{l,1} = \delta_l^{min} \quad , \quad \forall l \in L \tag{13}$$

$$\xi_{l,s}^{min} \leq \xi_{l,n,s} \leq \xi_{l,s}^{max} \quad , \quad \forall l \in L, \forall n \in N(l), \forall s \in S(l) \setminus \{1\} \tag{14}$$

$$\zeta_{l,s}^{min} \leq \zeta_{l,n,s} \leq \zeta_{l,s}^{max} \quad , \quad \forall l \in L, \forall n \in N(l), \forall s \in S(l) \tag{15}$$

Program (Q) is a minimax optimization problem (see Wald (*44*), Wald (*45*)) and ranks the dispatching time solutions on the basis of their worst-case outcomes. Therefore, the objective of the minimax problem is to find a dispatching time solution which performs best at a pessimistic scenario of worst-case travel time and dwell time noises, ξ, ζ. (We note that in (Q) the uncertain parameters (ξ, ζ) appear as variables).

Solution Existence and Reformulation

The optimization problem (Q) is difficult to solve numerically. Intuitively, the set $\mathscr{F}(\xi,\zeta)$ depends on the choice of the noise parameters, while the choice of the noise depends on the choice of x. This is a non-probabilistic decision-making approach based on common rules from game theory where the objective is to minimize the possible loss of the worst-case scenario. In this section, we formulate a *relaxed problem* of (Q) that can be solved using any optimization toolbox. Therefore, we analyze program (Q) in more detail.

Given ξ^0, ζ^0, the objective function $f(x,\xi^0,\zeta^0)$ is continuous and convex. Therefore, the optimization problem

$$(P(\xi^0,\zeta^0)) \qquad \min_x f(x,\xi^0,\zeta^0) \qquad \begin{array}{l} x \in \mathscr{F}(\xi^0,\zeta^0) \\ x \text{ satisfies } 13 \end{array} \tag{16}$$

can easily be solved to global optimality with parameters (ξ^0,ζ^0) if the corresponding feasible set $\mathscr{F}(\xi^0,\zeta^0)$ is non-empty (and compact). Let us now examine the "behavior" of the inequality constraints of Eq.7, 9, 10 and the equality constraints of Eq.13.

First, the equality constraints of Eq.13 can be always satisfied because the dispatching times of the first trips of the day are not constrained by the travel time/dwell time noise levels. This is not true though for the inequality, schedule sliding constraints of Eq.10 if the travel time noise is highly uncertain. That is, a solution that avoids schedule sliding for all travel time and dwell time noise instances might not exist. If this is the case, then there is at least a travel time and dwell time noise instance (ξ^0,ζ^0) for which the mathematical program (Q) does not have a feasible solution $x \in \mathscr{F}(\xi^0,\zeta^0)$. Hence, we propose to relax these constraints by introducing penalty terms in the objective function that penalize the value of the penalized objective function when (at least one) of the schedule sliding constraints is violated.

We therefore introduce the functions:

$$\varphi_{l,n}(x,\xi,\zeta) = \begin{cases} 0 \text{ if } x_{l,n} + \sum\limits_{s\in S(l)\setminus\{1\}} (t_{l,n,z}+\xi_{l,n,z}) + \sum\limits_{s\in S(l)} (k_{l,n,z}+\zeta_{l,n,z}) \le \delta_l^{max} \\ c_\varphi \Bigg(x_{l,n} + \sum\limits_{s\in S(l)\setminus\{1\}} (t_{l,n,z}+\xi_{l,n,z}) + \\ \qquad \sum\limits_{s\in S(l)} (k_{l,n,z}+\zeta_{l,n,z}) - \delta_l^{max} \Bigg)^2 \text{ otherwise} \end{cases} \quad \forall l \in L, n \in N(l) \tag{17}$$

where c_φ is a nonnegative constant with a *sufficiently high* value for ensuring that the satisfaction of constraints is prioritized. This sufficiently high value of c_φ is determined in practice by starting with a small value, minimizing the penalized objective function with this small value and then increasing this value incrementally until reaching solution stability. For any fixed noise (ξ^0,ζ^0), a penalty function $\varphi_{l,n}(x,\xi^0,\zeta^0)$ penalizes any dispatching time $x_{l,n}$ for which $x_{l,n} + \sum_{s\in S(l)\setminus\{1\}} (t_{l,n,z}+\xi_{l,n,z}) + \sum_{s\in S(l)} (k_{l,n,z}+\zeta_{l,n,z}) > \delta_l^{max}$ and is twice differentiable and convex. The penalty functions are structured in such a way that will strongly encourage the penalized objective function to choose the best solution which satisfies as many schedule sliding constraints as possible while the squared value of $\left(x_{l,n} + \sum_{s\in S(l)\setminus\{1\}} (t_{l,n,z}+\xi_{l,n,z}) + \sum_{s\in S(l)} (k_{l,n,z}+\zeta_{l,n,z}) - \delta_l^{max} \right)^2$ ensures that trips which are significantly prolonged beyond the time limit δ_l^{max} are penalized more severely than others which are close to δ_l^{max}.

Similarly, we also propose to relax the inequality constraints related to the synchronization of trips at transfer stops to avoid infeasibility issues by introducing the following penalty functions to the penalized objective function.

First, the inequality constraints:

$$0 \leq Y_{lj}^{bnm}\Bigg(\Big(x_{l,n}+\sum_{z=2}^{b}\left(t_{l,n,z}+\xi_{l,n,z}\right)+\sum_{z=1}^{b-1}\left(k_{l,n,z}+\zeta_{l,n,z}\right)\Big)-$$
$$\Big(x_{j,m}+\sum_{z=2}^{b}\left(t_{j,m,z}+\xi_{j,m,z}\right)+\sum_{z=1}^{b-1}\left(k_{j,m,z}+\zeta_{j,m,z}\right)\Big)\Bigg),\ \forall l,j\in L,\forall n\in N(l),\forall m\in N(j)\setminus\{1\},\forall b\in B_{lj}$$

are approximated by the penalty functions:

$$\mu_{lj}^{bnm}(x,\xi,\zeta)=\begin{cases}0 \text{ if } Y_{lj}^{bnm}=0 \text{ or } a_{l,n,b}\geq a_{j,m,b}\\ c_{\mu}\left(a_{j,m,b}-a_{l,n,b}\right)^2 \text{ otherwise}\end{cases}\quad \forall l,j\in L,\forall n\in N(l),\forall m\in N(j)\setminus\{1\},\forall b\in B_{lj} \tag{18}$$

where the arrival times $a_{l,n,b},a_{j,m,b}$ are given from Eq.1 and are the compact forms of $\Big(x_{l,n}+\sum_{z=2}^{b}\left(t_{l,n,z}+\xi_{l,n,z}\right)+\sum_{z=1}^{b-1}\left(k_{l,n,z}+\zeta_{l,n,z}\right)\Big)$ and $\Big(x_{j,m}+\sum_{z=2}^{b}\left(t_{j,m,z}+\xi_{j,m,z}\right)+\sum_{z=1}^{b-1}\left(k_{j,m,z}+\zeta_{j,m,z}\right)\Big)$ respectively whereas c_{μ} is a parameter with a sufficiently high value.

Then, the inequality constraints $a_{l,n,b}-a_{j,m,b}\leq\Delta t,\forall n\in N(l),\forall m\in N(j)\setminus\{1\},\forall b\in B_{lj}$ are approximated by the penalty functions:

$$\nu_{lj}^{bnm}(x,\xi,\zeta)=\begin{cases}0 \text{ if } Y_{lj}^{bnm}=0 \text{ or } a_{l,n,b}-a_{j,m,b}\leq\Delta t\\ c_{\nu}\left(a_{l,n,b}-a_{j,m,b}-\Delta t\right)^2 \text{ otherwise}\end{cases}\quad \forall l,j\in L,\forall n\in N(l),\forall m\in N(j)\setminus\{1\},\forall b\in B_{lj} \tag{19}$$

where c_{ν} is a sufficiently high value. Note that the penalty functions $\mu_{lj}^{bnm}(x,\xi,\zeta)$ and $\nu_{lj}^{bnm}(x,\xi,\zeta)$ increase the value of the penalized objective function every time a synchronization is missed (i.e., the transfer does not occur within the time interval $[0,\Delta t]$). In addition, for any given noise instance, (ξ^0,ζ^0), the functions $\mu_{lj}^{bnm}(x,\xi^0,\zeta^0)$ and $\nu_{lj}^{bnm}(x,\xi^0,\zeta^0)$ are convex because they are both piecewise linear and piecewise quadratic.

The penalized objective function now becomes:

$$\begin{aligned}\tilde{f}(x,\xi,\zeta)=&f(x,\xi,\zeta)+\sum_{l\in L}\sum_{n\in N(l)}\varphi_{l,n}(x,\xi,\zeta)+\\ &\sum_{l\in L}\sum_{j\in L}\sum_{b\in B_{lj}}\sum_{n\in N(l)}\sum_{m\in N(j)}\Big(\mu_{lj}^{bnm}(x,\xi,\zeta)+\nu_{lj}^{bnm}(x,\xi,\zeta)\Big)\end{aligned} \tag{20}$$

which maintains convexity for any noise instance of (ξ^0,ζ^0). The program (Q) can be approximated by the relaxed program $(\tilde{Q})$ that includes the penalized objective function $\tilde{f}(x,\xi,\zeta)$:

$$
\begin{aligned}
(\tilde{Q}):\quad & \min_{x}\ \max_{\xi,\zeta} && \tilde{f}(x,\xi,\zeta) && (21)\\
& \text{s.t.:} && \text{Equations } 9 && (22)\\
& && x_{l,1}=\delta_l^{min}\ ,\ \ \forall l\in L && (23)\\
& && \xi_{l,s}^{min}\le \xi_{l,n,s}\le \xi_{l,s}^{max}\ ,\ \ \forall l\in L,\forall n\in N(l),\forall s\in S(l)\setminus\{1\} && (24)\\
& && \zeta_{l,s}^{min}\le \zeta_{l,n,s}\le \zeta_{l,s}^{max}\ ,\ \ \forall l\in L,\forall n\in N(l),\forall s\in S(l) && (25)
\end{aligned}
$$

Note that the inequality constraints of Eq.7 and 10 are not included in the mathematical program $(\tilde{Q})$ because they are incorporated into the penalized objective function, $\tilde{f}(x,\xi,\zeta)$, with the use of penalty functions. Note also that the remaining inequality constraints are the layover constraints of Eq.9. From a mathematical perspective, we have relaxed program (Q) into an easier-to-study problem $(\tilde{Q})$. The following theorem (Theorem 3.1) shows that for a given noise instance (ξ^0,ζ^0), the corresponding optimization problem

$$(\tilde{P}(\xi^0,\zeta^0)) \qquad \min_x \tilde{f}(x,\xi^0,\zeta^0) \quad \text{s.t.} \quad x \text{ satisfies } 9,\ 13$$

is easy to solve and has always a feasible solution.

Theorem 3.1. *Given travel time and dwell time noise instance (ξ^0,ζ^0), the feasible set that corresponds to $(\tilde{P}(\xi^0,\zeta^0))$ is nonempty and has an optimal solution. If, in addition, for some optimal solution x^* it holds that $\tilde{f}(x^*,\xi^0,\zeta^0)=f(x^*,\xi^0,\zeta^0)$ then solution x^* is feasible for mathematical program (Q).*

Proof. Since the domain of the dispatching times of the first trips of the planning period is $\mathbb{R}^{|L|}$, there is always a set of values $x_{l,1},\ \forall l\in L$ for which $x_{l,1}=\delta_l^{min},\ \forall l\in L$ satisfying the equality constraints of Eq.13. In addition, the inequality layover constraints of Eq.9 can be always satisfied since there exists a $x_{l,n'}\in\mathbb{R}$ for which $\Phi^l_{n,n'}\Big(x_{l,n'}-\Big(x_{l,n}+\sum_{s\in S(l)\setminus\{1\}}(t_{l,n,z}+\xi_{l,n,z})+\sum_{s\in S(l)}(k_{l,n,z}+\zeta_{l,n,z})\Big)\Big)\ge \Phi^l_{n,n'}\psi_l\ ,\ \ \forall n,n'\in N(l),\forall l\in L$ since the value of $x_{l,n'}$, given that program $(\tilde{Q})$ does not include the inequality constraints of Eq.7, does not have a finite upper bound of $\delta_l^{max}-\Big(\sum_{s\in S(l)\setminus\{1\}}(t_{l,n',z}+\xi_{l,n',z})+\sum_{s\in S(l)}(k_{l,n',z}+\zeta_{l,n',z})\Big)$ anymore.

Proving that there is always a feasible solution for the program $(\tilde{Q})$, if x^* is such that $\tilde{f}(x^*,\xi^0,\zeta^0)=f(x^*,\xi^0,\zeta^0)$, then

$$\sum_{l\in L}\sum_{n\in N(l)}\varphi_{l,n}(x^*,\xi^0,\zeta^0)+\sum_{l\in L}\sum_{j\in L}\sum_{b\in B_{lj}}\sum_{n\in N(l)}\sum_{m\in N(j)}\Big(\mu_{lj}^{bnm}(x^*,\xi^0,\zeta^0)+\nu_{lj}^{bnm}(x^*,\xi^0,\zeta^0)\Big)=0$$

which, in addition to the fact that all penalty functions are non-negative by definition, means that $\varphi_{l,n}(x^*,\xi^0,\zeta^0)=0,\ \forall l\in L,n\in N(l)$ and thus $x^*_{l,n}\le\delta_l^{max},\ \forall l\in L,n\in N(l)$ which proves the satisfaction of all schedule sliding inequality constraints and, at the same time, $\mu_{lj}^{bnm}(x^*,\xi^0,\zeta^0)=\nu_{lj}^{bnm}(x^*,\xi^0,\zeta^0)=0,\forall n\in N(l),\forall m\in N(j)\setminus\{1\},\forall b\in B_{lj}$, which proves that all synchronizations at transfer stops are performed according to plan. Therefore, solution x^* satisfies all inequality constraints of the mathematical program (Q) including those of Eq.7 and 10 and is a feasible solution of (Q) for the given travel time and dwell time noise instance (ξ^0,ζ^0). □

SOLUTION METHOD

Alternating Optimization

The minimax problem of $(\tilde{Q})$ can be conceptualized as a two-player game where player 1 chooses his/her strategy (i.e., finds the values of the uncertain parameters ξ, ζ from the corresponding uncertainty sets for maximizing the penalized objective function $\tilde{f}(x^0, \xi, \zeta)$, given x^0) and player 2 selects the dispatching times, x, so that the function $f(x, \xi^0, \zeta^0)$ is minimized while satisfying the constraints for a fixed (ξ^0, ζ^0). In this way, the decisions of player 1 and player 2 are interrelated (a decision made by player 1 affects player 2 and vice versa).

Let $x^k,\ k \in \mathbb{N}$ be an *initial guess* of the dispatching time modifications that satisfies Eq.9 and 13. The worst-performing scenario for such solution can be determined from the maximization problem with parameter x^k:

$$\begin{aligned}
(\mathscr{P}(x^k)):\qquad & \max_{\xi,\zeta} && \tilde{f}(x^k, \xi, \zeta) \\
& \text{s.t.:} && \xi_{l,s}^{min} \leq \xi_{l,n,s} \leq \xi_{l,s}^{max} \quad, \quad \forall l \in L, \forall n \in N(l), \forall s \in S(l) \setminus \{1\} \\
& && \zeta_{l,s}^{min} \leq \zeta_{l,n,s} \leq \zeta_{l,s}^{max} \quad, \quad \forall l \in L, \forall n \in N(l), \forall s \in S(l)
\end{aligned} \tag{26}$$

By solving the maximization problem $(\mathscr{P}(x^k))$ the values of the uncertain parameters related to the travel and dwell times that result in a worst-performing scenario for the solution x^k can be determined as ξ^k and ζ^k.

Given the worst-case noise of the uncertainty values (ξ^k, ζ^k) for the realization of x^k, an updated solution, x^{k+1}, can be computed. The updated solution minimizes the optimization problem $(\tilde{P}(\xi^k, \zeta^k))$.

This alternating optimization continues iteratively until a termination criterion is satisfied. The termination criterion can be related to the stability of the solution performance. If, for instance, consecutive solutions $x^{k-q}, \ldots, x^{k-1}, x^k$ have stable worst-case performances, then this can be an indication that there are no further oscillations and the solutions have a relatively stable performance in worst-case scenarios.

This can be summarized in the following algorithm:

Step 0: Choose x^1 that satisfies Eq.9 and 13, set $k := 1$;

Step 1: Solve $\mathscr{P}(x^k)$ and obtain (ξ^k, ζ^k);

Step 2: Solve $\tilde{P}(\xi^k, \zeta^k)$ for (ξ^k, ζ^k) and obtain x^{k+1}, set $k := k+1$;

Step 3: If the performance of the most recent solutions is stabilized, STOP. Else, go to Step 1.

NUMERICAL EXPERIMENTS

Illustrative example in an idealized network

Figure 1 shows the idealized network under consideration. Even though the numerical experiment includes two bus lines, the analysis can be expanded to a full-scale city network without loss of generality.

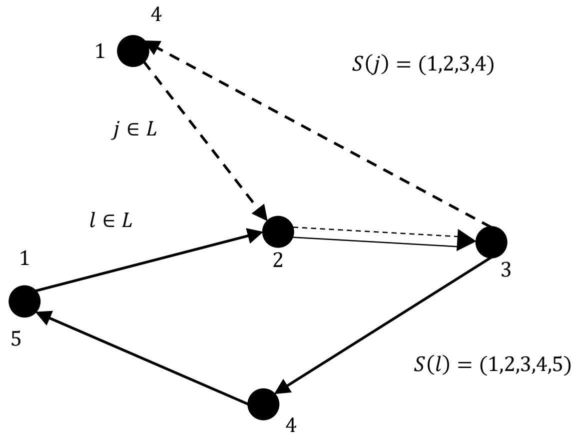

FIGURE 1 : Idealized bus network with two bus lines $l, j \in L$

The idealized bus network has two bus lines. Bus line l serves 5 stops, $S(l) = \{1,2,3,4,5\}$ and line j serves 4 stops, $S(j) = \{1,2,3,4\}$. The transfer stops of bus lines l, j are $B_{lj} = \{2,3\}$. Let assume that bus lines l, j operate three trips each, $N(l) = \{1,2,3\}$ and $N(j) = \{1,2,3\}$ during the planning period, where the first trip of bus line l should be dispatched at δ_l^{min} = 8:00 am (which, if we start counting seconds from the beginning of the day, is 28,800) and the first trip of bus line j at δ_l^{min} = 8:02 am (or 28,920 sec). Let also assume that all bus trips are operated by different buses.

Each trip of bus line l needs to synchronize its arrival time at stops $b \in B_{lj}$ with the arrival time of the corresponding trip of line j (6 synchronizations in total) where a synchronization is successful if the arrival time difference remains within the range $[0, \Delta t]$, where Δt = 300 sec. The ideal time headways between successive bus trips at bus stops are 8 minutes (or $h_l^* = 460$ sec) for line l and 10 minutes (or $h_j^* = 600$ sec) for line j. In addition, to prevent schedule sliding, all trips of bus lines l and j should have been completed before 10:00 am, thus $\delta_l^{max} = \delta_j^{max} = 36,000$ sec. Finally, the expected travel times, dwell times and the respective bounds of the travel and dwell time noises are presented in Table 1.

Note that we start the analysis from a mild scenario where we want to be robust for travel time deviations of up to 1 minute and dwell time deviations of up to 20 sec. Evidently, the bus operator might either prefer to maintain robustness in common-case scenarios where program $(\tilde{Q})$ is optimized considering tight travel time and dwell time noise bounds or might desire to ensure robustness in more extreme scenarios by increasing the travel time and dwell time noise bounds in the optimization (in the latter case, it is expected that the computed schedules will not perform as well as the former ones in practice if the disturbances in the actual operations are minor).

Continuing in the analysis, starting from a randomly selected dispatching time solution, x^1, we need to obtain the worst-case noise (ξ^1, ζ^1). An obvious choice for the initial dispatching time solution is the one that optimizes the normal case (i.e., the case where the travel time and dwell time noises are not considered). This solution can be easily obtained by solving the program $(\tilde{P})$ without the consideration of noise. Our initial dispatching time solution reads:

$$x^1 = \begin{cases} (x_{l,1} = 28800,\ x_{l,2} = 31000,\ x_{l,3} = 33200) \text{ in sec} \\ (x_{j,1} = 28920,\ x_{j,2} = 31120,\ x_{j,3} = 33320) \text{ in sec} \end{cases}$$

TABLE 1 : Travel time and dwell time values for the idealized bus network in seconds

bus line l						bus line j					
Trip	Stop	$t_{l,n,s}$	$k_{l,n,s}$	$[\xi_{l,s}^{min},\xi_{l,s}^{max}]$	$[\zeta_{l,s}^{min},\zeta_{l,s}^{max}]$	Trip	Stop	$t_{j,m,z}$	$k_{j,m,z}$	$[\xi_{j,z}^{min},\xi_{j,z}^{max}]$	$[\zeta_{j,z}^{min},\zeta_{j,z}^{max}]$
n=1	s=1	na	0	na	[0,+20]	m=1	z=1	na	0	na	[0,+20]
	s=2	450	20	[-60, +60]	[-10,+20]		z=2	610	38	[-60,+60]	[-10,+20]
	s=3	445	22	[-60,+60]	[-10,+20]		z=3	480	33	[-60,+60]	[-10,+20]
	s=4	450	20	[-60,+60]	[-10,+20]		z=4	710	26	[-60,+60]	[-10,+20]
	s=5	452	25	[-60,+60]	[-10,+20]						
n=2	s=1	na	0	na	[0,+20]	m=2	z=1	na	0	na	[0,+20]
	s=2	450	24	[-60,+60]	[-10,+20]		z=2	590	46	[-60,+60]	[-10,+20]
	s=3	460	25	[-60,+60]	[-10,+20]		z=3	490	32	[-60,+60]	[-10,+20]
	s=4	445	22	[-60,+60]	[-10,+20]		z=4	745	22	[-60,+60]	[-10,+20]
	s=5	450	25	[-60,+60]	[-10,+20]						
n=3	s=1	na	0	na	[0,+20]	m=3	z=1	na	0	na	[0,+20]
	s=2	450	24	[-60,+60]	[-10,+20]		z=2	630	35	[-60,+60]	[-10,+20]
	s=3	462	22	[-60,+60]	[-10,+20]		z=3	480	41	[-60,+60]	[-10,+20]
	s=4	450	31	[-60,+60]	[-10,+20]		z=4	770	27	[-60,+60]	[-10,+20]
	s=5	448	28	[-60,+60]	[-10,+20]						

To obtain the worst-case noise (ξ^1,ζ^1) for x^1, we need to solve the maximization problem of the program $(\mathscr{P}(x^1))$. The maximization program of $(\mathscr{P}(x^1))$ can be transformed to an equivalent minimization problem:

$$\begin{aligned}(\tilde{\mathscr{P}}(x^1)): \quad & \min_{\xi,\zeta} \quad -\tilde{f}(x^1,\xi,\zeta) \\ & \text{s.t.:} \quad \xi_{l,s}^{min} \leq \xi_{l,n,s} \leq \xi_{l,s}^{max} \quad , \quad \forall l \in L, \forall n \in N(l), \forall s \in S(l) \setminus \{1\} \\ & \qquad\;\; \zeta_{l,s}^{min} \leq \zeta_{l,n,s} \leq \zeta_{l,s}^{max} \quad , \quad \forall l \in L, \forall n \in N(l), \forall s \in S(l)\end{aligned} \tag{27}$$

where $\tilde{\mathscr{P}}(x^1)$ seeks to minimize a concave function over a convex polyhedron. This problem can be solved to optimality using an algorithm for nonlinear optimization such as the Limited-memory Broyden-Fletcher-Goldfarb-Shanno algorithm that accepts Bounds (L-BFGS-B) and uses derivatives of the penalized objective function as a key driver of the algorithm to identify the direction of steepest descent, and also to form an estimate of the Hessian matrix (see Byrd et al. (*46*)). An implementation of this algorithm in Python 3.4 via SciPy yields travel time and dwell time noise solutions with a penalized objective function value of 1.727E+10 which indicates that if the bus operator uses the dispatching times x^1, at the worst-case scenario of travel time and dwell time noise: (i) the (average) excessive waiting time of a typical passenger at each stop will be 2.66 minutes; (ii) 6 out of the 6 synchronizations will be missed where all three trips of line l will arrive earlier to transfer stops $B_{lj}=\{2,3\}$ than the corresponding trips of line j by 430, 633, 410, 612, 450 and 629 sec, respectively; and (iii) no schedule sliding violations will occur.

Solving program $(\tilde{P}(\xi^1,\zeta^1))$ for the worst-case noise that was computed above for x^1 will give us a new solution x^2 and this procedure can continue iteratively until convergence as shown in Fig.2. From Fig.2 one can observe that after some initial oscillations, a robust solution is obtained in iteration 12. For this dispatching time solution, the worst-case value of the penalized objective function is $0.701E+10$ (a 59% improvement from the worst-case performance of the initial dispatching time solution). After iteration 12, this solution is not improved any further because the minimax game between the two players has reached an equilibrium (neither the player that controls the dispatching times nor the player that controls the travel and dwell time disturbances is willing to change strategy because there is no foreseeable payoff and both players act rationally).

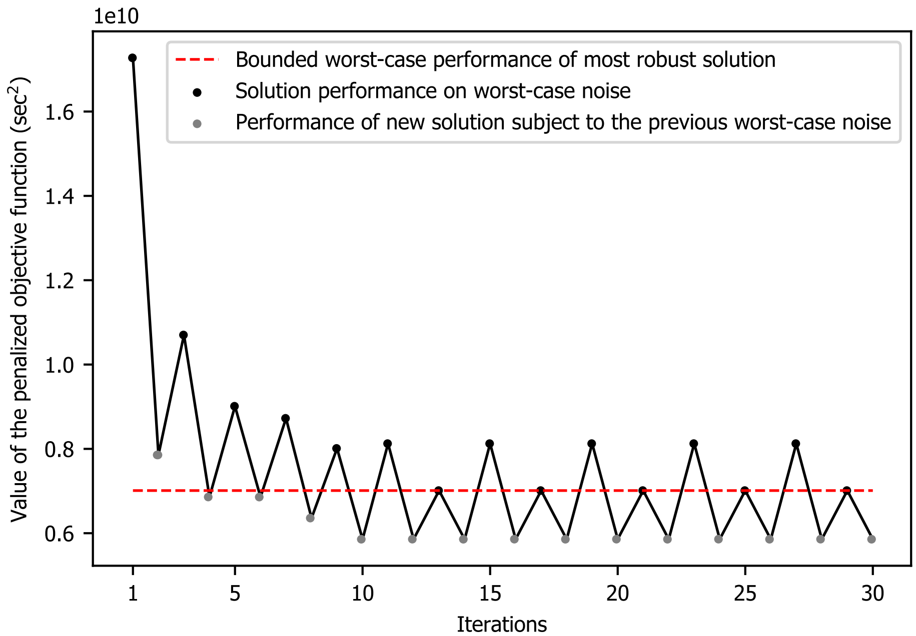

FIGURE 2 : Convergence of the alternating optimization. The robust solution reduces the worst-case penalized objective function value from $1.727E+10$ to $0.701E+10$

The robust solution for the travel time and dwell time noise scenario of Table 1 that appears for the first time at the 12-th iteration in Fig.2 is:

$$x^* = \begin{cases} (x_{l,1} = 28800,\ x_{l,2} = 30835,\ x_{l,3} = 32616) \text{ in sec} \\ (x_{j,1} = 28920,\ x_{j,2} = 30661,\ x_{j,3} = 32406) \text{ in sec} \end{cases}$$

At the presence of the worst-case travel time, the robust solution results in an (average) excessive waiting time for a typical passenger of 2.125 minutes and allows the synchronization of the second trip of line l with the second trip of line j at both transfer stops. In addition, the deviation of the arrival times of trips at transfer stops are closer to the synchronization ranges (the worst-case deviation reduces from 430+633+410+612+450+629=3164 sec to 430+633+0+0+117+319=1499 sec).

Application for two bi-directional lines in Stockholm

In this application, we solve the robust synchronization for two bi-directional bus lines in Stockholm (l is bus line 1 and j is bus line 4). As illustrated in Fig.3, bus line l comprises of direction 1 (Essingetorget to Stockholm Frihamnen) and direction 2 (Stockholm Frihamnen to Essingetorget).

Bus line j comprises of direction 1 (Gullmarsplan to Radiohuset) and direction 2 (Radiohuset to Gullmarsplan).

The planning period of this experiment is from 2:00pm to 7:30pm because this is one of the four periods of the day where a uniform frequency is set. There are five transfer stops between the two bi-directional services: {Västerbroplan, Mariebergsgatan, Fridhemsplan T-bana, Jungfrugatan, Värtavägen}.

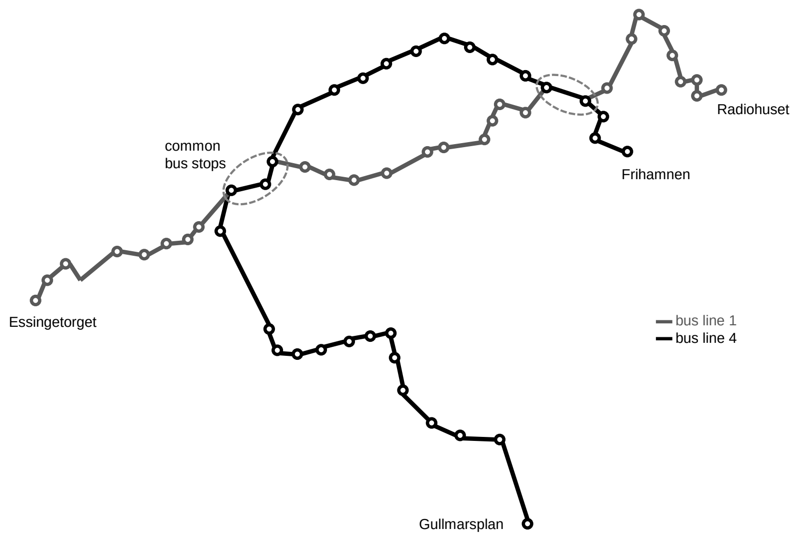

FIGURE 3 : Bus lines 1 and 4 in Stockholm

Note that our robust optimization method can be applied even if the historical travel and dwell times do not follow a *specific probability distribution*. Consequently, we can directly use historical data as input in our minimax problem without defining the respective probability distributions. In an illustrative example, we present the historical travel times of a trip of line 1 between the first two bus stops and the resulting Tukey boxplot McGill et al. (*47*) in Fig.4.

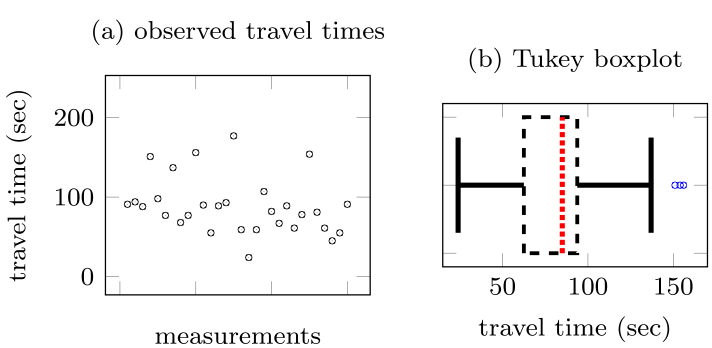

FIGURE 4 : (a) Example of interstation travel time observations and (b) resulting Tukey boxplot

In Fig.4b, the red line in the boxplot is the median of the dataset. The bounds of the box are the lower and upper quartile Q_1 and Q_3. The lower and upper whiskers are the minimum (lowest datum still within 1.5 the interquartile range (IQR) of the first quartile), and the maximum (highest datum still within 1.5 IQR of the third quartile). In addition, the travel time observations in blue are outliers.

Defining realistic lower and upper limits for the travel time and dwell time noises, $(\xi_{l,s}^{min}, \xi_{l,s}^{max})$, $(\zeta_{l,s}^{min}, \zeta_{l,s}^{max})$, plays an important role in finding robust designs. By definition, a robust design has the best performance at the worst-case scenario. The worst-case scenario depends on the adversary (in our case, the travel and dwell time noise). If we impose strict limitations on our adversary (i.e., consider that the travel and dwell times are always equal to their expected values), this will result in designs that perform well on average, but are not able to cope with changes. In contrary, if our adversary is unlimited (i.e., the travel times are allowed to take unrealistically high values), our robust design will perform the best at scenarios that never occur in practice whereas it might underperform in common-case scenarios.

To examine the importance of the limits of the adversary in a robust design, we consider three scenarios:

(i) the adversary is inactive. I.e., we consider only the expected travel and dwell times (red lines in Tukey boxplot)

(ii) the environmental variables of the adversary are allowed to take any value from the lower to the upper quartile $Q_1 - Q_3$

(iii) the environmental variables of the adversary are allowed to take any value from the lower to the upper whisker

Scenario (i) does not yield a robust design because we do not have an adversary (there is no travel or dwell time noise). Scenarios (ii) and (iii) are robust designs where the adversary is more limited in (ii) and has more loose restrictions in (iii).

To investigate the performance of different robust designs in realistic operations, we sample actual values from the observed travel and dwell times using 1 month of Automated Vehicle Location (AVL) and Automated Passenger Count (APC) data (15 Nov 2011-15 Dec 2011). For each one of the 30-day travel time and dwell time scenarios that are based on real data, we evaluate the performance of designs (i), (ii), (iii). After applying designs (i), (ii), (iii) at each day, the results in terms of (average) passenger excessive waiting times and waiting times at transfer stops due to missed synchronizations are presented in Table 2. The improvement of the median (that indicates the potential performance improvement on the average case) and the upper whisker (that indicates the potential performance improvement at the worst-case scenario) when applying designs (i), (ii) and (iii) is summarized in Fig.5.

TABLE 2 : Validation results

	Average Excessive Waiting Time per passenger (min)				
	lower whisker	Q_1	median	Q_3	upper whisker
Design (i)	1.443	1.583	1.626	1.673	1.810
Robust Design (ii)	1.372	1.495	1.542	1.587	1.710
Robust Design (iii)	1.442	1.622	1.654	1.661	1.681
	Average Waiting Time for Transferring (min)				
	lower whisker	Q_1	median	Q_3	upper whisker
Design (i)	1.888	2.229	2.381	2.579	2.910
Robust Design (ii)	1.511	1.692	1.710	1.984	2.811
Robust Design (iii)	1.819	2.231	2.403	2.468	2.581

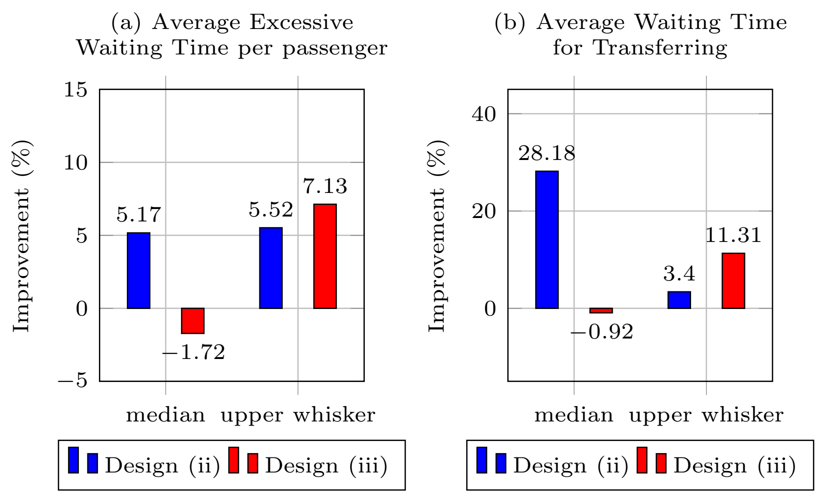

FIGURE 5 : Validation results: investigating the potential improvement of robust designs (ii) and (iii) compared to design (i)

Fig.5 indicates that design (i) is inferior to the robust designs by 3.4%-11.31% in extreme scenarios. This is in line with the results reported from the daily operations of schedules that are optimized for the average case without considering potential travel/dwell time fluctuations Gkiotsalitis and Maslekar (*14*).

Allowing the adversary to take more extreme values (i.e., design (iii)) will result in a robust design that:

- performs better at extreme scenarios (upper whisker improved by 7.13% and 11.31%, respectively);
- might exhibit similar performances to design (i) on the average-case (median deterioration of 1.72% and 0.92% for excessive waiting time and transfer waiting time, respectively).

In contrast, allowing the adversary to take more common values (design (ii)) will result in a robust design that:

- performs better in common-case scenarios (significant median improvement by 5.17% and 28.18%, respectively);
- yields improvements in extreme scenarios (upper whisker improved by 5.52% and 3.40%, respectively).

Therefore, it is evident that the limits of the adversary when we determine a robust design play an important role in the performance of our design in real operations. This can be exploited by bus operators who might prefer robust designs that perform better in common case scenarios or robust designs that are more resilient to severe disruptions.

CONCLUDING REMARKS

This study formulated the multi-line synchronization problem considering the potential variability in the travel and dwell times of daily trips, the regularity of individual bus lines and the operational regulatory constraints such as schedule sliding prevention and layover time limits. After proving that for some travel and dwell time noise levels schedule sliding and missed synchronizations cannot be prevented, a flexible problem formulation was introduced that incorporated the constraint violations with the use of penalties.

Solving the resulting mathematical program in a small-scale, idealized network with alternating optimization, it was demonstrated that at some point the minimax problem reaches an equilibrium for which neither the "decision-maker" that selects the dispatching times of trips, nor the "decision-maker" that selects the travel and dwell time disturbances is willing to change strategy. In a further application in two bus lines in Stockholm with five transfer stops, it is clear that there is a trade-off between: (a) robust designs that impose stricter limits to the adversary and result in solutions that perform better at common-case scenarios, and (b) robust designs that prepare for a wide-range of values of the adversary and overperform at extreme-case scenarios. This sensitivity of the generated robust designs to the limitations of the adversary can be exploited by bus operators to generate designs that fit their specific needs/preferences.

In future studies, a broader set of robust timetables can be examined by solving the mathematical program $(\tilde{Q})$ for different percentages of travel and dwell time deviations from the average case and selecting the "dominant" solution(s) that yield the highest payoffs in terms of service regularity and synchronization improvements at both the common-case scenarios and the abnormal ones.

Author Contribution Statement

The authors confirm contribution to the paper as follows: study conception and design: K. Gkiotsalitis, O. Cats; data collection: O. Cats; analysis and interpretation of results: K. Gkiotsalitis, O.A.L. Eikenbroek, O. Cats; draft manuscript preparation: K. Gkiotsalitis, O.A.L. Eikenbroek, O. Cats. All authors reviewed the results and approved the final version of the manuscript.

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