

Path Graphs

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ABSTRACT

The concept of a line graph is generalized to that of a path graph. The path graph $P_k(G)$ of a graph G is obtained by representing the paths P_k in G by vertices and joining two vertices whenever the corresponding paths P_k in G form a path P_{k+1} or a cycle C_k . P_3 -graphs are characterized and investigated on isomorphism and traversability. Trees and unicyclic graphs with hamiltonian P_3 -graphs are characterized.

1. INTRODUCTION

We refer to Harary [2] for terminology, but we shall speak of vertices and edges instead of points and lines. Accordingly, we denote the edge set of a graph G by $E(G)$. An edge is called *endedge* if it is incident with an endvertex. We maintain the notion of line graph. $L(G)$ is a graph-valued function mapping a graph G on a graph $L(G)$ by representing edges by vertices and joining two of these vertices whenever the edges they represent are adjacent in G . This way of describing a line graph stresses the adjacency concept. However, we may also say that the paths P_2 in G are represented by vertices and that two vertices are adjacent whenever the paths they represent form a P_3 in G . This stresses the concept of *path generation* by consecutive paths. This can be generalized in the following way. We denote by $\Pi_k(G)$ the set of all paths of G on k vertices ($k \geq 1$).

Definition 1.1. The *path graph* $P_k(G)$ of a graph G has vertex set $\Pi_k(G)$ and edges joining pairs of vertices that represent two paths P_k , the union of which forms either a path P_{k+1} or a cycle C_k in G . A graph is called a P_k -*graph* if it is isomorphic to $P_k(H)$ for some graph H .

There are two remarks that should be made here. The first remark is that the definition might be restricted by admitting only edges corresponding to the formation of paths P_{k+1} . The possibility of edges in case of formation of cycles C_k has been included in view of the results derived later on. The second remark is

that $P_1(G) = G$ and $P_2(G) = L(G)$ as G is a graph and not a multigraph, and hence contains no cycles of length 2.

We shall restrict ourselves to P_3 -graphs mainly. A few examples should give the reader a feeling for the structure of P_3 -graphs.

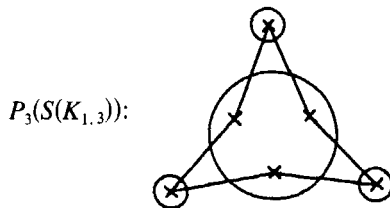
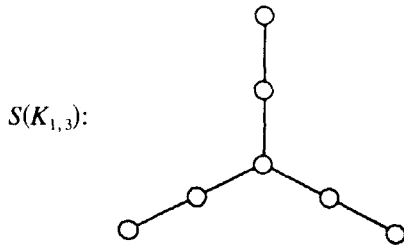
Example 1.2.

- $P_3(C_l) = C_l, 3 \leq l$: Cycles give cycles. This is also true for $k > 3$ as $P_k(C_l) = C_l (k \leq l)$.
- $P_3(K_{1,d}) = \binom{d}{2}K_1$: The length of a path is the number of edges in it. All paths of length 2 in a graph G with a common middle vertex of degree d form an independent set of order $\binom{d}{2}$ in $P_3(G)$. These sets will be called *binomial sets*.
- $P_3(P_l) = P_{l-2}, 3 \leq l$: Paths are shortened by two vertices.

It is well known that $L(K_{1,3}) = L(K_3) = K_3$, and that $K_{1,3}$ and K_3 are the only pair of connected nonisomorphic graphs with the same line graph. For P_3 -graphs we have more than one pair of connected nonisomorphic graphs that yield the same path graph. The *subdivision graph* $S(G)$ of a graph G is the graph resulting from G by subdividing every edge of G . The graph $S(K_{1,3}) - v$, where v is an endvertex, is denoted by Y . In the figures, circles will indicate vertices of a graph G and crosses will indicate vertices of a graph $P_3(G)$. Circles around vertices indicate binomial sets.

Example 1.3.

- $P_3(S(K_{1,3})) = C_6$ and $P_3(C_6) = C_6$.



— $P_3(Y) = P_5$ and $P_3(P_7) = P_5$. ■

In Section 2, preliminary results will be derived. Section 3 contains a discussion of isomorphisms of path graphs. In Section 4 the characterization of P_3 -graphs is discussed. Properties of P_3 -graphs with respect to traversability are discussed in Section 5. Section 6 contains some miscellaneous results and open problems.

2. PRELIMINARY RESULTS

Paths P_3 in G as well as vertices of a path graph $P_3(G)$ will be represented by triples vmw , where m is the middle vertex of a path P_3 in G from v to w and $vmw = wmv$. $N(v)$ denotes the set of neighbors of v in G .

Lemma 2.1. Let $G = (V, E)$ be a graph, $P_3(G) = (V', E')$ its path graph, and let $I(G)$ be the set of vertices of G with degree greater than 1. Then

$$|V'| = \sum_{v \in I(G)} \binom{\deg v}{2}$$

$$|E'| = \frac{1}{2} \sum_{v \in V} [(\deg v - 1) \sum_{u \in N(v)} (\deg u - 1)].$$

Proof. Only vertices $v \in I(G)$ can be middle vertices of paths P_3 in G . Obviously there are $\binom{\deg v}{2}$ paths P_3 with v as middle vertex. This gives the formula for $|V'|$.

To derive $|E'|$ we consider an edge uv of G . There are $(\deg v - 1)$ paths P_3 with representation xvu . These form paths P_4 or cycles C_3 with the $(\deg u - 1)$ paths P_3 with representation vuy , depending on whether x and y are different or not. On summation over all $v \in V$, pairs of paths are counted twice. The formula now follows. ■

If in Definition 1.1 we would not have allowed that two paths P_3 form a triangle, in the formula for $|E'|$ we would have had to subtract 3 for each triangle present in G . With the definition as given, we also have that for a vertex $x = vmw$ of $P_3(G)$

$$\deg x = \deg v + \deg w - 2,$$

irrespective of whether $vw \in E(G)$ or $vw \notin E(G)$. Note that $\deg v$ and $\deg w$ are degrees in G , whereas $\deg x$ is the degree of a vertex of $P_3(G)$.

Yet another advantage of Definition 1.1 is that $P_3(C_3) = C_3$, in line with the fact that cycles give cycles. This would not be so if two vertices in $P_3(G)$ were nonadjacent if the represented paths P_3 formed a triangle in G . In fact, one could then easily prove that $P_3(G)$ would not contain any triangles at all.

Several definitions and results concerning line graphs have counterparts for path graphs. The *iterated* path graph is $P_k^n(G) = P_k(P_k^{n-1}(G))$ just like the iterated line graph is $L^n(G) = L(L^{n-1}(G))$. Every cut vertex of $L(G)$ represents a bridge of G that is not an endedge, and conversely. In the context of P_3 -graphs the analog of a bridge, or separating edge of a P_2 as we may see it, is formed by the two edges and the middle vertex of a P_3 vmw with $\deg m = 2$, the removal of which from G creates two nontrivial components. This will be called a *bridge path*. We have the following result:

Lemma 2.2. If vmw is a bridge path in a graph G , then vmw is a cut vertex in $P_3(G)$.

Proof. Let vmw be a bridge path of G . Any vertex of $P_3(G)$ adjacent to vertex vmw must have a name xvm or a name mwy , where x ranges over $N(v) - \{m\}$ and y ranges over $N(w) - \{m\}$. No pair of such vertices are adjacent. If xvm and mwy are two of these vertices that are not separated on deletion of vertex vmw , there exists a path in $P_3(G)$ connecting these two vertices. As m has degree 2, no vertex of this path has m as middle vertex. Neighboring vertices of this path represent overlapping paths P_3 in G . These do not necessarily form a path in G , but in the union of these paths P_3 there exists a path in G connecting v and w . This contradicts the fact that vmw is a bridge path of G . So vmw is a cut vertex of $P_3(G)$. ■

The converse of this lemma does not hold. The example in Figure 1 shows a graph G without bridge paths and its path graph $P_3(G)$ with three cut vertices.

3. ISOMORPHISMS OF PATH GRAPHS

For line graphs there are two well-known results concerning isomorphisms:

- (1) A connected graph is isomorphic to its line graph if and only if it is a cycle.

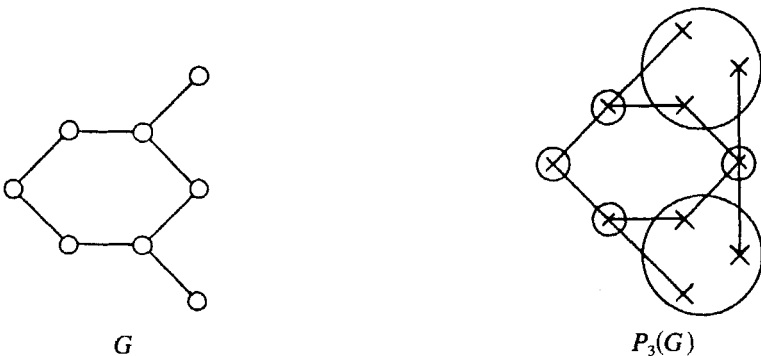


FIGURE 1.

- (2) If G and G' are connected and have isomorphic line graphs, then G and G' are isomorphic unless one is $K_{1,3}$ and the other is K_3 .

The second result is due to Whitney [6].

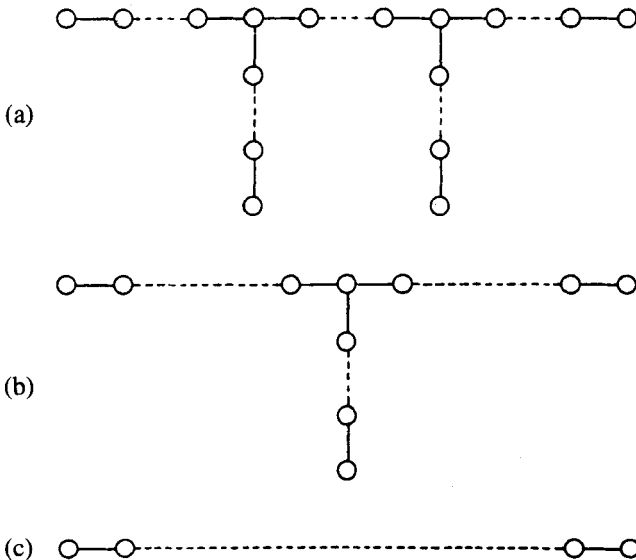
Theorem 3.1. A connected graph G is isomorphic to its path graph $P_3(G)$ if and only if G is a cycle.

Proof. In Example 1.2 we have seen that the “if” part holds.

Let G have n vertices. Then $P_3(G)$ must have n vertices too, so G should have exactly n subgraphs P_3 .

As G is connected, it has a spanning tree T . Let T have degree sequence (d_1, d_2, \dots, d_n) and let the highest degree d_1 of vertex v_1 be greater than 2. If T is transformed into a tree T^* by removing one of the subtrees pendant from v_1 and adding it to an endvertex w of the resulting tree, then in T^* the vertex v_1 has degree $d_1 - 1$ and w has degree 2. As $d_1 \geq 3$ and $\binom{d_1}{2}$ paths P_3 in T have v_1 as middle vertex, the number of paths P_3 in T^* is lower than that in T by $\binom{d_1}{2} - \binom{d_1-1}{2}$ and higher by $1 - 0$, as w can now be middle vertex of precisely one P_3 . The change is $1 - (d_1 - 1) = 2 - d_1$, which is negative.

By repetition of the transformation, every tree T can be transformed into P_n , which has $n - 2$ subgraphs P_3 . If T is to have no more than n subgraphs P_3 it cannot therefore have a vertex v of degree 4 (or more) as two transformations make v into a vertex of degree 2 with a change of 3 in the number of P_3 's and T , and thus G , would have at least $(n - 2) + 3 = n + 1$ subgraphs P_3 . Similarly, T cannot have three or more vertices of degree 3. The remaining possible structures of the spanning tree of G are



In case (a) the number of subgraphs P_3 is equal to the number of vertices. However, G cannot be this tree as $P_3(G)$ contains isolated vertices if a vertex of degree 3 is adjacent to two vertices of degree 1, and more than two vertices of degree at least 3 in the other cases.

In case (b) an edge has to be added to obtain a graph with at least n paths of length 2. However, then at least two subgraphs P_3 are added to the $n - 1$ present in the spanning tree T and $P_3(G)$ would have at least $n + 1$ vertices.

In case (c) addition of an edge leads to a unicyclic graph G . If the number of vertices of degree 3 is two, G contains $n + 2$ subgraphs P_3 , and if this number is one, then G contains $n + 1$ subgraphs P_3 . The only possibility left is that the added edge is adjacent to the two endvertices of T and G is a cycle. ■

The same result for line graphs is much easier to prove as $G \cong L(G)$ must be a unicyclic graph, and if G contains a vertex of degree 3, $L(G)$ would contain a triangle, next to the cycle G has, and therefore could not be isomorphic to G .

The important issue on isomorphism is whether the graphs C_6 and P_5 , given in Example 1.3, are the only two connected graphs for which there exist connected nonisomorphic graphs with the same P_3 -graph. This appears to be untrue. We briefly describe two infinite classes of pairs of nonisomorphic connected graphs that have isomorphic connected P_3 -graphs. These classes are schematically shown in Figures 2 and 3, respectively.

To see that $T_{k,l}$ and $T'_{k,l}$ have isomorphic P_3 -graphs, consider a function $\varphi: \Pi_3(T_{k,l}) \rightarrow \Pi_3(T'_{k,l})$ defined by

$$\left\{ \begin{array}{l} \varphi(v_i x t) = v_i u_i t_i, \quad i = 1, \dots, k; \\ \varphi(z x t) = z y p; \\ \varphi(p_r r_i q_i) = q_i y p, \quad i = 1, \dots, l \quad (\text{if } l > 0); \\ \varphi(abc) = abc \text{ for all other } abc \in \Pi_3(T_{k,l}). \end{array} \right.$$

It is easy to check that φ is a bijection preserving adjacencies of P_3 's, and that $P_3(T_{k,l}) \cong P_3(T'_{k,l})$.

To see that U_k and U'_k have isomorphic P_3 -graphs, consider a function $\psi: \Pi_3(U_k) \rightarrow \Pi_3(U'_k)$ defined by

$$\left\{ \begin{array}{l} \psi(v_i x t) = v_i u_i t_i, \quad i = 1, \dots, k; \\ \psi(w_1 x t) = w_1 w_2 p; \\ \psi(w_3 x t) = w_3 w_2 p; \\ \psi(abc) = abc \text{ for all other } abc \in \Pi_3(U_k). \end{array} \right.$$

It is easy to check that ψ is a bijection preserving adjacencies of P_3 's, and that $P_3(U_k) \cong P_3(U'_k)$.

These classes of graphs show that Whitney's result on line graphs has no similar counterpart with respect to P_3 -graphs. At the moment we do not know, however, whether there exist triples of mutually nonisomorphic connected

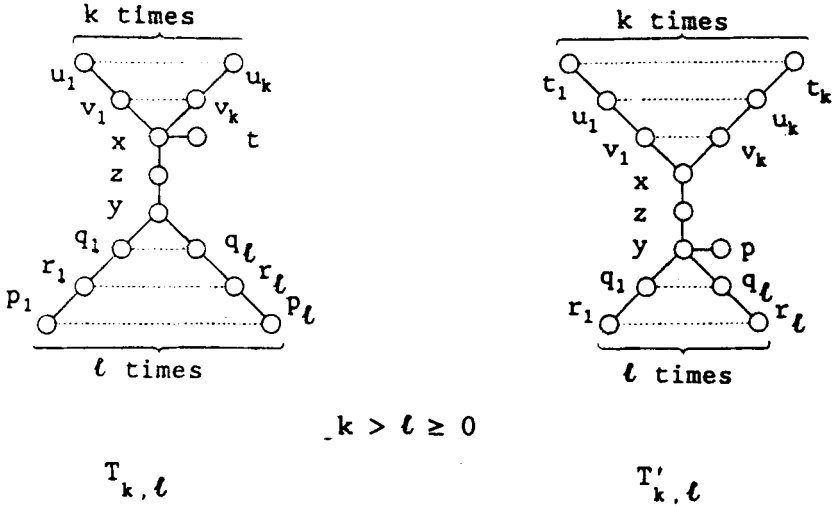


FIGURE 2. Pairs of nonisomorphic trees $T_{k,l}$ and $T'_{k,l}$ with isomorphic P_3 -graphs.

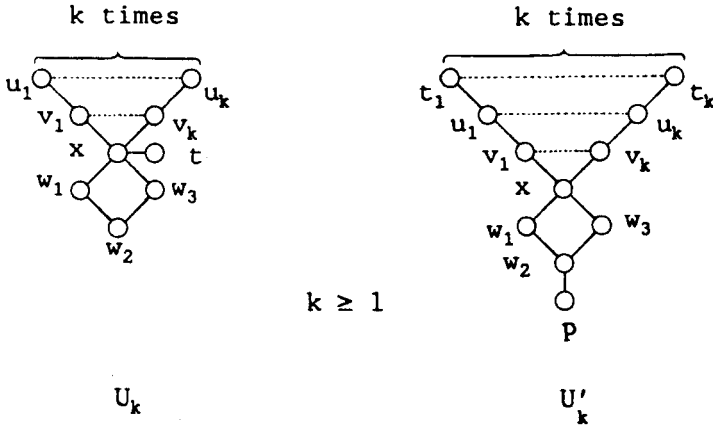


FIGURE 3. Pairs of nonisomorphic unicyclic graphs U_k and U'_k with isomorphic P_3 -graphs.

graphs with isomorphic connected P_3 -graphs. It is also an open problem to characterize all pairs of nonisomorphic connected graphs with isomorphic connected P_3 -graphs.

4. CHARACTERIZATION OF P_3 -GRAPHS

A triangle T of G is called odd if there is a vertex of G adjacent to an odd number of its vertices. Otherwise T is called even. For line graphs the following result is known:

Theorem 4.1. The following statements are equivalent:

- (1) G is a line graph.
- (2) The edges of G can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of the subgraphs (Krausz [3]).
- (3) G does not have $K_{1,3}$ as an induced subgraph, and if two odd triangles have a common edge, then the subgraph induced by their vertices is K_4 (Van Rooij and Wilf [4]).
- (4) None of a set of nine graphs is an induced subgraph of G (Beineke [1]).

The most elegant of these three characterizations is that in terms of forbidden subgraphs. The situation is somewhat different for P_3 -graphs. An induced subgraph of a P_3 -graph may not be a P_3 -graph. There are four connected graphs on four vertices that are not P_3 -graphs, as will be shown later in this section. They are depicted in Figure 4.

In Figure 5 a graph S_3 and its path graph $P_3(S_3)$ are given. The path graph contains a $K_{1,3}$ and a $K_{1,3} + e$.

There are, however, induced graphs that are forbidden in a P_3 -graph. First, we derive the following result:

Lemma 4.2. In a P_3 -graph no vertex belongs to more than one triangle.



FIGURE 4.

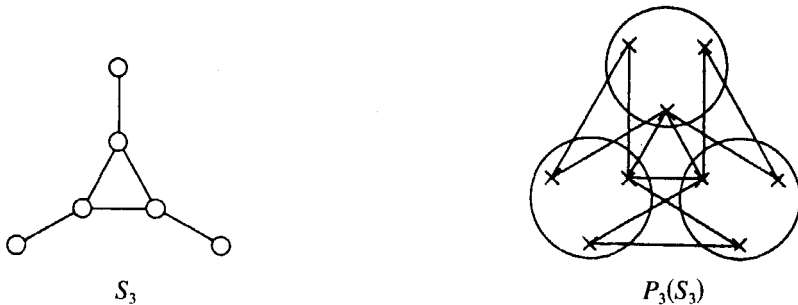


FIGURE 5.

Proof. If $P_3(G)$ contains a triangle, its pairs of vertices represent pairs of paths P_3 that form a P_4 or a C_3 in G . If two paths form a P_4 , a third path P_3 cannot form a P_4 or a C_3 with both of them. Triangles in $P_3(G)$ are therefore necessarily images of triangles in G . A vertex abc of $P_3(G)$ that belongs to a triangle forces the other vertices of the triangle to be vertices bca and cab . The result follows. ■

As a consequence of this lemma, $K_4 - e$ and K_4 are indeed forbidden induced subgraphs for P_3 -graphs, as is the graph consisting of two triangles with only one vertex in common.

$K_{1,3}$ and $K_{1,3} + e$ are not P_3 -graphs, but these graphs can occur in a P_3 -graph in the proper "context." A characterization in terms of forbidden subgraphs would ask for a detailed treatment of this context. A minor result in this direction is the following. A *bull* is a graph with five vertices consisting of a triangle and two nonadjacent edges incident with two vertices of the triangle. This name was introduced by Chvátal.

Lemma 4.3. Every induced subgraph of $P_3(G)$ isomorphic to $K_{1,3} + e$ is a subgraph of an induced bull.

Proof. From Lemma 4.2 we know that the triangle in $P_3(G)$ with vertices a , b , and c is the image of three paths P_3 in G that form a triangle in G . Let vertex d be adjacent to vertex b and not to vertices a and c . Then in G there must be a path P_3 that overlaps with the original of b . This means that in G there is a vertex adjacent to a vertex of the triangle, i.e., G contains a subgraph $K_{1,3} + e$. The image of this subgraph is an induced subgraph in $P_3(G)$ that is a bull. ■

In the proof of this lemma we saw that a not necessarily induced subgraph of G gave an induced subgraph in $P_3(G)$. This holds in general.

Lemma 4.4. If H is a subgraph of G , then $P_3(H)$ is an induced subgraph of $P_3(G)$.

Proof. It is obvious that $P_3(H)$ is a subgraph of $P_3(G)$. Suppose that the vertices of $P_3(H)$ induce a subgraph of $P_3(G)$ with more edges than $P_3(H)$ has. Adjacent vertices of $P_3(G)$ correspond to adjacent middle vertices of paths P_3 in G that overlap. Any edge e not in $P_3(H)$ but in the graph induced by the vertices of $P_3(H)$ implies that two middle vertices of paths P_3 in H are adjacent. But then e must be an edge of $P_3(H)$. ■

This result can also be seen as a consequence of the fact that each vertex of degree d in G gives rise to a binomial set of $\binom{d}{2}$ independent vertices in $P_3(G)$. If $V(H)$ induces a subgraph of G with an edge uv more than H has, the images of paths P_3 with u and v as middle vertices are other vertices in the two binomial sets than the vertices that belong to $P_3(H)$.

If two adjacent vertices u and v have degrees $\deg u$ and $\deg v$ in G , then in $P_3(G)$ there will be edges between $\deg u - 1$ vertices of the binomial set corresponding to vertex u and $\deg v - 1$ vertices of the other binomial set, corresponding to vertex v , that form a complete bipartite graph $K_{\deg u - 1, \deg v - 1}$. As a path P_3 in G has two edges, its image vertex in $P_3(G)$ will be part of at most two of these complete bipartite graphs. These obviously necessary conditions for a graph to be a P_3 -graph are also sufficient.

Theorem 4.5. A graph K is the P_3 -graph of a graph G if and only if

- (i) the vertices of K can be partitioned into binomial sets of independent vertices in such a way that
- (ii) the edges of K can be partitioned into sets of edges that induce complete bipartite graphs B with classes each belonging to one binomial set with order $d - 1$ if the order of the binomial set is $\binom{d}{2}$
- (iii) the vertices of K belong to at most two of the graphs B , and
- (iv) no vertex of K belongs to more than one triangle.

Proof. The only if-part is clear from the preceding discussion.

The if-part follows from the fact that the formed partition of the vertices determines the vertices of a graph G^- , one vertex for each binomial set, and the found complete bipartite graphs B determine which of these vertices of G^- are adjacent. The resulting graph G^- is extended to a graph G with extra vertices adjacent to the vertices of G^- to obtain the degrees corresponding to the orders of the binomial sets.

G has indeed graph K as its P_3 -graph, due to the fact that the third condition holds. ■

This simple characterization is very similar to that of Krausz for line graphs, formulated in Theorem 4.1(2), be it that now complete bipartite graphs instead of complete graphs are considered. We give some examples of applications of Theorem 4.4.

Examples

- (a) $K_{1,3}$ has four vertices that can be partitioned in two ways into binomial sets of independent vertices. One way is to partition into four sets of one vertex. The bipartite graphs B are graphs $K_{1,1}$. The central vertex belongs to three of these graphs B . Condition (iii) is violated. The other way is to partition into one binomial set of three and one of one vertex. The set of one vertex must be the central vertex. The complete bipartite graph is $K_{1,3}$, which should be a graph $K_{1,2}$ according to condition (ii). $K_{1,3}$ is not a P_3 -graph.
- (b) $K_{1,3} + e$, $K_4 - e$, and K_4 have four vertices, too, which can only be partitioned into four binomial sets of one vertex as the independence num-

ber is 2 or 1. However, then the bipartite graphs are graphs $K_{1,1}$, and there are vertices that belong to three of them. None of these three graphs is a P_3 -graph.

- (c) C_6 has six vertices and can only have partitionings of its vertices into sets of order 1 or 3. If all binomial sets have one vertex, the bipartite graphs B have one edge and each vertex belongs to two of them. C_6 is therefore a P_3 -graph, namely of C_6 . If two binomial sets of three are taken for the partitioning, which is possible, there should be a bipartite graph $K_{2,2}$, but there are six edges. The partitioning into three sets of one vertex and one of three vertices leads to a partitioning of the edges into three bipartite graphs $K_{1,2}$, as must be, and each vertex belongs to at most two of these bipartite graphs. The graph G^- has four vertices, one of degree 3 and three of degree 1. The bipartite graphs show that $G^- = K_{1,3}$. Extending G^- to G by addition of three extra vertices, one adjacent to each vertex of degree 1, leads to $G = S(K_{1,3})$ as original of C_6 .
- (d) P_5 also allows two partitionings of its five vertices. One is into five binomial sets of one vertex and leads to P_7 as original. The other is into two sets of one vertex and one set of three vertices, namely the two end vertices and the middle vertex of the path. The graph G^- is $K_{1,2}$ and this graph is to be extended so that the vertex of degree 2 gets degree 3 and the vertices of degree 1 get degree 2. The result is the graph Y .
- (e) $\overline{K_6}$ has six independent vertices. Partitioning into six sets of one leads to an original $6P_3$, consisting of six components P_3 . Partitioning into three sets of one and one set of three vertices gives original $3P_3 \cup K_{1,3}$. Partitioning into two sets of three leads to original $2K_{1,3}$, and finally, partitioning into one binomial set of six independent vertices leads to the only connected original $K_{1,4}$. ■

Example (e) especially shows that restriction to connected P_3 -graphs means hiding an important feature of these graphs. In general, there will be more originals than one. This is a serious difficulty for generalizing Whitney's result.

5. TRAVERSABILITY OF PATH GRAPHS

We consider eulerian tours and hamiltonian cycles in path graphs.

Lemma 5.1. If all vertices of a graph G have even degree or all vertices have odd degree, then the components of the path graph $P_k(G)$ ($k = 2, 3$) are eulerian.

Proof. If x is a vertex of $P_k(G)$ ($k = 2, 3$) representing a path P_k in G with endvertices v and w , then we have

$$\deg x = \deg v + \deg w - 2.$$

If all vertices of G have even or all have odd degree, $\deg x$ is even in $P_k(G)$ and it follows that the components of $P_k(G)$ are eulerian. ■

The converse of this lemma does not hold. A path P_k has image $P_k(P_k) = K_1$ and, less pathologically, $P_3(S(K_{1,3})) = C_6$, see Example 1.3.

More interesting is the hamiltonicity of path graphs. In the line graph $L(G)$ of a hamiltonian graph G the image of the hamiltonian cycle is a cycle C of the same length. As the vertices of $L(G)$ are grouped in cliques of vertices that are either on C or adjacent to vertices of C , all vertices can be included to form a hamiltonian cycle for $L(G)$. Because of Theorem 4.5 we have a rather similar situation for P_3 -graphs that we consider separately.

Lemma 5.2. If a graph G is 3-regular and hamiltonian, then $P_3(G)$ is hamiltonian.

Proof. The hamiltonian cycle of G is mapped on a cycle C of equal length in $P_3(G)$. By Theorem 4.5, $P_3(G)$ has binomial sets of three independent vertices as G is 3-regular. The edges of $P_3(G)$ can be partitioned into complete bipartite graphs $K_{2,2}$. The cycle C contains exactly one of the four edges of each $K_{2,2}$. Replacing each edge of C by a path P_4 traversing the graphs $K_{2,2}$ as illustrated in Figure 6, C is changed into a hamiltonian cycle of $P_3(G)$. ■

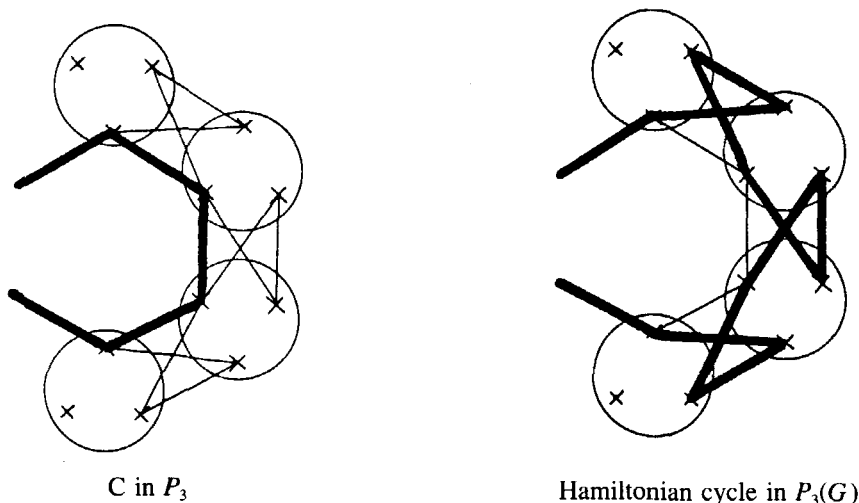


FIGURE 6.

For generalization of Lemma 5.2 to graphs that are d -regular, we remark that the cycle constructed in the graph $P_3(G)$ corresponds to a closed walk in G in which some edges are traversed three times, namely those of the hamiltonian cycle, and the others are traversed zero times.

Definition 5.3. A connected graph G is called *almost eulerian* if there exists a tour with the property that at most one edge incident with a vertex is not contained in the tour.

It will be clear that in an almost eulerian graph G only the vertices of odd degree can have an incident edge not contained in the tour. It is an obvious question as to whether d -regular connected graphs, with d odd, are almost eulerian. Let $o(G - S)$ denote the number of odd components of $G - S$, where S is a subset of $V(G)$.

Lemma 5.4. A connected d -regular graph, with d odd, is almost eulerian only if $o(G - S) \leq |S|$ for all $S \subseteq V(G)$.

Proof. G is almost eulerian only if G has a perfect matching. The tour determines a perfect matching. The result follows from the characterization of connected graphs with a perfect matching, given by Tutte [5]. ■

Theorem 5.5. If G is an almost eulerian d -regular graph, then $P_3(G)$ is hamiltonian.

Proof. Let d be even. Then G is eulerian and the eulerian tour is mapped on a cycle C in $P_3(G)$. Each binomial set of $\binom{d}{2}$ independent vertices is passed $d/2$ times by this cycle. We shall show that C can be transformed into a hamiltonian cycle. Like in the proof of Lemma 5.2, this is done by traversing edges of the complete bipartite graphs $K_{d-1, d-1}$ between two binomial sets.

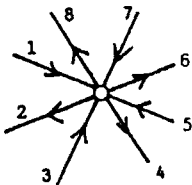
On passing a binomial set once, $d - 1$ vertices are to be included in the hamiltonian cycle. An edge of C will be replaced by a path passing the vertices of a graph $K_{d/2, d/2}$, that is, a subgraph of a graph $K_{d-1, d-1}$. We now show how this is done.

The eulerian tour in G determines an ordering $1, 2, \dots, d$ on the edges incident with a vertex of G . Here the numbers indicate ingoing and outgoing edges on the first up to the $d/2$ th time a vertex of G is passed in the eulerian tour that is traversed in a fixed manner, starting at an arbitrary edge. The eulerian tour thus determines a set of d "directions" in each vertex. These directions are used as labels of a $d \times d$ matrix M with entries zero or one.

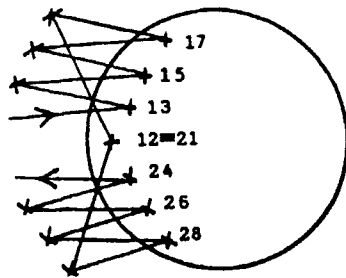
For $d = 8$ the matrix M looks as follows:

$$M = \begin{array}{c|cccc|cccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 3 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 5 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 6 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 7 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 8 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Along the diagonal this matrix has submatrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In the upper triangle, submatrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are found and in the lower triangle submatrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are found. The rows of this matrix are used to indicate the vertices of $P_3(G)$ that are to be included in the cycle C each time a vertex of G is passed by the eulerian tour. Note that these directions determine different P_3 's for different binomial sets. Let row i of M contain an entry 1 in column j . Then the P_3 with middle vertex v and endvertices in the directions i and j from v is to be included in the cycle C at the moment the eulerian tour arrives in v or departs from v using the edge with direction i . For example, each vertex of an 8-regular graph with a eulerian tour is passed 4 times. On first entrance the direction pairs 12, 13, 15, and 17 from the first row of M are used to determine the vertices, representing the paths P_3 , to be included in the cycle. On first exit the direction pairs 21 \equiv 12, 24, 26, and 28 from the second row of M are used. On second entrance the direction pairs 31, 34, 35, and 37 of the third row of M are used, etcetera. For one vertex the situation is sketched below.



vertex v in G



binomial set in $P_3(G)$
corresponding to paths
with middle vertex v

By doing this for all vertices of G , the cycle C is turned into a hamiltonian cycle for $P_3(G)$.

For d odd the $\binom{d}{2}$ vertices are to be included in C in $(d - 1)/2$ traversals of the binomial set. Each time d vertices are therefore to be included. Next to the vertex that is the image of a P_3 that is part of the tour in G , $d - 1$ other vertices are chosen, $(d - 1)/2$ from one class of one $K_{d-1, d-1}$ and $(d - 1)/2$ from one class of another $K_{d-1, d-1}$, both classes part of the binomial set. The problem is again to describe the pattern of choosing these vertices with respect to an ordering of the edges in G as induced by the tour. As d is odd one edge, the d th edge in the ordering is not traversed in G . All paths P_3 containing this edge have images to be included during the $(d - 1)/2$ traversals of the binomial set. We again use a $(0, 1)$ matrix to describe the choices. The pattern is illustrated for $d = 9$.

$$M = \begin{array}{c|cccc|cccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ \hline 3 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ \hline 5 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 6 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ \hline 7 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 8 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

By giving the pattern of choice for d odd it follows that it is possible to turn C into a hamiltonian cycle of $P_3(G)$. ■

6. MISCELLANEOUS RESULTS

Trees with Hamiltonian P_3 -Graphs

Definition 6.1. A tree T is a 1-2-tree if $\Delta(T) = 3$ and if every vertex with degree 1 has a neighbor with degree 2 and vice versa.

Let T be a 1-2-tree. Then Definition 6.1 implies that every vertex of T with degree 2 has a neighbor with degree 3. Otherwise $T \cong P_3$ or $T \cong P_4$, a contradiction. Furthermore Definition 6.1 implies that $|V(T)| \geq 7$ and that $S(K_{1,3})$ is the only 1-2-tree on 7 vertices. Suppose $T \neq S(K_{1,3})$ is a 1-2-tree. Then T contains a vertex v with degree 3 and with two neighbors u_1 and u_2 with degree 2. Let the neighbors of u_1 and u_2 with degree 1 be w_1 and w_2 , respectively. Then $T - \{w_1, u_2, w_2\}$ is again a 1-2-tree. This shows that every 1-2-tree can be reduced to $S(K_{1,3})$ by deleting vertices repeating this procedure. Conversely, every 1-2-tree can be obtained from $S(K_{1,3})$ by repeatedly applying the reverse procedure.

The following result on 1-2-trees is stated without proof.

Theorem 6.2. If T is a tree with $\Delta(T) \leq 3$, then $P_3(T)$ is hamiltonian if and only if T is a 1-2-tree.

We conjecture that the 1-2-trees are precisely the trees with hamiltonian P_3 -graphs.

Conjecture 6.3. If T is a tree with $\Delta(T) \geq 4$, then $P_3(T)$ is not hamiltonian.

We checked that the conjecture is true for $\Delta(T) = 4$.

Unicyclic Graphs with Hamiltonian P_3 -Graphs

Let G be unicyclic graph containing the cycle C . Then $G - V(C)$ is a forest. Every component of $G - V(C)$ has a unique neighbor on the cycle called its *source*. A component of $G - V(C)$ together with its source and the edge joining the source to a vertex of the component is called a *beam*. A beam with one edge is a *1-beam*; a beam is a *2-beam* if it is isomorphic to P_3 or if it can be obtained from P_3 by repeatedly applying the following procedure: for some end-vertex u other than the source with neighbor v , add new vertices u' , v' , and v'' and edges uu' , vv' , and $v'v''$ (2-beams are constructible from P_3 in the same way as 1-2-trees are constructible from $S(K_{1,3})$). If $C = v_1v_2 \dots v_kv_1$, then a *2-interval* of C is a sequence $\{v_i, v_{i+1}, \dots, v_j\}$ such that all v_r , $i \leq r \leq j$, are sources of a 2-beam, and v_{i-1} and v_{j+1} are no sources of a 2-beam (indices modulo k).

Definition 6.4. A unicyclic graph G with a cycle $C = v_1v_2 \dots v_kv_1$ and $\Delta(G) \leq 3$ is a *1-2-corona* if all beams are 1-beams or 2-beams, sources of 1-beams are not adjacent to vertices of C that are no sources, and if every 2-interval $I = \{v_i, v_{i+1}, \dots, v_j\}$ has the following property: If $|I|$ is odd, then precisely one of v_{i-1} and v_{j+1} is a source of a 1-beam ($v_{i-1} \neq v_{j+1}$); if $|I|$ is even, then either both v_{i-1} and v_{j+1} are sources of a 1-beam, or both v_{i-1} and v_{j+1} are no sources of a 1-beam (indices modulo k).

Note that a cycle is a 1-2-corona without beams; a unicyclic graph consisting of a cycle C and one 2-beam at every vertex of C is a 1-2-corona; a unicyclic graph consisting of a cycle C and one 1-beam at every vertex of C is also a 1-2-corona. We refer to the latter two types of unicyclic graphs as *2-coronas* and *1-coronas*, respectively.

Theorem 6.5. Let G be a unicyclic graph with $\Delta(G) \leq 3$. Then $P_3(G)$ is hamiltonian if and only if G is a 1-2-corona.

The proof is lengthy and tedious, and is therefore omitted.

There exist unicyclic graphs with maximum degree 4 and hamiltonian P_3 -graphs. Consider, for instance, graphs consisting of a cycle C and exactly two

2-beams at every vertex of C . Without proof we state that these graphs have hamiltonian P_3 -graphs. However, we do not believe there exist unicyclic graphs with maximum degree exceeding 4 and having hamiltonian P_3 -graphs.

Conjecture 6.6. If G is a unicyclic graph with $\Delta(G) \geq 5$, then $P_3(G)$ is not hamiltonian.

Folding and Unfolding

We finish this section by describing two interesting operations. We can define *folding* as the identification of two endedges of a graph in the following sense: Let G be a graph and let u_1 and u_2 be two distinct endvertices of G with distinct nonadjacent neighbors v_1 and v_2 , respectively. Let G' be the graph obtained from $G - \{u_1, u_2\}$ by adding the edge v_1v_2 . Then we say that G' is obtained from G by *folding the edges* u_1v_1 and u_2v_2 . A graph G is said to be *folded* into a graph H if H can be obtained from G by successively folding pairs of endedges. *Unfolding* works the other way around: Let G be a graph with a nonempty edge set, and let G' be obtained from G by deleting an edge $v_1v_2 \in E(G)$ and by adding two new vertices u_1, u_2 and the edges v_1u_1 and v_2u_2 . Then we say that G' is obtained from G by *unfolding the edge* v_1v_2 . A graph G is said to be *unfolded* into a graph H if H can be obtained from G by successively unfolding edges.

These operations are interesting because of the following results that are stated without proofs.

Lemma 6.7. If a graph G is folded into a graph H , then $P_3(G)$ is isomorphic to a spanning subgraph of $P_3(H)$.

Lemma 6.7 implies, for instance, that any graph that can be unfolded into a 1-2-tree or 1-2-corona has a hamiltonian P_3 -graph. More generally we have

Theorem 6.8. If a graph H is unfolded into a graph G where $P_3(G)$ is hamiltonian, then $P_3(H)$ is hamiltonian.

We note that Lemma 5.2 follows immediately from Theorems 6.5 and 6.8, by unfolding chords of the hamiltonian cycle of G .

Obviously, every connected graph can be unfolded into a tree or a unicyclic graph, by unfolding edges of cycles as long as this is necessary. It is not true, however, that every graph with a hamiltonian P_3 -graph can be unfolded into a 1-2-tree or a 1-2-corona. This is shown, e.g., by all 4-regular graphs. These graphs have hamiltonian P_3 -graphs by Theorem 5.5, but cannot be unfolded into a 1-2-tree or a 1-2-corona because vertices of degree 4 keep degree 4 in the unfolding process.

P_3 -Graphs of Subdivision Graphs

In this section we state without proof that the P_3 -graph of the subdivision graph of a graph G is isomorphic to the subdivision graph of the line graph of G .

Theorem 6.9. Let G be a graph with a nonempty edge set. Then $P_3(S(G)) \cong S(L(G))$.

This result has some interesting consequences related to previously discussed problems with respect to isomorphism and traversability. We omitted the proofs.

Corollary 6.10. Let $G_1 = S(H_1)$ and $G_2 = S(H_2)$ be connected graphs with isomorphic P_3 -graphs. Then G_1 and G_2 are isomorphic unless one is $S(K_{1,3})$ and the other is C_6 .

Corollary 6.11. Let $G = S(H)$ be a graph. Then $P_3(G)$ is hamiltonian if and only if G is an even cycle or $G = S(K_{1,3})$.

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