An exact method for real-time rescheduling after disturbances in metro lines

Konstantinos Gkiotsalitis
Assistant Professor
University of Twente

Center for Transport Studies (CTS)
Department of Civil Engineering
P.O. Box 217
7500 AE Enschede
The Netherlands
Email: k.gkiotsalitis@utwente.nl

Oskar A.L. Eikenbroek
PhD Candidate
University of Twente

Center for Transport Studies (CTS)
Department of Civil Engineering
P.O. Box 217
7500 AE Enschede
The Netherlands
Email: o.a.l.eikenbroek@utwente.nl

Oded Cats
Associate Professor
Delft University of Technology

Department of Transport and Planning
P.O. Box 5048
2600 GA Delft
The Netherlands
Email: o.cats@tudelft.nl

99th Annual Meeting of the Transportation Research Board

Washington D.C., USA
January 2020

ABSTRACT
This study proposes an exact model for timetable recovery after disturbances. Our model is applicable to high frequency services that operate under frequencies of at least 5 trips per hour. The objective of our model is the minimization of the deviation between the actual headways and their planned (target) values - a typical objective in high frequency services that indicates the service regularity. In the formulation of the timetable recovery model, we focus on metro lines with stable dwell times at stations that are not sensitive to changes in passenger demand. The resulting model is nonlinear and non-smooth; thus, it cannot be solved to optimality. To rectify this, we propose a model reformulation using slack variables. The reformulated program is equivalent to the original one and can be solved to global optimality in real time with exact optimization methods for quadratic programming. With our model, we investigate how many upstream trips should be rescheduled to respond to a service disturbance using real data from the red metro line in Washington D.C. Our experiments demonstrate an improvement potential of the service regularity by up to 30% if we reschedule the five upstream trips of a disrupted train.

Keywords: rescheduling; high-frequency services; disturbance management; metro recovery; regularity-based services

INTRODUCTION

The planning of metro lines consists of the strategic stage (determination of stops and routes), tactical stage (frequency settings, timetable design, crew and vehicle schedules), and the operational stage (timetable rescheduling, holding, short-turning/stop-skipping). Metro line operators in dense urban areas operate under regularity-based schemes that aim at maintaining the headways among successive trips Zhu and Goverde (*1*). To achieve this, timetables are developed considering robustness to travel time variations and are frequently rescheduled to adapt to operational disturbances (i.e., unexpected travel or dwell times). The real-time rescheduling of timetables is classified under the category of *disturbance* management and differs from *disruption* management which deals with large incidents, like station closures, and requires further control measures such as the cancellation of trips, re-routing or short-turnings Jespersen-Groth et al., Cacchiani et al. (*2, 3*).

In Zilko et al. (*4*) and Ghaemi et al. (*5*), methods for the reliable disruption length estimation were proposed to reduce the negative impact inflicted by a disruption. Zhu and Goverde (*1*) proposed a mixed integer linear programming (MILP) model that allows to skip/add stops or perform short-turnings when a disruption occurs. Other studies propose contingency plans based on the original timetable to deal with disruptions, e.g. Chu and Oetting (*6*). Since extensive disruptions and disruption management are not the primary focus of this work, we refer the interested reader to the comprehensive literature study of Ghaemi et al. (*7*) on disruption management.

In this study, we focus on the problem of disturbance management to produce an efficient model that can react to disturbances and re-time the dispatching times of trips in real-time. Solutions to this problem commonly adopt local re-timing to adjust the timetable, see D'Ariano et al., Corman et al., Meng and Zhou (*8, 9, 10*). D'Ariano et al. (*8*) aimed at improving the punctuality of trains by routing and sequencing trains in an iterative manner - first, an optimal train sequencing was produced for the given train routes, and then this solution was improved by locally rerouting some trains. Their solution method was based on local search and Branch and Bound (B&B) given the discrete nature of the problem. This work was extended in Corman et al. (*9*) incorporating effective rescheduling algorithms and local rerouting strategies in a Tabu search scheme. Corman et al. (*9*) alternated between a fast heuristic and a truncated B&B algorithm for computing train schedules within a short computation time, without guaranteeing the convergence to a globally optimal solution.

Pellegrini et al. (*11*) aimed at minimizing delays after an unexpected disturbance perturbs the operations by seeking the best train routing and scheduling. The proposed model was a mixed integer linear program, representing the infrastructure with fine granularity. Solving the model of Pellegrini et al. (*11*) to global optimality is not feasible in most of the cases, and the reported computational costs are typically beyond one and a half minute, even with the use of heuristics.

Most relevant to our work, Krasemann (*12*) focuses solely on the timetable rescheduling after a disturbance. In Krasemann (*12*), the rescheduling problem was not solved to global optimality. Instead, a greedy heuristic was introduced to ensure that a (hopefully) good-enough solution is obtained within a short time (within 30 seconds). To this end, Krasemann (*12*) introduced directly a heuristic solution method without modeling the timetable rescheduling problem as a mathematical program.

Apart from the works on disturbances at rail operations, several algorithms for the recovery of timetables have been developed for bus operations, which are inherently prone to disturbances given that they typically operate in mixed-traffic environments, e.g. Gkiotsalitis and Cats, Gkiotsalitis et al., Gkiotsalitis and Van Berkum (*13, 14, 15*). Bus operators apply dynamic control strategies such as stop-skipping (Sun and Hickman, Liu et al., Chen et al., Gkiotsalitis (*16, 17, 18, 19*)), bus holding (Newell, Hernández et al., Wu et al., Gavriilidou and Cats, Gkiotsalitis and Cats (*20, 21, 22, 23, 24*)) or rescheduling (Adamski and Turnau, Strathman et al. (*25, 26*)) to improve the service regularity. Nevertheless, rescheduling methods typically resort to heuristics, such as evolutionary optimization approaches, to obtain a solution in real-time given the computational complexity of the problem Gkiotsalitis and Alesiani (*27*).

Unlike past works that propose complex mixed integer programs that cannot be easily solved in real-time or heuristics that do not return a globally optimal solution, in this study we propose a novel quadratic programming formulation which is proved to be convex and can reschedule a timetable every time a disturbance occurs. The contributions of our work to the state of the art are:

- a novel mathematical program for train rescheduling that can be solved in real-time;
- problem reformulation and a solution approach that guarantees convergence towards a globally optimal solution.

The remainder of this study is structured as follows: in section 2, we formulate our problem and we introduce the objectives and constraints of our main mathematical model. The mathematical model is presented in section 3 and is reformulated to ensure its feasibility after relaxing its soft constraints. This mathematical model is non-smooth and its objective function is not differentiable at every point of its domain - prohibiting the application of an exact solution method. To rectify this, in section 4 we propose a model reformulation with the introduction of slack variables. The reformulated program is proved to have a globally optimal solution and can be easily solved with exact optimization methods. A detailed demonstration of our model in a toy network is presented in section 5. This demonstration facilitates the reproduction of our work. In addition, the application of our approach to the red metro line in Washington D.C. is presented in the same section demonstrating that we can achieve a significant benefit if we reschedule the dispatching times of up to 5 upstream trips of a train that exhibits a disturbance. Finally, section 6 provides the concluding remarks and discusses the future direction.

PROBLEM DEFINITION

Trip re-indexing after a disturbance

We consider a metro line operating in a loop that serves a set of ordered stations $S = \langle 1, 2, ..., s, ... \rangle$ with 1 being the dispatching station (see Fig.1). The ordered set of daily trips operating in this line is $\mathbf{N}$. When the operations of a trip $m \in \mathbf{N}$ are disturbed, the set of its following up trips (henceforth referred as *upstream* trips) is $\mathbf{N}_m = \{m+1, m+2, ...\}$ with $\mathbf{N}_m \subset \mathbf{N}$. To alleviate the effects of the disturbance, the dispatching times of all trips $j \in \mathbf{N}_m$ are modified following a headway-based optimization process.

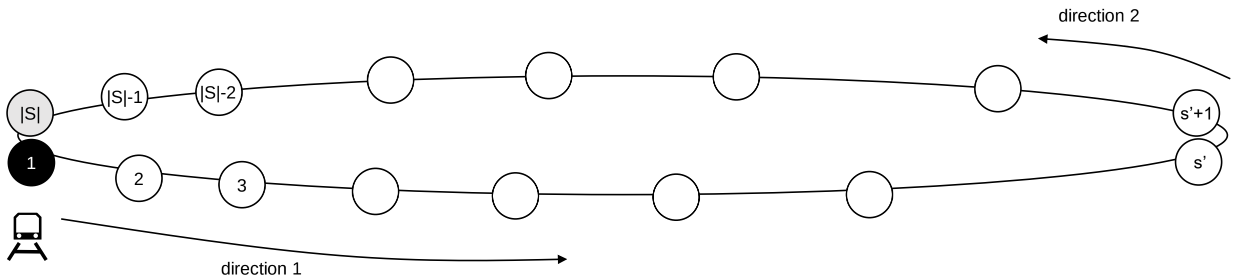

FIGURE 1 : Illustration of loop-formed metro line

The choice of the length of set $\mathbf{N}_m$ has significant practical implications. For instance, if $\mathbf{N}_m = \{m+1, m+2, ...\}$ contains all remaining daily trips the computational cost of rescheduling their dispatching times increases. In addition to the increase of the computational costs, modifying the dispatching times of trips that are expected to be dispatched in the far future might be proved irrelevant if further disturbances occur in the near future and we need to re-optimize. For this reason, the sensitivity of the solution performance to the number of rescheduled upstream trips, $\mathbf{N}_m$, is investigated in our case study.

Let us re-index set $\mathbf{N}_m = \{m+1, m+2, ...\}$ into $\mathbf{N}_m = \{0, 1, ..., n\}$ where trip 0 has already been dispatched and exhibits a disturbance, and n the last trip in $\mathbf{N}_m$. Thus, trips 0 and $n+1$ are the "boundaries" of our problem because their dispatching times cannot be modified.

Let $\delta_j \in \mathbb{R}_{\geq 0}$ denote the originally planned dispatching time of each trip $j \in \{1, 2, ..., n\}$ (the time reported at the timetable). Then, $\delta_1 < \delta_2 < ... < \delta_j < ... < \delta_n$. The decision variable is an n-valued vector $\mathbf{x} = \{x_1, x_2, ..., x_n\}$ which expresses the dispatching time modification (offset) of each trip $j \in \mathbf{N}_m$, where $\mathbf{x} \in \mathbb{R}^n$. Thus, the adjusted dispatching times of trips $\{1, 2, ..., n\}$ are $\{\delta_1 + x_1, \delta_2 + x_2, ..., \delta_n + x_n\}$.

Before introducing the vehicle motion law that determines the train trajectories, we list the main assumptions from past literature that also apply to our study:

1. Dwell times are pre-determined at the station level and are not influenced by the passenger demand due to service provider policies or opening/closing the doors automatically.

2. Service supply is determined at the frequency settings stage and ensures that the passenger demand can be accommodated even at the maximum load point Marguier, Lin and Wilson, Eberlein et al., Daganzo (*28, 29, 30, 31*). That is, dispatching time changes will not lead to overcrowding because the service supply can accommodate the demand.

Vehicle trajectories

With assumptions 1-2, we can outline a set of rules governing the vehicle movements. For this, we briefly introduce the notation in Table 1.

Sets	
$S=\{1,...,s,...\}$	ordered set of consecutive stations;
$\mathbf{N}_m=\{1,2,...,n\}$	ordered set of future trips under consideration;
Indices	
j	index of vehicles;
s	index of stations;
Parameters	
h_{min}, h_{max}	Minimum and maximum headways (agency's requirements) between two successive dispatches from the first station
β_j	latest possible dispatching time of any trip $j \in N_m$ to avoid schedule sliding and/or disrupting crew schedules
π_j	the earliest possible dispatching time of any trip $j \in N_m$ to ensure vehicle circulation
δ_j	originally planned dispatching time of trip $j \in N_m$
h^*_{js}	scheduled (target) headway of trip $j \in N_m$ at station $s \in \{2,3,...,\lvert S\rvert-1\}$
$t_{j,s}$	expected inter-station travel time of trip $j \in N_m$ from station s to station $s+1$
$k_{j,s}$	pre-determined dwell time of trip $j \in N_m$ at station $s \in \{2,3,...,\lvert S\rvert\}$
$\bar{a}_{0s}$	the realized arrival times of trip 0 at stations $s=\{2,3,...,\lvert S\rvert\}$
$\bar{\delta}_0$	the realized dispatching time of trip 0
Decision Variables	
$\{x_1,...,x_n\}$	the dispatching time offsets of trips $j \in \{1,2,...,n\}$ from δ_j;
Variables	
$a_{j,s}$	arrival time of trip j at station s, where $s \in \{2,3,...,\lvert S\rvert\}$
h_{js}	headway between trips j and $j-1$ at station s.

TABLE 1 : Nomenclature

The expected arrival time $a_{j,s}$ of a trip $j \in \{1,2,...,n\}$ at station $s \in \{2,3,...,|S|\}$ is

$$a_{j,s} := (\delta_j + x_j) + \sum_{\phi=1}^{s-1} \tau_{j,\phi} + \sum_{\phi=2}^{s-1} k_{j,\phi} \tag{1}$$

where δ_j is the planned dispatching time, x_j the dispatching offset, $\tau_{j,\phi}$ the travel time from station ϕ to $\phi+1$ and $k_{j,\phi}$ the dwell time at station ϕ. In addition, $\sum_{\phi=1}^{s-1}\tau_{j,\phi}$ is the total travel time from the first station until station s and $\sum_{\phi=2}^{s-1}k_{j,\phi}$ the accumulated dwell times from stations $2,3,...,s-1$. Note that the dwell times are aggregated starting from station 2.

From Eq.(1), the arrival time of each trip $j \in N_m$ at each station s varies according to the decision variable values of x_j. Therefore, Eq.(1) can be succinctly written as:

$$a_{j,s} := x_j + c_{j,s},\ \forall j \in N_m, \forall s \in \{2,3,...,|S|\} \tag{2}$$

where

$$c_{j,s} := \delta_j + \sum_{\phi=1}^{s-1} \tau_{j,\phi} + \sum_{\phi=2}^{s-1} k_{j,\phi},\ \forall j \in N_m, \forall s \in \{2,3,...,|S|\} \quad (3)$$

The time headway $h_{j,s}$ of two successive trips $j-1$ and $j \in N_m \setminus \{1\}$ at the time of their arrival at station $s \in S \setminus \{1\}$ is defined as

$$h_{j,s} := a_{j,s} - a_{j-1,s} = (x_j + c_{j,s}) - (x_{j-1} - c_{j-1,s}),\ \ \forall j \in \{2,3,...,n\}, \forall s \in \{2,3,...,|S|\} \quad (4)$$

Boundary conditions
Eq.(1) and (4) are interrelated and express the dynamic equations of the movement of trains. To apply them in practice, initial conditions are required. The first initial condition is that the arrival times $\bar{a}_{0,s}$ of trip 0 at stations $s \in \{2,\dots,|S|\}$ are not affected by the decision variables since trip 0 has already been dispatched.

Incorporating this initial condition, the time headways between trip 1 and its preceding one, 0, are:

$$h_{1,s} := x_j + c_{j,s} - \bar{a}_{0,s}\ , \forall\ s \in \{2,3,...,|S|\} \quad (5)$$

Thus, Eq.(5) links the time headways between trip 1 and 0 with the dispatching time offset of trip 1, x_1.

Constraints
A first constraint is imposed by the latest possible dispatching time of a trip. While some works compute the vehicle and crew schedules together with the departure times of trips Walker et al. (*32*), most works treat them separately Krasemann (*12*). Therefore, it is not practical to delay the dispatching time of a trip further than a certain threshold because that would lead to schedule sliding. If β_j is that pre-planned threshold, then

$$\delta_j + x_j \leq \beta_j,\ \ \forall j \in N_m \quad (6)$$

Additionally, agencies have specific requirements on the minimum and maximum allowable dispatching headway, h_{min}, h_{max}, to ensure a minimum level of service Ceder (*33*). This constraint can be expressed as:

$$h_{min} \leq (\delta_j + x_j) - (\delta_{j-1} + x_{j-1}) \leq h_{max}\ \ \forall j \in \{2,3,...,n\} \quad (7)$$

and, in the boundary case where $j = 1$,

$$h_{min} \leq (\delta_j + x_j) - \bar{\delta}_0 \leq h_{max} \quad (8)$$

where $\bar{\delta}_0$ is the realized dispatching time of trip 0.

Lastly, to ensure the circulation of vehicles, each trip j should depart after time π_j which is the time that the vehicle and driver assigned to perform trip j have completed their previous trip and are ready to start trip j. The vehicle circulation constraint is modeled as:

$$\delta_j + x_j \geq \pi_j \;\; \forall j \in \{1,2,3,...,n\} \tag{9}$$

Objective Function

In high-frequency services, each one of the trips $\{1,2,...,n\}$ has a target headway $h^*_{j,s}$ with its leading train at any station $s \in \{2,...,|S|-1\}$ Gkiotsalitis et al. (*34*). The target headway is determined at the tactical planning stage and should be maintained during the daily operations Trompet et al. (*35*).

When striving to maintain the target (ideal) headways, the objective is to minimize the headway variance around the target values. To achieve that, the optimal dispatching offset $x = \{x_1, x_2, ..., x_n\}$ should be the solution of:

$$\min_{\mathbf{h}} \sum_{s=2}^{|S|-1} \sum_{j=1}^{n} \left(h_{j,s} - h^*_{j,s}\right)^2 \tag{10}$$

which expresses the variance of headways around their target values which, in the ideal case, can be equal to zero.

Eq.(10) can be equivalently expressed as:

$$\begin{aligned} \min_{x} f(x) := &\sum_{s=2}^{|S|-1} \left((x_1 + c_{1,s}) - \bar{a}_{0,s} - h^*_{1,s}\right)^2 + \\ &\sum_{s=2}^{|S|-1} \sum_{j=2}^{n} \left((x_j + c_{j,s}) - (x_{j-1} + c_{j-1,s}) - h^*_{j,s}\right)^2 \end{aligned} \tag{11}$$

MATHEMATICAL PROGRAM

Combining the expected trajectories of future trips and the objective function yields our main mathematical program which can be written as:

$$
\begin{aligned}
(Q):\min_{x} & \sum_{s=2}^{|S|-1}\big((x_1+c_{1,s})-\bar{a}_{0,s}-h^*_{1,s}\big)^2+\sum_{s=2}^{|S|-1}\sum_{j=2}^{n}\big((x_j+c_{j,s})-(x_{j-1}+c_{j-1,s})-h^*_{j,s}\big)^2\\
\text{s.t. } & h_{min}\leq(\delta_j+x_j)-(\delta_{j-1}+x_{j-1}),\forall j\in\{2,3,...,n\}\\
& (\delta_j+x_j)-(\delta_{j-1}+x_{j-1})\leq h_{max},\forall j\in\{2,3,...,n\}\\
& h_{min}\leq(\delta_1+x_1)-\bar{\delta}_0\\
& (\delta_1+x_1)-\bar{\delta}_0\leq h_{max}\\
& (\delta_j+x_j)\leq\beta_j,\ \forall j\in\{1,2,...,n\}\\
& (\delta_j+x_j)\geq\pi_j,\ \forall j\in\{1,2,...,n\}\\
& c_{j,s}=\delta_j+\sum_{\phi=1}^{s-1}\tau_{j,\phi}+\sum_{\phi=2}^{s-1}k_{j,\phi},\ \forall j\in N_m,\forall s\in\{2,3,...,|S|\}\\
& x_j\in\mathbb{R},\ \forall j\in\{1,2,...,n\}
\end{aligned}
\tag{12}
$$

Infeasibility and hard/soft constraints

Program (Q) can be succinctly written as:

$$
\begin{aligned}
(Q):\quad & \min_{x}\ f(x)\\
& \text{s.t. } x\in\mathscr{F}:=\{x\mid x \text{ satisfy Eq.(3),(6)-(9),(11)}\}
\end{aligned}
\tag{13}
$$

where $\mathscr{F}$ is the feasible set. Note that from the above constraints, the equality constraints of Eq.(3),(11) should be always satisfied because they are physical (hard) constraints that set the values of arrival times, headways and the objective function. Additionally, the circulation inequality constraints of Eq.(9) are also hard constraints because a trip cannot start if a vehicle/driver is not yet available. In contrast, the inequality constraints of Eq.(6),(7),(8) cannot be always satisfied at the same time and some of them should be prioritized in the expense of others. Indeed, program (Q) might not have a feasible solution for some values of the inequality constraints of Eq.(6),(7),(8) yielding an empty feasibility set $\mathscr{F}$ (refer to Lemma .1 in the Appendix).

Therefore, we relax the inequality constraints of schedule sliding presented in Eq.(6) which become soft constraints and are allowed to be violated under certain circumstances. Soft constraints are typically treated as penalty terms and are added to the objective function (see Li and Manya (*36*)). In this way, program (Q) that, under certain circumstances has no feasible solution, can be transformed to program $(\bar{Q})$ by relaxing the inequality constraints of Eq.(6) and adding a violation penalty to the objective function. This approach will ensure that the constraints of Eq.(6) are satisfied when possible, or violated as little as possible when there is no feasible solution. To this end, their relative importance is weighted by introducing a very large number $M\in\mathbb{R}_{\geq 0}$ which ensures that the satisfaction of the schedule sliding constraints prioritized over the objective function:

$$(\bar{Q}): \quad \min_{x} \; f(x) + \sum_{j \in N_m} M \max(\delta_j + x_j - \beta_j, 0) \\ \text{s.t. } \; x \in \mathscr{F} := \{x \mid x \text{ satisfy Eq.(3),(7)-(9),(11)}\} \tag{14}$$

The penalty term $M\max(\delta_j + x_j - \beta_j, 0)$ ensures that the soft constraint $\delta_j + x_j \leq \beta_j$ is prioritized over $f(x)$. Indeed, if $\delta_j + x_j \leq \beta_j$ for some x_j, then x_j does not add any penalty to the objective function since $M\max(\delta_j + x_j - \beta_j, 0) = 0$. In reverse, when $\delta_j + x_j > \beta_j$ for some x_j, then the penalty term penalizes the objective function by a very large number $M\max(\delta_j + x_j - \beta_j)$ and directs the solution search towards another solution that reduces the value of $M\max(\delta_j + x_j - \beta_j, 0)$.

Program $\bar{Q}$ is a nonlinear programming (NLP) problem. Additionally, our new objective is a non-smooth function because of the non-smooth term $\sum_{j \in N_m} M\max(\delta_j + x_j - \beta_j, 0)$; hence, the objective function of $\bar{Q}$ is not differentiable at every point of its domain. This results in a nonlinear, non-convex function that cannot be solved to global optimality with exact optimization methods. As a remedy, we propose a reformulation to cast the problem as an easier-to-solve quadratic program.

REFORMULATION TO A QUADRATIC PROGRAM AND EXACT SOLUTION

The "max" term of $\sum_{j \in N_m} M\max(\delta_j + x_j - \beta_j, 0)$ makes the objective function of program $\bar{Q}$ non-smooth. To rectify this, we implement the "max" penalty by introducing a new set of variables $\nu_j,\ j \in N_m$ that, due to their bounds and the direction of optimization, will take the value $\sum_{j \in N_m} M\max(\delta_j + x_j - \beta_j, 0)$ at the solution. The reformulated program is:

$$(\tilde{Q}): \quad \min_{x,\nu} \; f(x) + \sum_{j \in N_m} M\nu_j \\ \text{s.t. } \; x \in \mathscr{F} := \{x \mid x \text{ satisfy Eq.(3),(7)-(9),(11)}\} \\ \nu_j \geq 0, \; \forall j \in N_m \\ \nu_j \geq \delta_j + x_j - \beta_j, \; \forall j \in N_m \tag{15}$$

which is reduced to a quadratic program (QP). As shown in the Appendix (Theorem .2), program $(\tilde{Q})$ is strictly convex and can be easily solved to global optimality since any locally optimal solution returned by a quadratic programming solver is also a globally optimal one.

NUMERICAL EXPERIMENTS

Demonstration for a toy network

To demonstrate the application of our mathematical program $(\tilde{Q})$ on timetable recovery, we introduce a small-scale idealized scenario (toy scenario). Trip 0 has already been dispatched at time $\bar{d}_0 = 0$ s and has exhibited a disturbance. Its arrival times in our 4-stop toy network are $a_{0,2} = 900$ s and $a_{0,3} = 1600$ s. Note that we only report the arrival times at the 2nd and the 3rd station because the service regularity in $f(x)$ is not measured at the first/last station (Fig.2).

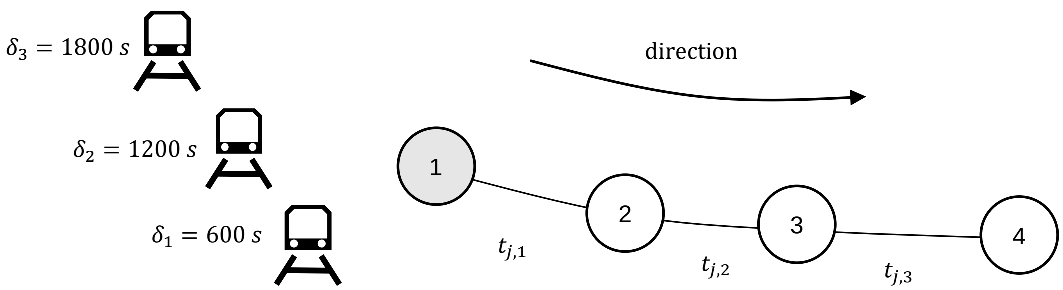

FIGURE 2 : Toy metro line

To proceed with the timetable recovery, we are allowed to modify the dispatching times of its three following trips (namely 1, 2 and 3). The originally planned dispatching times of trips 1, 2 and 3 are: $\delta_1 = 600$ s, $\delta_2 = 1200$ s and $\delta_3 = 1800$ s.

The expected inter-station travel times of trips 1, 2 and 3 are:

$$(\tau_{1,1}, \tau_{1,2}, \tau_{1,3}) = (900, 720, 800) \text{ s}$$
$$(\tau_{2,1}, \tau_{2,2}, \tau_{2,3}) = (920, 700, 800) \text{ s}$$
$$(\tau_{3,1}, \tau_{3,2}, \tau_{3,3}) = (880, 640, 800) \text{ s}$$

The target time headways are 10 minutes, thus $h^*_{j,s} = 600$ s, $\forall j \in \{1,2,3\}$, $\forall s \in \{2,3\}$. In addition, the minimum and maximum dispatching headways are $(h_{min}, h_{max}) = (300$ s, 900 s$)$. The pre-determined times a metro train remains at the station for boardings and alightings is $k_{j,s} = 30$ s $\forall j \in \{1,2,3\}, \forall s \in \{2,3\}$. Due to the vehicle circulation, the earliest possible dispatching times of trips 1,2,3 are $(\pi_1, \pi_2, \pi_3) = (600$ s,1220 s,1820 s$)$. Finally, to avoid schedule sliding, the latest dispatching times of our trips are $(\beta_1, \beta_2, \beta_3) = (660, 1260, 1860)$ s.

Our mathematical model $(\tilde{Q})$ is programmed in Python 3.7 and the experimental tests are performed in a general-purpose computer with Intel Core i7-455 7700HQ CPU @ 2.80GHz and 16 GB RAM. To solve our model to global optimality, we use Gurobi. To facilitate the reproduction of our work, our source code is publicly released at Gkiotsalitis (*37*). Starting from an initial solution guess, Gurobi converged to a globally optimal solution in 10 iterations (see Table 2).

TABLE 2 : Iterations until convergence and computational cost of obtaining the optimal dispatching times with Gurobi

Iteration	$\tilde{f}(x,v)$	Computational Time
0	1.70E+09	0 s
1	7.79E+07	0 s
2	1.48E+07	0 s
3	1.41E+05	0 s
4	3.71E+04	0 s
5	1.92E+04	0 s
6	1.04E+04	0 s
7	8.23E+03	0 s
8	8.08E+03	0 s
9	8.08E+03	0 s
10	8.08E+03	0 s
Value of global minimum:	8.08E+03	

The globally optimal solution is:

$$x^* = \{x_1 = 2.5, x_2 = 20, x_3 = 60\} \text{ s}$$

$$v^* = \{v_1 = 0, v_2 = 0, v_3 = 0\} \text{ s}$$

In Fig.3 we show how this solution is expected to improve the squared headway deviations at stations 2 and 3 from their target values. Fig.3 demonstrates the theoretical improvement in terms of service regularity compared to the do-nothing case where rescheduling is not applied.

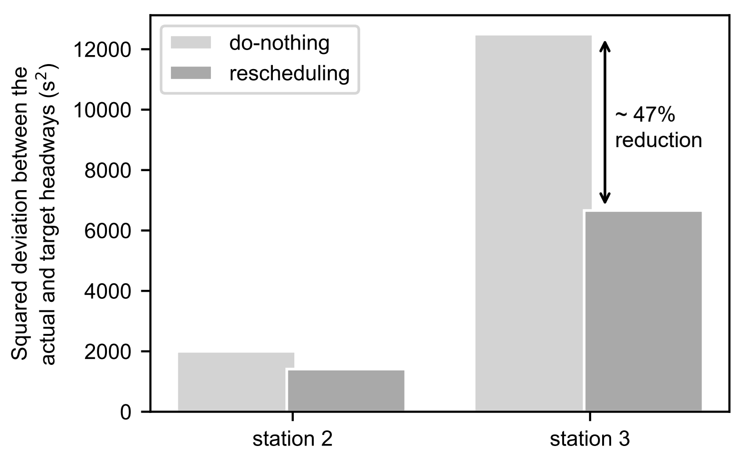

FIGURE 3 : Squared deviation between the actual and target headways at stations 2 and 3 when our rescheduling solution is applied and when it is not (do-nothing case).

We finally note that if we do not consider the schedule sliding constraints (that is, $\beta_j = +\infty, \forall j \in \{1,2,3\}$), the globally optimal solution is

$$x^* = \{x_1 = 2.5, x_2 = 20, x_3 = 90\} \text{ s}$$

$$v^* = \{v_1 = 0, v_2 = 0, v_3 = 0\} \text{ s}$$

with a solution performance of 6.28E+03. Hence, if we had additional resources (i.e., trains) to perform the next trips when our dispatching time adjustments result in schedule sliding, our service regularity would have been improved by 22.3%.

Finally, to demonstrate how our mathematical model treats the case where schedule sliding cannot be avoided, let us consider the same scenario with $(\beta_1, \beta_2, \beta_3) = (600, 1200, 1800)$ s. Obviously, for such values of β_j the schedule will slide because the earliest possible dispatching times of trips to ensure vehicle circulation are $(\pi_1, \pi_2, \pi_3) = (600, 1220, 1820)$ s. Hence, our program returns a globally optimal solution that slides the schedule as little as possible:

$$x^* = \{x_1 = 0, x_2 = 20, x_3 = 20\} \text{ s}$$

$$v^* = \{v_1 = 0, v_2 = 20, v_3 = 20\} \text{ s}$$

with a performance of 4.02E+06. Note that the schedule sliding is indicated by the positive values of v_2, v_3 which are always equal to zero when a feasible solution of (Q) exists.

Case study and sensitivity analysis

Our case study is the red line in Washington D.C. The red line is a rapid transit line of the Washington Metro system, consisting of 27 stations in Montgomery County, Maryland, and Washington, D.C., in the United States. It is a primary line through downtown Washington and forms a long, narrow "U", capped by its terminal stations at Shady Grove and Glenmont. Its topology in the metro network is presented in Fig.4.

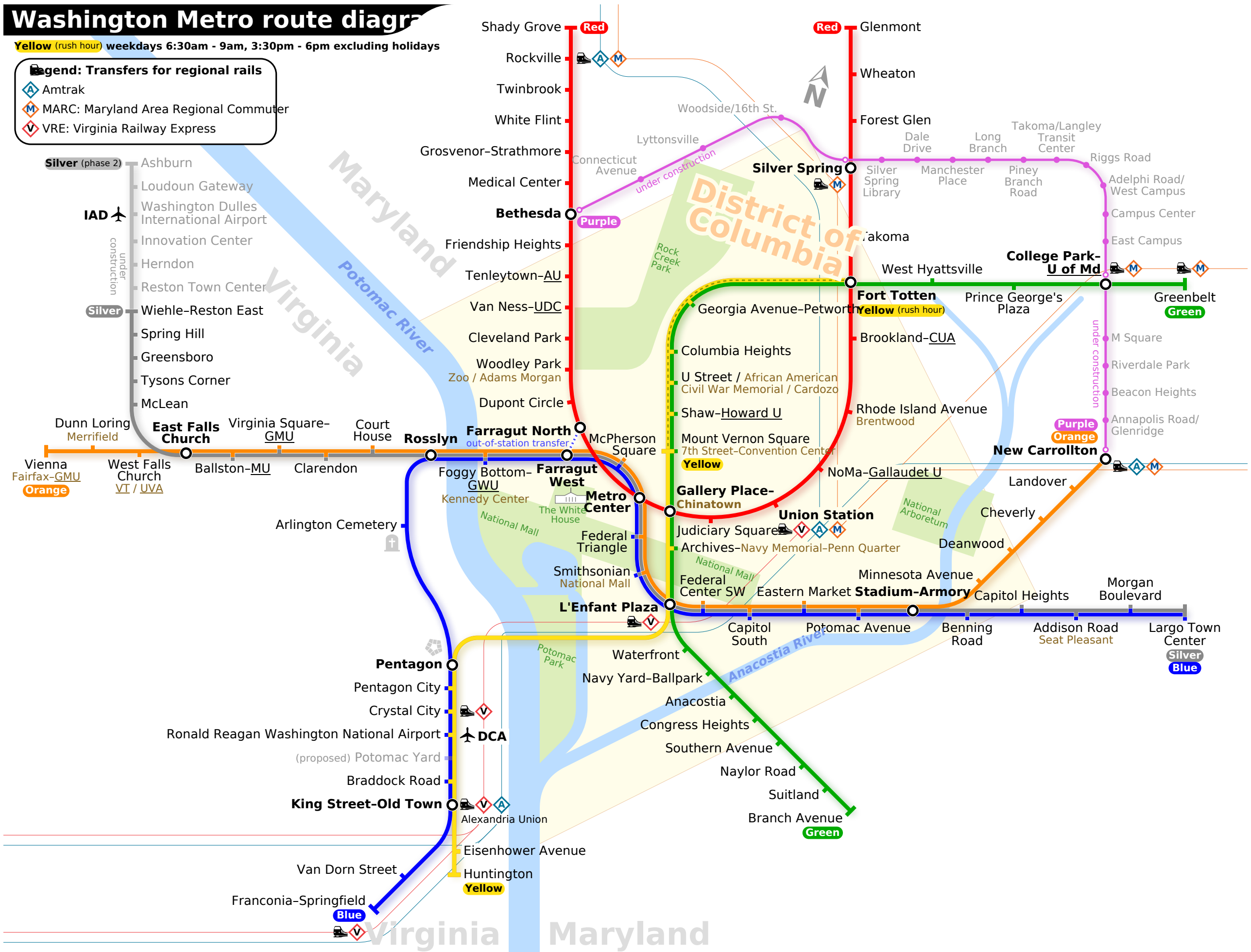

FIGURE 4 : Red line in the Washington Metro route diagram. Source: en.wikipedia.org/wiki/Red_Line_(Washington_Metro)#/media/File:Washington_Metro_diagram_sb.svg

Our data covers the period from 11/3/2018 until 29/3/2018 and includes the arrival time and departure time from each station together with the time period between opening and closing the doors. From this data, we can easily derive the dwell time at each station, which exhibits a slight variation from the median as presented in Fig.5. Fig.5 presents the observed dwell times in this time period using the Tukey boxplot convention McGill et al. (*38*). The upper and lower boundaries of the boxes indicate the upper and lower quartiles (i.e. 75th and 25th percentiles denoted as Q3 and Q1, respectively). The black lines vertical to the boxes (whiskers) show the maximum and minimal values that are not outliers. The whiskers are determined by plotting the lowest datum still within 1.5 the interquartile range (IQR) Q3-Q1 of the lower quartile, and the highest datum still within 1.5 IQR.

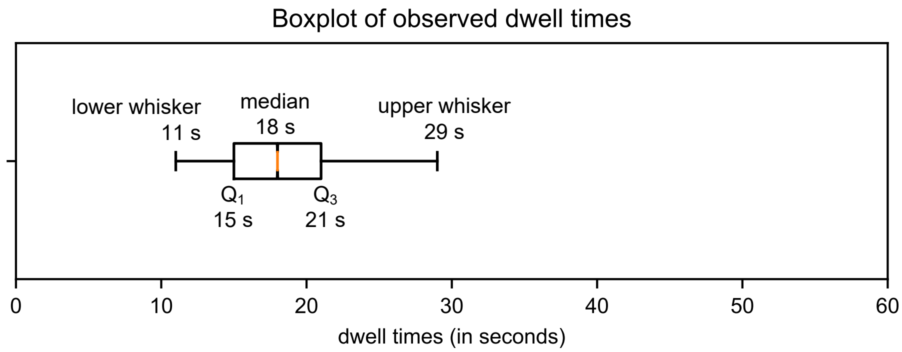

FIGURE 5 : Tukey boxplot of observed dwell times at all stations from 11/3/2018 until 29/3/2018.

From Fig.5 one can note that the median dwell time at all stations is 18 s and the interquartile range between the 25th and the 75th percentile is only 6 s. Additionally, the mean and standard deviation of our observed dwell times is 18.38 s and 4.16 s, respectively resulting in a coefficient of variation (CV) equal to 0.22. Therefore, the red metro line is suitable for the application of our model because of its relatively stable dwell times.

The target headways of the red metro line at each station vary between peak and off-peak hours and between weekdays and weekends. Table 3 summarizes the expected headways which the operator strives to meet to perform a regular service.

TABLE 3 : Target headways at different days of the week and peak/off-peak hours

		Weekdays		
AM Rush (5am-9:30am)	Midday (9:30am-3pm)	PM Rush (3pm-7pm)	Evening (7pm-9:30pm)	Late Night (9:30pm-close)
4 min	6 min	4 min	10 min	15 min
Saturday			**Sunday**	
Daytime (7am-9:30pm)	Late Night (9:30pm-close)		Evening (8am-9:30pm)	Late Night (9:30pm-close)
6 min	15 min		8 min	15 min

In our experiments, we focus on the PM rush hours of one weekday where the target headways at the metro stations are 4 minutes. To investigate the improvement potential of our method, we compare the current regularity of services, as is obtained by the actual data, and the regularity when using our dispatching time modifications. Our day of interest is the 11th of March 2018, and our time period of interest is 3pm-7pm.

In addition, we perform a sensitivity analysis of the effect of the number of trips, $\mathbf{N}_m$, that we are allowed to modify their dispatching times after a disturbance to the service regularity. On one hand, a limited number of dispatching time modifications is likely to be sufficient to smoothen the operations after a disturbance. On the other hand, if we are allowed to modify the dispatching time of a single trip only (i.e., the trip that follows the train that exhibits a disturbance), the positive impact on the service regularity might be limited. In our experiments, we use realistic travel times and dwell times from the actual data. Our assumption is that if we modify the dispatching time of a trip, its travel times between stations remain unchanged.

In our experiments, all trips that operate from 3pm until 7pm depart according to their actual departure times. Those dispatching times are allowed to be modified by our model to improve the service regularity when we reschedule from 1 trip at a time ($\mathbf{N}_m = \{1\}$) to 12 trips at a time ($\mathbf{N}_m = \{1,2,...,12\}$). That is to say, we perform 12 different experiments to investigate the importance of considering fewer or more "upstream" trips for which we modify their dispatching times with our model. The results from this analysis are presented in Fig.6.

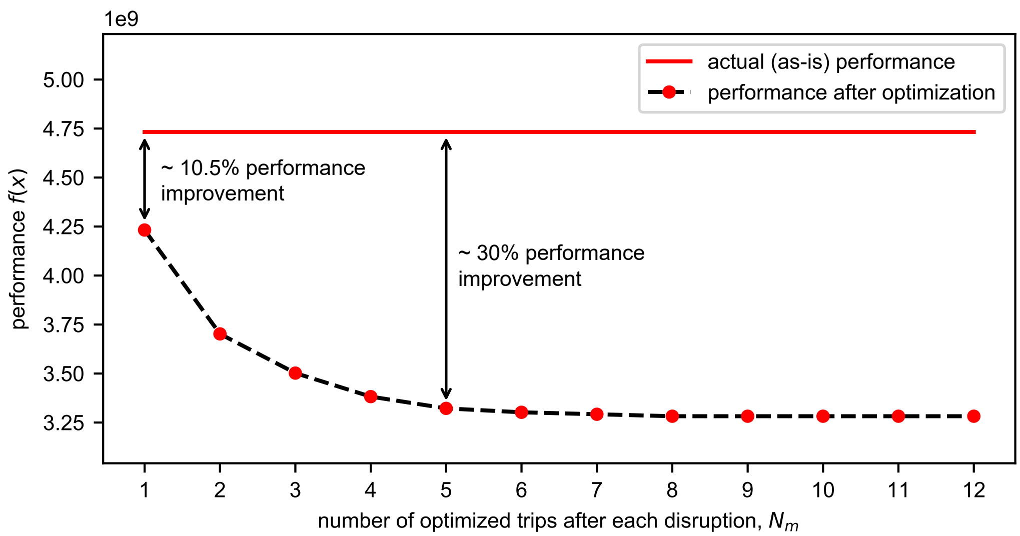

FIGURE 6 : Improvement of the service regularity expressed by $f(x)$ between 3pm and 7pm on the 11th of March when applying our model after each disturbance to upstream trips $\mathbf{N}_m$, ranging from 1 to 12.

From Fig.6 one can note that if we only modify the dispatching time of the following trip when a disturbance occurs ($\mathbf{N}_m = 1$), the improvement of the service regularity compared to the actual operations (do-nothing case) is $\approx 10.5\%$. If we modify the dispatching times of more upstream trips, $\mathbf{N}_m$, our performance improvement in terms of regularity increases. If we change the dispatching times of 5 upstream trips, $\mathbf{N}_m = \{1,2,3,4,5\}$, every time a disturbance occurs, the potential benefit rises to $\approx 30\%$. After that, further regularity improvements are marginal.

CONCLUDING REMARKS

In this study, we proposed a timetable recovery model that modifies the dispatching times of following trips after disturbances. In pursuit of a model that can be solved exactly and applied in real time, we studied the timetabling problem and introduced a quadratic model reformulation with penalty terms. This reformulated model was then proved to be solved to global optimality.

With this model, we investigated how many trips, $\mathbf{N}_m$, should we consider for modifying their dispatching times after a disturbance occurs. This is instrumental in the understanding of the practical use of the model because it might not be prudent to reschedule the dispatching times of all daily trips every time a disturbance occurs. This investigation was performed in a case study using actual data from the red metro line in Washington D.C. From our experimentation with the use of realistic data, we showed that one can consider the dispatching time modification of up to

5 upstream trips to smoothen the headways of a metro line after a disturbance. If more trips are considered, the benefits are marginal and do not justify the inherent disutility of implementing such changes.

To ensure that this work is reproduced appropriately, we hereby list its main limitations:

- Our approach is suitable for correcting the effects of mild disturbances to the service regularity by modifying the dispatching times of upstream trips. In the case of severe disruptions, metro operators should consider changes in the planned service provision (i.e. rescheduling, trip cancellation, short-turning, expressing);
- Our approach is designed for optimizing regularity-based services that operate in high-frequencies (more than 5 trips per hour). In the case of low frequencies, the objective function of our problem is no longer valid because in that case the objective is punctuality-oriented;
- Our approach is suitable in the context where the dwell times at stations are relatively stable and do not vary significantly with passenger demand changes. This makes our approach particularly suitable for automated public transport systems where the process of opening/closing the door channels is automated.

In future research, some of the limitations of this study can be lifted. The most prominent one is the application to services with stable dwell times that do not deviate significantly with changes in passenger demand. This will allow the extension of this work to other public transport services, such as bus operations which are prone to dwell time variations.

Author Contribution Statement

The authors confirm contribution to the paper as follows: study conception and design: K. Gkiotsalitis, O. Cats; data collection: O. Cats; analysis and interpretation of results: K. Gkiotsalitis, O.A.L. Eikenbroek, O. Cats; draft manuscript preparation: K. Gkiotsalitis, O.A.L. Eikenbroek, O. Cats. All authors reviewed the results and approved the final version of the manuscript.

REFERENCES

[1] Zhu, Y. and R. M. Goverde, Railway timetable rescheduling with flexible stopping and flexible short-turning during disruptions. *Transportation Research Part B: Methodological*, Vol. 123, 2019, pp. 149–181.

[2] Jespersen-Groth, J., D. Potthoff, J. Clausen, D. Huisman, L. Kroon, G. Maróti, and M. N. Nielsen, Disruption management in passenger railway transportation. In *Robust and online large-scale optimization*, Springer, 2009, pp. 399–421.

[3] Cacchiani, V., D. Huisman, M. Kidd, L. Kroon, P. Toth, L. Veelenturf, and J. Wagenaar, An overview of recovery models and algorithms for real-time railway rescheduling. *Transportation Research Part B: Methodological*, Vol. 63, 2014, pp. 15–37.

[4] Zilko, A. A., D. Kurowicka, and R. M. Goverde, Modeling railway disruption lengths with Copula Bayesian Networks. *Transportation Research Part C: Emerging Technologies*, Vol. 68, 2016, pp. 350–368.

[5] Ghaemi, N., A. A. Zilko, F. Yan, O. Cats, D. Kurowicka, and R. M. Goverde, Impact of railway disruption predictions and rescheduling on passenger delays. *Journal of Rail Transport Planning & Management*, Vol. 8, No. 2, 2018, pp. 103–122.

[6] Chu, F. and A. Oetting, Modeling capacity consumption considering disruption program characteristics and the transition phase to steady operations during disruptions. *Journal of Rail Transport Planning & Management*, Vol. 3, No. 3, 2013, pp. 54–67.

[7] Ghaemi, N., O. Cats, and R. M. Goverde, Railway disruption management challenges and possible solution directions. *Public Transport*, Vol. 9, No. 1-2, 2017, pp. 343–364.

[8] D'Ariano, A., F. Corman, D. Pacciarelli, and M. Pranzo, Reordering and local rerouting strategies to manage train traffic in real time. *Transportation science*, Vol. 42, No. 4, 2008, pp. 405–419.

[9] Corman, F., A. DâĂŹAriano, D. Pacciarelli, and M. Pranzo, A tabu search algorithm for rerouting trains during rail operations. *Transportation Research Part B: Methodological*, Vol. 44, No. 1, 2010, pp. 175–192.

[10] Meng, L. and X. Zhou, Simultaneous train rerouting and rescheduling on an N-track network: A model reformulation with network-based cumulative flow variables. *Transportation Research Part B: Methodological*, Vol. 67, 2014, pp. 208–234.

[11] Pellegrini, P., G. Marlière, and J. Rodriguez, Optimal train routing and scheduling for managing traffic perturbations in complex junctions. *Transportation Research Part B: Methodological*, Vol. 59, 2014, pp. 58–80.

[12] Krasemann, J. T., Design of an effective algorithm for fast response to the re-scheduling of railway traffic during disturbances. *Transportation Research Part C: Emerging Technologies*, Vol. 20, No. 1, 2012, pp. 62–78.

[13] Gkiotsalitis, K. and O. Cats, Reliable frequency determination: Incorporating information on service uncertainty when setting dispatching headways. *Transportation Research Part C: Emerging Technologies*, Vol. 88, 2018, pp. 187–207.

[14] Gkiotsalitis, K., O. A. Eikenbroek, and O. Cats, Robust network-wide bus scheduling with transfer synchronizations. *IEEE transactions on intelligent transportation systems*, 2019.

[15] Gkiotsalitis, K. and E. Van Berkum, An exact method for the bus dispatching problem in rolling horizons. *Transportation Research Part C: Emerging Technologies*, Vol. 110, 2020, pp. 143–165.

[16] Sun, A. and M. Hickman, The real–time stop–skipping problem. *Journal of Intelligent Transportation Systems*, Vol. 9, No. 2, 2005, pp. 91–109.

[17] Liu, Z., Y. Yan, X. Qu, and Y. Zhang, Bus stop-skipping scheme with random travel time. *Transportation Research Part C: Emerging Technologies*, Vol. 35, 2013, pp. 46–56.

[18] Chen, X., B. Hellinga, C. Chang, and L. Fu, Optimization of headways with stop-skipping control: a case study of bus rapid transit system. *Journal of advanced transportation*, Vol. 49, No. 3, 2015, pp. 385–401.

[19] Gkiotsalitis, K., Robust stop-skipping at the tactical planning stage with evolutionary optimization. *Transportation research record*, Vol. 2673, No. 3, 2019, pp. 611–623.

[20] Newell, G. F., Control of pairing of vehicles on a public transportation route, two vehicles, one control point. *Transportation Science*, Vol. 8, No. 3, 1974, pp. 248–264.

[21] Hernández, D., J. C. Muñoz, R. Giesen, and F. Delgado, Analysis of real-time control strategies in a corridor with multiple bus services. *Transportation Research Part B: Methodological*, Vol. 78, 2015, pp. 83–105.

[22] Wu, W., R. Liu, and W. Jin, Modelling bus bunching and holding control with vehicle overtaking and distributed passenger boarding behaviour. *Transportation Research Part B: Methodological*, Vol. 104, 2017, pp. 175–197.

[23] Gavriilidou, A. and O. Cats, Reconciling transfer synchronization and service regularity: real-time control strategies using passenger data. *Transportmetrica A: Transport Science*, 2018, pp. 1–29.

[24] Gkiotsalitis, K. and O. Cats, Multi-constrained bus holding control in time windows with branch and bound and alternating minimization. *Transportmetrica B: Transport Dynamics*, Vol. 7, No. 1, 2019, pp. 1258–1285.

[25] Adamski, A. and A. Turnau, Simulation support tool for real-time dispatching control in public transport. *Transportation Research Part A: Policy and Practice*, Vol. 32, No. 2, 1998, pp. 73–87.

[26] Strathman, J., K. Dueker, T. Kimpel, R. Gerhart, K. Turner, P. Taylor, S. Callas, D. Griffin, and J. Hopper, Automated bus dispatching, operations control, and service reliability: Baseline analysis. *Transportation Research Record: Journal of the Transportation Research Board*, , No. 1666, 1999, pp. 28–36.

[27] Gkiotsalitis, K. and F. Alesiani, Robust timetable optimization for bus lines subject to resource and regulatory constraints. *Transportation Research Part E: Logistics and Transportation Review*, Vol. 128, 2019, pp. 30–51.

[28] Marguier, P., *Bus route performance evaluation under stochastic conditions*. Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, MA, 1985.

[29] Lin, T.-m. and N. H. Wilson, Dwell time relationships for light rail systems. *Transportation Research Record*, , No. 1361, 1992.

[30] Eberlein, X. J., N. H. Wilson, and D. Bernstein, The holding problem with real–time information available. *Transportation science*, Vol. 35, No. 1, 2001, pp. 1–18.

[31] Daganzo, C. F., A headway-based approach to eliminate bus bunching: Systematic analysis and comparisons. *Transportation Research Part B: Methodological*, Vol. 43, No. 10, 2009, pp. 913–921.

[32] Walker, C. G., J. N. Snowdon, and D. M. Ryan, Simultaneous disruption recovery of a train timetable and crew roster in real time. *Computers & Operations Research*, Vol. 32, No. 8, 2005, pp. 2077–2094.

[33] Ceder, A., *Public transit planning and operation: Modeling, practice and behavior*. CRC press, 2016.

[34] Gkiotsalitis, K., Z. Wu, and O. Cats, A cost-minimization model for bus fleet allocation featuring the tactical generation of short-turning and interlining options. *Transportation Research Part C: Emerging Technologies*, Vol. 98, 2019, pp. 14–36.

[35] Trompet, M., X. Liu, and D. Graham, Development of key performance indicator to compare regularity of service between urban bus operators. *Transportation Research Record: Journal of the Transportation Research Board*, , No. 2216, 2011, pp. 33–41.

[36] Li, C. M. and F. Manya, MaxSAT, Hard and Soft Constraints. *Handbook of satisfiability*, Vol. 185, 2009, pp. 613–631.

[37] Gkiotsalitis, K., *Timetable recovery model*. https://github.com/KGkiotsalitis/Timetable_recovery, 2019.

[38] McGill, R., J. W. Tukey, and W. A. Larsen, Variations of box plots. *The American Statistician*, Vol. 32, No. 1, 1978, pp. 12–16.

APPENDIX

Lemma .1. *For* $\beta_1 < h_{min} + \bar{\delta}_0$, $\mathscr{F} = \emptyset$.

Proof. Let assume that for $\beta_1 < h_{min} + \bar{\delta}_0$, $\mathscr{F} \neq \emptyset$. Then, $\neg \exists\, x^0 \mid (x^0, a, h)$ satisfy Eq.(6)-(8). To satisfy constraint Eq.(8), $h_{min} \leq \delta_1 + x_1^0 - \bar{\delta}_0 \Rightarrow \delta_1 + x_1^0 \geq h_{min} + \bar{\delta}_0$. In addition, to satisfy constraint Eq.(6), $\delta_1 + x_1^0 \leq \beta_1$. Thus, β_1 should be greater than or equal to $h_{min} + \bar{\delta}_0$ and we reached a contradiction. This proves that

$\mathscr{F}$ can be an empty set if the inequality constraints of Eq.(6)-(8) are binding (i.e., must be satisfied in all cases). □

Theorem .2. *A local optimum of program $(\tilde{Q})$ is also a globally optimal solution*

Proof. A local minimizer of $\tilde{Q}$ is the global minimizer of $\tilde{Q}$ if the objective function is strictly convex and the feasible region is a convex set. The feasible region is defined by linear (in)equalities (affine functions) and is a polyhedron. Thus, it is also a convex set. Further, we prove that the objective function $f(x)+\sum_{j\in N_m} M\nu_j$ is strictly convex with respect to x,ν.

Let $\tilde{f}(x,\nu) := f(x) + \sum_{j\in N_m} M\nu_j$. Then, the Hessian matrix of $\tilde{f}(x,\nu)$ is a matrix $\mathbf{H}\in\mathbb{R}^{2n\times 2n}$ with elements:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_1^2} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_1 \partial x_n} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_1 \partial \nu_1} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_1 \partial \nu_n} \\ \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_2 \partial x_1} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_2^2} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_2 \partial x_n} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_2 \partial \nu_1} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_2 \partial \nu_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_n \partial x_1} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_n^2} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_n \partial \nu_1} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x_n \partial \nu_n} \\ \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_1 \partial x_1} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_1 \partial x_2} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_1 \partial x_n} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_1^2} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_1 \partial \nu_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_n \partial x_1} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_n \partial x_2} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_n \partial x_n} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_n \partial \nu_1} & \dots & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu_n^2} \end{bmatrix} \tag{16}$$

The gradient of $\tilde{f}(x,\nu)$ is a $\mathbb{R}^{2n}$ vector:

$$\begin{aligned} \nabla \tilde{f}(x,\nu) = \Big(& \sum_{s=2}^{|S|-1} (4x_1 - 2x_2 + \rho_1), \sum_{s=2}^{|S|-1} (4x_2 - 2x_1 - 2x_3 + \rho_2), \\ & \dots, \sum_{s=2}^{|S|-1} (4x_{n-1} - 2x_{n-1} - 2x_n + \rho_{n-1}), \\ & \sum_{s=2}^{|S|-1} (2x_n - 2x_{n-1} + \rho_n), \underbrace{1,\dots,1}_{\text{n}} \Big) \end{aligned} \tag{17}$$

where $\rho_1,\rho_2,\dots,\rho_n$ are parameter values consisting of travel times, dwell times and target headways which do not vary with x or ν.

This yields the Hessian

$$\mathbf{H} = \begin{bmatrix} 4(|S|-2) & -2(|S|-2) & 0 & \dots & 0 & 0 & \dots & 0 \\ -2(|S|-2) & 4(|S|-2) & -2(|S|-2) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2(|S|-2) & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \tag{18}$$

$\tilde{f}(x,\nu)$ is strictly convex if the Hessian matrix is positive definite. That is, $\mathbf{z}^\intercal\mathbf{H}\mathbf{z} > 0$ for any non-zero vector $\mathbf{z}\in\mathbb{R}^{2n}\setminus 0$.

To simplify the notation, let us set $\zeta := |S|-2$ with $\zeta > 0$. Then, $\mathbf{z}^\intercal\mathbf{H}\mathbf{z}$ becomes:

$$
\begin{aligned}
\mathbf{z}^\intercal \mathbf{H}\mathbf{z} &= \begin{bmatrix} z_1 & z_2 & \dots & z_{n-1} & z_n \end{bmatrix} \mathbf{H} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} (4\zeta z_1 - 2\zeta z_2) & (-2\zeta z_1 + 4\zeta z_2 - 2\zeta z_3) & \dots & (-2\zeta z_{n-2} + 4\zeta z_{n-1} - 2\zeta z_n) & (-2\zeta z_{n-1} + 2\zeta z_n) \end{bmatrix} \mathbf{z} \\
&= \zeta \left(4z_1^2 - 2z_1 z_2 - 2z_1 z_2 + 4z_2^2 - 2z_2 z_3 - \dots - 2z_{n-2} z_{n-1} + 4z_{n-1}^2 - 2z_{n-1} z_n - 2z_{n-1} z_n + 2z_n^2 \right) \\
&= \zeta \left(4z_1^2 - 4z_1 z_2 + 4z_2^2 - 4z_2 z_3 + 4z_3^2 - \dots + 4z_{n-2}^2 - 4z_{n-2} z_{n-1} + 4z_{n-1}^2 - 4z_{n-1} z_n + 2z_n^2 \right) \\
&= \zeta \left(2z_1^2 + (z_1\sqrt{2} - z_2\sqrt{2})^2 + (z_2\sqrt{2} - z_3\sqrt{2})^2 + \dots + (z_{n-2}\sqrt{2} - z_{n-1}\sqrt{2})^2 + (z_{n-1}\sqrt{2} - z_n\sqrt{2})^2 \right)
\end{aligned}
$$

Hence, $\mathbf{z}^\intercal \mathbf{H}\mathbf{z} > 0$ for any $\mathbf{z} \in \mathbb{R}^{2n} \setminus 0$ and thus $\tilde{f}(x, \nu)$ is strictly convex. This proves that a local optimum of the reformulated program $(\tilde{Q})$ is also its *unique* global minimizer. □