The average tree value for hypergraph games*

Liying Kang[†] Anna Khmelnitskaya[‡] Erfang Shan[§]
Dolf Talman[¶] Guang Zhang[‖]

January 14, 2020

Abstract

We consider transferable utility cooperative games (TU games) with limited cooperation introduced by hypergraph communication structure, the so-called hypergraph games. A hypergraph communication structure is given by a collection of coalitions, the hyperlinks of the hypergraph, for which it is assumed that only coalitions that are hyperlinks or connected unions of hyperlinks are able to cooperate and realize their worth. We introduce the average tree value for hypergraph games, which assigns to each player the average of the player's marginal contributions with respect to a particular collection of rooted spanning trees of the hypergraph. We also provide axiomatization of the average tree value for hypergraph games on the subclasses of cycle-free hypergraph games, hypertree games and cycle hypergraph games.

Keywords: TU game; hypergraph communication structure; average tree value; component fairness

JEL Classification Number: C71

Mathematics Subject Classification 2000: 91A12, 91A43

1 Introduction

In classical cooperative game theory it is assumed that any coalition of players may form and realize its worth, and fair distribution of total rewards among the players

*The research of Anna Khmelnitskaya was supported by RFBR (Russian Foundation for Basic Research) grant #18-01-00780. Her research was done partially during her stay at the University of Twente, whose hospitality is highly appreciated.

[†]L. Kang, Department of Mathematics, Shanghai University, 200444, Shanghai, P.R.China, e-mail: lykang@i.shu.edu.cn

[‡]A.B. Khmelnitskaya, Saint-Petersburg State University, Faculty of Applied Mathematics, Universitetskii prospekt 35, 198504, Peterhof, Saint-Petersburg, Russia, e-mail: a.b.khmelnitskaya@utwente.nl

[§]E. Shan, School of Management, Shanghai University, 200444, Shanghai, P.R.China, e-mail: efshan@shu.edu.cn

[¶]A.J.J. Talman, CentER, Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands, e-mail: talman@tilburguniversity.edu

[‖]G. Zhang, Business School, University of Shanghai for Science and Technology, 200093, Shanghai, P.R.China, e-mail: g.zhang@usst.edu.cn

takes into account capacities of all coalitions. For example, the most prominent solution of cooperative games with transferable utility, or TU games, the Shapley value, cf. Shapley (1953), assigns to each player as a payoff the average of the player's marginal contributions to his predecessors with respect to all possible linear orderings of the players. However, in many practical situations the collection of feasible coalitions is restricted by some social, economical, communication, or technical structure. The study of transferable utility games with limited cooperation introduced by means of an undirected communication graph, called for brevity graph games, is initiated in Myerson (1977). Assuming that only connected players can cooperate, the Myerson value for graph games is defined as the Shapley value of the so-called restricted game for which the worth of each coalition is equal to the sum of the worths of its connected components in the graph. Lately several other solutions for graph games based also on Myerson's assumption that only connected players can cooperate are proposed, in particular, the average tree solution, introduced by Herings, van der Laan, and Talman, cf. Herings et al. (2008), for cycle-free graph games and generalized by Herings, van der Laan, Talman, and Yang, cf. Herings et al. (2010), for the class of all graph games. In comparison to the Myerson value the average tree solution is stable on the subclass of superadditive cycle-free graph games and for cycle-free graph games the order of computational complexity of the average tree solution is linear in the number of players, while it is exponential for the Myerson value.

Yet, the communication graphs reflect only bilateral communication between the players. The idea of consideration of cooperative games with a more general communication structure, allowing to represent communication within sets of more than two players appears first in Myerson (1980), where NTU games with conference structure are investigated. In fact a conference in terms of Myerson coincides with a hyperlink of a hypergraph. TU games with hypergraph communication structure, called for brevity hypergraph games, are formally introduced by van den Nouweland, Borm, and Tijs, cf. van den Nouweland et al. (1992), where also the Myerson and position values[1] for hypergraph games are defined and axiomatized.

The goal of this paper is to extend the average tree solution for graph games to a value for hypergraph games. We introduce the average tree value for hypergraph games, which assigns to each player the average of the player's marginal contributions with respect to a particular collection of rooted spanning trees of the hypergraph. On the subclass of cycle-free hypergraph games the average tree value for hypergraph games, similar to the average tree solution for cycle-free graph games, is characterized by component efficiency and component fairness. However, while in a cycle-free graph removing a link between two nodes results in two components, in case of removing a hyperlink from a cycle-free hypergraph, the number of components can be more. We also provide axiomatic characterizations of the average tree value for hypergraph games on the subclasses of hypertree games and cycle hypergraph games.

The paper is organized as follows. Basic definitions and notation are given in Section 2. In Section 3 we introduce the set of admissible spanning trees of arbitrary hypergraph and define the average tree value for hypergraph games. Section 4 is devoted to axiomatic characterization of the average tree value for hypergraph games on three subclasses of hypergraph games – cycle-free hypergraph games, hypertree

[1] The position value for graph games iss first defined in Meessen (1988) and later studied and axiomatized by Borm, Owen, and Tijs, cf. Borm et al. (1992).

games, and cycle hypergraph games. Logical independence of the axioms in the axiomatizations obtained in Section 4 is shown by means of examples in Section 5.

2 Preliminaries

A *cooperative game with transferable utility*, or *TU game*, is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, with $v(\emptyset) = 0$, assigning to every *coalition* $S \subseteq N$ its *worth* $v(S)$, which can be freely distributed as payoff among the members of S. By $\mathcal{G}_N$ we denote the set of TU games with fixed player set N. For simplicity of notation and if no ambiguity appears we write v instead of (N, v). A game $v \in \mathcal{G}_N$ is *superadditive* if $v(S \cup Q) \geq v(S) + v(Q)$ for all $S, Q \subseteq N$ satisfying $S \cap Q = \emptyset$. For a finite set S, $|S|$ denotes the cardinality of S. The *unanimity game* with respect to coalition $S \in 2^N \setminus \{\emptyset\}$ is the game $u_S \in \mathcal{G}_N$ defined by $u_S(Q) = 1$ if $S \subseteq Q$ and 0 otherwise. The unanimity games $\{u_S\}_{\substack{S \subseteq N \\ S \neq \emptyset}}$ form a basis in $\mathcal{G}_N$, i.e., every game $v \in \mathcal{G}_N$ can be uniquely presented in the linear form

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta_v(S) u_S, \tag{1}$$

where $\Delta_v(S) \in \mathbb{R}$ is the *dividend* of coalition $S \in 2^N \setminus \{\emptyset\}$ in game v given by

$$\Delta_v(S) = \sum_{Q \subseteq S} (-1)^{|S|-|Q|} v(Q). \tag{2}$$

A communication structure on a set of nodes (players) N is specified by a hypergraph on N. A *hypergraph* on N is a set $H \subseteq \{e \in 2^N : |e| \geq 2\}$ of *hyperlinks*. H is *r-uniform* if $|e| = r$ for all $e \in H$. A 2-uniform hypergraph is an (undirected) *graph*. For $i \in N$, $H_{-i} = \{e \in H : i \notin e\}$ is the set of hyperlinks in H not containing player i, $H_i = \{e \in H : i \in e\}$ is the set of hyperlinks in H containing i, and $|H_i|$ is the *degree* of i in H. A player j is *adjacent* to player i in H if $\{i, j\} \subseteq e$ for some $e \in H$. A sequence $(i_1, e_1, i_2, e_2, \ldots, i_{k-1}, e_{k-1}, i_k)$, with $k \geq 2$, is a *chain* in H between player i_1 and player i_k if it satisfies the following conditions: (i) $i_1, \ldots, i_{k-1}$ are distinct players in N, (ii) $i_2, \ldots, i_k$ are distinct players in N, (iii) $e_1, \ldots, e_{k-1}$ are distinct hyperlinks in H, and (iv) $i_{t+1}, i_t \in e_t$ for all $t \in \{1, \ldots, k-1\}$. H is *connected* if n=1 or there is a chain in H between any two distinct players in N. For $S \subseteq N$, $H|_S = \{e \in H : e \subseteq S\}$ is the *subhypergraph* of H induced by S. $S \subseteq N$ is *connected* in H if $H|_S$ is connected. $C^H(S)$ denotes the set of subsets of $S \subseteq N$ that are connected in H. For $S \subseteq N$, Q is a *component* of S in H, if Q is a maximal connected subset of S in H. S/H denotes the set of components of $S \subseteq N$ in H. H is *linear* if $|e \cap e'| \leq 1$ for any two distinct $e, e' \in H$. A chain $(i_1, e_1, i_2, e_2, \ldots, i_{k-1}, e_{k-1}, i_k)$ in H is a *cycle* in H if $k \geq 3$ and $i_1 = i_k$. H is *cycle-free* if there is no cycle in H. If H is cycle-free, then H is linear, since $\{i_1, i_2\} \subseteq e_1 \cap e_2$, $e_1 \neq e_2$, implies that $(i_1, e_1, i_2, e_2, i_1)$ is a cycle. H is a *hypertree* if H is both connected and cycle-free. A hyperlink $e \in H$ is a *bridge* in H if $|K/(H \setminus \{e\})| > 1$, where $K \in N/H$ is such that $e \in H|_K$.

A *rooted tree* on a component $K \in N/H$ of N in a hypergraph H on N is a set $T \subseteq \{(i, j) : i, j \in K, i \neq j\}$ of *directed links* with one player $r(T)$, the *root* of T, satisfying that $(i, r(T)) \notin T$ for all $i \in K$ and for every $i \in K$, $i \neq r(T)$, there is a unique *directed path* $(i_1,\dots,i_k)$ in T from i_1 to i_k, where $i_1 = r(T)$, $i_k = i$, and $(i_h, i_{h+1}) \in T$ for all $h \in \{1,\dots,k-1\}$. If there exists a directed path in T from i to j, then j is a *successor* of i and i is an *predecessor* of j in T, and if $(i,j) \in T$, then j is an *immediate successor* of i and i is an *immediate predecessor* of j in T. For $i \in K$, S_i^T and $\widehat{S}_i^T$ denote the set of successors and the set of immediate successors of i in T, respectively, and $\bar{S}_i^T = S_i^T \cup \{i\}$. T is a *rooted spanning tree* of the subhypergraph $H|_K$ if $(i,j) \in T$ implies $\{i,j\} \subseteq e$ for some $e \in H|_{\bar{S}_i^T}$.

A *game with hypergraph communication structure*, or *hypergraph game*, is a triple (N, v, H), or shortly (v, H), where $v \in \mathcal{G}_N$ is a TU game and H is a hypergraph on N. In particular, if H is a graph, then (v, H) is a *graph game*. For fixed player set N, $\mathcal{G}_N^{\mathcal{H}}$ denotes the set of hypergraph games, $\mathcal{G}_N^{\mathcal{H}^c}$ the set of connected hypergraph games, $\mathcal{G}_N^{\mathcal{H}^{cf}}$ the set of cycle-free hypergraph games, and $\mathcal{G}_N^{\mathcal{H}^t}$ the set of hypertree games. The *hypergraph-restricted* game of a hypergraph game $(v, H) \in \mathcal{G}_N^{\mathcal{H}}$ is the TU game v^H, where $v^H(S) = \sum_{Q \in S/H} v(Q)$ for all $S \in 2^N$. A *payoff vector* is a vector $x \in \mathbb{R}^n$ that assigns payoff x_i to player $i \in N$. For a subset of hypergraph games $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, a *value* on $\mathcal{G}$ is a mapping $\xi : \mathcal{G} \to \mathbb{R}^n$ that assigns to every $(v, H) \in \mathcal{G}$ a payoff vector $\xi(v, H) \in \mathbb{R}^n$ with $\xi_i(v, H)$ as the payoff to player $i \in N$.

Following Myerson (1980) it is assumed that in a game with hypergraph communication structure each player can communicate with himself and all other players in a hyperlink he belongs to, moreover, all players of a hyperlink have to be present before communication between its members can take place. Therefore, only coalitions that are connected in the hypergraph are able to communicate in order to cooperate and realize their worth. A connected coalition in a hypergraph is either a singleton player or a single hyperlink or the connected union of two or more hyperlinks in the hypergraph. The set of connected coalitions in a hypergraph is a building set[2], cf. Koshevoy and Talman (2014). Note that different hypergraphs may have the same set of connected coalitions.

3 The average tree value for hypergraph games

In this section we introduce the average tree value for hypergraph games which generalizes the average tree solution for graph games, cf. Herings et al. (2008) and Herings et al. (2010). For any connected graph game, the average tree solution assigns as a payoff to each player the average of the player's marginal contributions to his successors in all admissible rooted spanning trees of the given communication graph. A rooted spanning tree of a graph is admissible if every player has in each component of the set of his successors in the tree exactly one immediate successor. For a hypergraph game we first determine a set of admissible spanning trees on each component in the hypergraph and then define the average tree value as the average of all marginal contributions with respect to these trees.

A rooted spanning tree on a component in a hypergraph is admissible if each component of the set of successors of a player in the tree consists of one of the player's immediate successors in the tree together with the successors in the tree of this immediate successor.

[2]A collection of coalitions $\mathcal{B}$ on N is a *building set* on N if (i) for any $S, Q \in \mathcal{B}$ such that $S \cap Q \neq \emptyset$ it holds that $S \cup Q \in \mathcal{B}$, and (ii) $\{i\} \in \mathcal{B}$ for all $i \in \mathcal{B}$, and therefore, it is also a union stable system, cf. Algaba et al. (2001).

Definition 1 Given a hypergraph H on N and component $K \in N/H$, a rooted spanning tree T of the subhypergraph $H|_K$ is *admissible* if $S_i^T/H = \{\bar{S}_j^T : j \in \widehat{S}_i^T\}$ for all $i \in K$.

The definition implies that an admissible rooted spanning tree on a component corresponds to a partial ordering on the set of players in the component. In such a tree the root can transmit information to every other player through a sequence of adjacent players in an efficient way, in the sense that every player is transmitting information in each component of his set of successors to only one adjacent player.

An admissible rooted spanning tree of a (connected) hypergraph H on N having player $r \in N$ as root can be constructed by the following procedure. Remove all hyperlinks in H containing player r, then the resulting hypergraph H_{-r} consists of one or more components, each component containing at least one player adjacent to r. In an admissible rooted spanning tree having player r as the root, one of these players is an immediate successor of the root. Then we remove all hyperlinks containing these players, and so on, until no hyperlinks are left. This procedure is illustrated by the following example.

Example 1 Consider the hypergraph $H = \{e_1, \ldots, e_5\}$ on a set of 8 players, where $e_1 = \{1, 2, 3\}$, $e_2 = \{3, 4, 7\}$, $e_3 = \{1, 5, 6\}$, $e_4 = \{5, 6, 7\}$, $e_5 = \{7, 8\}$, as depicted in Figure 1(a), and take $r = 1$. Player 1 (underlined) is contained in the hyperlinks e_1 and e_3 (underlined).

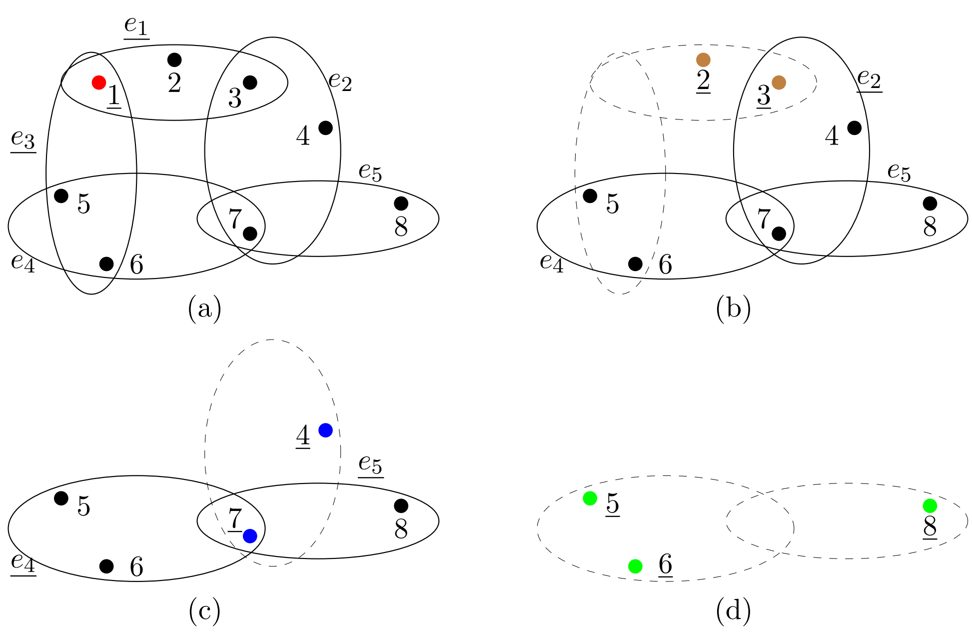


Figure 1: Illustration of the procedure to construct an admissible rooted spanning tree in Example 1.

Figure 1(b) shows the components in H_{-1} after removing the hyperlinks containing player 1. Player 2 is adjacent to player 1 in the component $\{2\}$ and players 3, 5, and 6 are adjacent to player 1 in the component $\{3, 4, 5, 6, 7, 8\}$. Figure 1(c) shows the components in the resulting hypergraph after removing in H_{-1} the hyperlink e_2 containing player 3. Player 4 is adjacent to player 3 in the component $\{4\}$ and only player 7 is adjacent to player 3 in the component $\{5, 6, 7, 8\}$. Figure 1(d) shows the three singleton players in the resulting hypergraph after removing next the hyperlinks e_4 and e_5 containing player 7.

The procedure of subsequently removing hyperlinks as depicted in Figure 1 leads to the admissible rooted spanning tree

$$T_1^1 = \{(1, 2), (1, 3), (3, 4), (3, 7), (7, 5), (7, 6), (7, 8)\}.$$

When taking in Figure 1(b) player 5 instead of player 3 we obtain the admissible rooted spanning tree

$$T_1^2 = \{(1, 2), (1, 5), (5, 6), (5, 7), (7, 3), (7, 4), (7, 8)\}$$

and when taking player 6 instead of player 3 we obtain the admissible rooted spanning tree

$$T_1^3 = \{(1, 2), (1, 6), (6, 5), (6, 7), (7, 3), (7, 4), (7, 8)\}.$$

There are no other admissible rooted spanning trees having player 1 as the root. The three admissible rooted spanning trees having player 1 as the root are depicted in Figure 2.

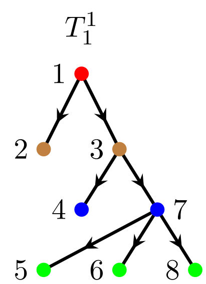


Figure 2: The three admissible rooted spanning trees in Example 1 having player 1 as the root.

For a component $K \in N/H$ of a hypergraph H on N we denote by $\mathcal{T}^H(K)$ the set of admissible rooted spanning trees of the subhypergraph $H|_K$ and by $\mathcal{T}_r^H(K)$ the ones having player $r \in K$ as the root.

Distinct hypergraphs having the same set of connected coalitions may not have the same sets of admissible rooted spanning trees as illustrated by the following example.

Example 2 Consider the hypergraphs $H = \{e_1, e_2\}$ and $H' = \{e_1, e_2, e_3\}$ on a set of 3 players, where $e_1 = \{1, 2\}$, $e_2 = \{2, 3\}$, $e_3 = \{1, 2, 3\}$, as depicted in Figure 3. For both H and H' the set of connected coalitions is equal to H'. Note that H is cycle-free, but H' is not, whereas H' is a building set, but H is not.

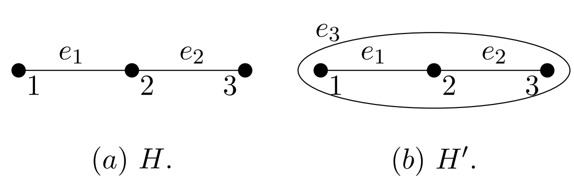
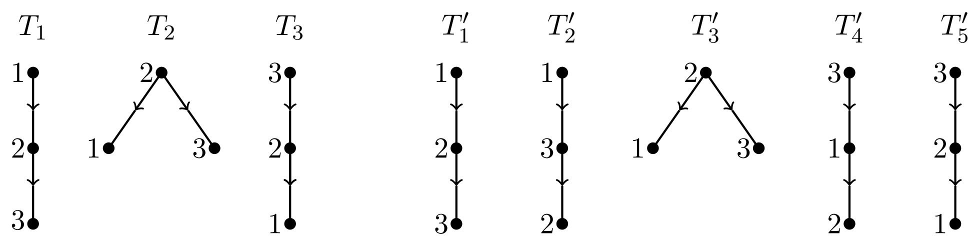


Figure 3: The hypergraphs of Example 2.

In Figure 4 the admissible rooted spanning trees of H and H' are depicted. H has three admissible rooted spanning trees, T_1, T_2, and T_3, and H' has five admissible rooted spanning trees, T'_1, T'_2, T'_3, T'_4, and T'_5. T'_2 and T'_4 are admissible rooted spanning trees of H' but not of H, because 1 and 3 do not both belong to any hyperlink in H, while both belong to the hyperlink e_3 in H'.



(*a*) $\mathcal{T}^H(N)$. (*b*) $\mathcal{T}^{H'}(N)$.

Figure 4: The admissible rooted spanning trees of Example 2.

The following lemma shows that any component in a cycle-free hypergraph has precisely one admissible rooted spanning tree with a given player as the root.

Lemma 1 *For a cycle-free hypergraph H on N it holds that $|\mathcal{T}_r^H(K)| = 1$ for all $K \in N/H$ and $r \in K$.*

Proof. Take any $K \in N/H$ and $r \in K$. From the procedure above it follows that $|\mathcal{T}_r^H(K)| \neq \emptyset$. Suppose $T, T' \in \mathcal{T}_r^H(K)$ and $T \neq T'$. Then some $k \in K \setminus \{r\}$ has distinct immediate predecessors in T and T'. Let $(r, \dots, i, j, \dots, k)$ be the directed path in T from r to k and $(r, \dots, i, j', \dots, k)$ the directed path in T' from r to k, where the indices up to i are the same in both sequences and $j' \neq j$. Clearly, $S_i^T = S_i^{T'}$, $\bar{S}_j^T \in S_i^T/H$, and $\bar{S}_{j'}^{T'} \in S_i^{T'}/H$. Since $k \in \bar{S}_j^T$ and $k \in \bar{S}_{j'}^{T'}$, this implies $\bar{S}_j^T = \bar{S}_{j'}^{T'}$. This set is connected in H and contains both j and j'. Therefore, there exists a chain $(\ell_1, e_1, \ell_2, e_2, \dots, \ell_{t-1}, e_{t-1}, \ell_t)$ in $H|_{\bar{S}_j^T}$ between $\ell_1 = j$ and $\ell_t = j'$. Since T is a rooted spanning tree of $H|_K$ and $(i, j) \in T$, there exists $e \in H|_{\bar{S}_i^T}$ such that $\{i, j\} \subseteq e$. Similarly, there exists $e' \in H|_{\bar{S}_i^{T'}}$ such that $\{i, j'\} \subseteq e'$. Since $i \notin \bar{S}_j^T$ and $i \notin \bar{S}_{j'}^{T'}$, it holds that $e \neq e_h$ and $e' \neq e_h$ for all $h \in \{1, \dots, t-1\}$. If $e = e'$, then $(j, e_1, \ell_2, e_2, \dots, \ell_{t-1}, e_{t-1}, j', e, j)$ is a cycle in H, contradicting that H is cycle-free. If $e \neq e'$, then $(j, e_1, \ell_2, e_2, \dots, \ell_{t-1}, e_{t-1}, j', e', i, e, j)$ is a cycle in H, contradicting again that H is cycle-free. Hence, $|\mathcal{T}_r^H(K)| = 1$. ∎

The lemma implies that in a cycle-free hypergraph H the number of admissible rooted spanning trees of the subhypergraph induced by any component is equal to the size of the component, i.e., $|\mathcal{T}^H(K)| = |K|$ for all $K \in N/H$.

To each admissible rooted spanning tree on a component in a hypergraph game corresponds the marginal contribution of any player in the component, being his contribution in worth when he joins (the components of the set of) his successors in the tree. More precisely, for a hypergraph game $(v, H) \in \mathcal{G}_N^{\mathcal{H}}$ and component

$K \in N/H$, the *marginal contribution* of player $i \in K$ corresponding to admissible rooted spanning tree $T \in \mathcal{T}^H(K)$ is given by

$$m_i^T(v,H) = v(\bar{S}_i^T) - \sum_{Q \in S_i^T/H} v(Q).$$

Since, by definition of admissibility of a rooted spanning tree, $Q \in S_i^T/H$ if and only if Q consists of one of the immediate successors of player i and the successors of this player in the tree, it holds that

$$m_i^T(v,H) = v(\bar{S}_i^T) - \sum_{j \in \widehat{S}_i^T} v(\bar{S}_j^T), \tag{3}$$

being player i's contribution in worth to his immediate successors and their successors in the tree.

The average tree value of a hypergraph game assigns to every player the average of his marginal contributions corresponding to all admissible rooted spanning trees of the subhypergraph induced by the component to which the player belongs.

Definition 2 The *average tree value* for hypergraph games assigns to every hypergraph game $(v,H) \in \mathcal{G}_N^{\mathcal{H}}$ a payoff vector $AT(v,H)$ given by

$$AT_i(v,H) = \frac{1}{|\mathcal{T}^H(K)|} \sum_{T \in \mathcal{T}^H(K)} m_i^T(v,H), \quad i \in K,\, K \in N/H.$$

If the underlying hypergraph is a graph the average tree value of a hypergraph game reduces to the average tree solution for graph games. Similar to the position value, but not to the Myerson value, the average tree value may differ for different hypergraph games that have the same set of connected coalitions. This is because different hypergraphs with the same set of connected coalitions may have different sets of admissible rooted spanning trees and therefore different marginal contributions of the players.

The following example illustrates the computation procedure of the average tree value for hypergraph games.

Example 3 Consider the hypergraph game $(v,H) \in \mathcal{G}_N^{\mathcal{H}}$ on a set of 5 players with $v = u_{\{1,5\}}$ and $H = \{e_1, e_2, e_3\}$, where $e_1 = \{1,2,3\}$, $e_2 = \{2,3,4\}$, $e_3 = \{4,5\}$, as depicted in Figure 5. Note that H is not cycle-free.

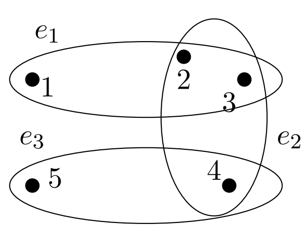


Figure 5: The hypergraph H of Example 3.

H contains eight admissible rooted spanning trees depicted in Figure 6. T_1^1 and T_1^2 have player 1 as the root, T_2 has player 2 as the root, T_3 has player 3 as the root, T_4^1 and T_4^2 have player 4 as the root, and T_5^1 and T_5^2 have player 5 as the root.

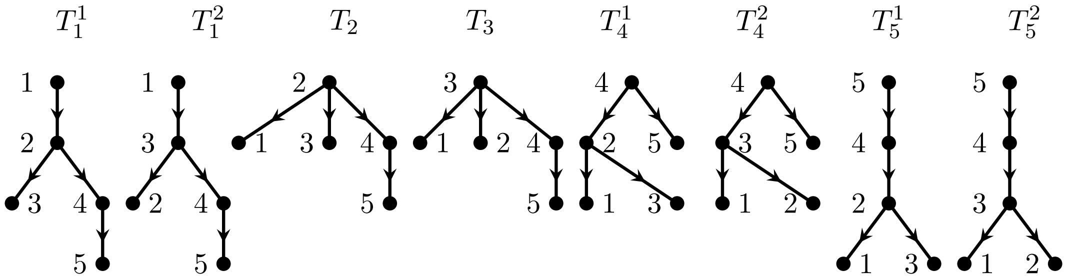


Figure 6: The eight admissible rooted spanning trees of Example 3.

From (3) we obtain that the marginal contribution vectors are given by

$$\begin{aligned}
&m^{T_1^1}(v,H) = m^{T_1^2}(v,H) = (1,0,0,0,0),\\
&m^{T_2}(v,H) = (0,1,0,0,0),\\
&m^{T_3}(v,H) = (0,0,1,0,0),\\
&m^{T_4^1}(v,H) = m^{T_4^2}(v,H) = (0,0,0,1,0),\\
&m^{T_5^1}(v,H) = m^{T_5^2}(v,H) = (0,0,0,0,1).
\end{aligned}$$

The average tree value of (v,H) is the average of these eight vectors:

$$AT(v,H) = (\frac{1}{4},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4}).$$

Next we study the core stability of the average tree value for hypergraph games. The core of a hypergraph game consists of all payoff vectors at which every component of the hypergraph gets its worth and each connected coalition gets at least its worth. Formally, the *core* of a hypergraph game $(v,H) \in \mathcal{G}_N^{\mathcal{H}}$ is given by

$$C(v,H) = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} \sum_{i\in K} x_i = v(K), & \text{for all } K \in N/H, \\ \sum_{i\in S} x_i \geq v(S), & \text{for all } S \in C^H(N) \end{array} \right\}.$$

The following theorem shows that on the class of cycle-free hypergraph games the average tree value is an element of the core if the underlying game is superadditive.

Theorem 1 *If v is superadditive, then for any $(v,H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$ it holds that $AT(v,H) \in C(v,H)$.*

Proof. For $Q \in 2^N \setminus \{\emptyset\}$, let $v|_Q$ denote the *subgame* of v on Q, where $v|_Q(S) = v(S)$ for all $S \subseteq Q$. Since $S \in C^H(N)$ if and only if $S \in C^{H|_K}(K)$ for some $K \in N/H$, it holds that $x \in C(v,H)$ if and only if $(x_i)_{i\in K} \in C(v|_K, H|_K)$ for all $K \in N/H$. To show that $(AT_i(v,H))_{i\in K} \in C(v|_K, H|_K)$ for all $K \in N/H$, we first prove that for every $K \in N/H$ and $T \in \mathcal{T}^H(K)$ it holds that $(m_i^T(v,H))_{i\in K} \in C(v|_K, H|_K)$.

Take any $K \in N/H$ and $T \in \mathcal{T}^H(K)$. From (3) it immediately follows that $\sum_{i\in K} m_i^T(v,H) = v(K)$. Let $S \in C^{H|_K}(K)$. We show that $\sum_{h\in S} m_h^T(v,H) \geq v(S)$. Since S is connected in $H|_K$ and T is admissible, there exists a unique $i \in S$ such that $S \subseteq \bar{S}_i^T$.

Let $\widehat{S}_S^T = \{j \in K \setminus S : (h,j) \in T, h \in S\}$ be the set of immediate successors of S in T. Since H is cycle-free, it holds that $\bar{S}_i^T$ is partitioned by the connected coalitions S and $\bar{S}_j^T$, $j \in \widehat{S}_S^T$. Therefore, we have

$$\begin{aligned}
\sum_{h\in S} m_h^T(v,H) &= \sum_{h\in S}\big(v(\bar{S}_h^T) - \sum_{j\in\widehat{S}_h^T} v(\bar{S}_j^T)\big) \\
&= v(\bar{S}_i^T) - \sum_{j\in\widehat{S}_S^T} v(\bar{S}_j^T) \\
&= v(S \cup (\bigcup_{j\in\widehat{S}_S^T} \bar{S}_j^T)) - \sum_{j\in\widehat{S}_S^T} v(\bar{S}_j^T) \\
&\geq v(S),
\end{aligned}$$

where the first equality follows from (3), the second equality follows because for every $h \in S \setminus \{i\}$ the first term cancels, the third equality follows from the fact that $\bar{S}_i^T = S \cup (\bigcup_{j\in\widehat{S}_S^T} \bar{S}_j^T)$, and the inequality follows from repeated application of superadditivity of v and the fact that $\bar{S}_i^T$ is partitioned by S and $\bar{S}_j^T$, $j \in \widehat{S}_S^T$. Together with $\sum_{h\in K} m_h^T(v,H) = v(K)$, we obtain $(m_i^T(v,H))_{i\in K} \in C(v|_K, H|_K)$.

Since, for every $K \in N/H$, $C(v|_K, H|_K)$ is a convex set and $(AT_i(v,H))_{i\in K}$ is a convex combination of $(m_i^T(v,H))_{i\in K}$ over all $T \in \mathcal{T}^H(K)$, we obtain that $(AT_i(v,H))_{i\in K} \in C(v|_K, H|_K)$ for all $K \in N/H$. ∎

4 Axiomatizations

This section provides several characterizations of the average tree value on some subclasses of hypergraph games. In the following three subsections, we characterize the average tree value on the class of cycle-free hypergraph games, on the class of hypertree games, and on the class of cycle hypergraph games. The theorems in this section generalize the corresponding results for graph games obtained in Herings et al. (2008), Mishra and Talman (2010), and Selçuk et al. (2013).

4.1 Cycle-free hypergraph games

In this subsection, we show that on the class of cycle-free hypergraph games the average tree value can be characterized by component efficiency and component fairness, where the latter property is generalized from graph games to hypergraph games.

First, we introduce a standard property, called component efficiency, on any subclass of hypergraph games.

- A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$ is *component efficient* if for every $(v,H) \in \mathcal{G}$ it holds that

$$\sum_{h\in K} \xi_h(v,H) = v(K), \qquad \text{for all } K \in N/H.$$

Component efficiency requires that each component distributes its worth among its members.

The following property is an extension of component fairness introduced in Herings et al. (2008) for cycle-free graph games and deals with the payoff changes when a hyperlink is removed.

- A value ξ on $\mathcal{G}_N^{\mathcal{H}^{cf}}$ is *component fair* if for every $(v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$ and $e \in H$ it holds that

$$\frac{1}{|K|}\sum_{h\in K}\big(\xi_h(v,H)-\xi_h(N,v,H\setminus\{e\})\big)=\frac{1}{|K'|}\sum_{h\in K'}\big(\xi_h(v,H)-\xi_h(N,v,H\setminus\{e\})\big),$$

for all distinct $K, K' \in N/(H \setminus \{e\})$ satisfying $K \cap e \neq \emptyset$ and $K' \cap e \neq \emptyset$.

Component fairness states that if a hyperlink is deleted, the average payoff difference is the same for each resulting component. Note that in a cycle-free hypergraph more than two components may result by deleting a hyperlink, while in a cycle-free graph always two components result.

Lemma 2 *The average tree value satisfies component efficiency on the class of hypergraph games and satisfies component fairness on the class of cycle-free hypergraph games.*

Proof. To show component efficiency on $\mathcal{G}_N^{\mathcal{H}}$, take any $(v, H) \in \mathcal{G}_N^{\mathcal{H}}$ and $K \in N/H$. From (3) it follows that $\sum_{h\in K} m_h^T(v,H) = v(K)$ for all $T \in \mathcal{T}^H(K)$. Hence, we have

$$\begin{aligned}\sum_{h\in K} AT_h(v,H) &= \sum_{h\in K}\frac{1}{|\mathcal{T}^H(K)|}\sum_{T\in\mathcal{T}^H(K)} m_h^T(v,H)\\ &= \frac{1}{|\mathcal{T}^H(K)|}\sum_{T\in\mathcal{T}^H(K)}\sum_{h\in K} m_h^T(v,H)\\ &= \frac{1}{|\mathcal{T}^H(K)|}\sum_{T\in\mathcal{T}^H(K)} v(K) \;=\; v(K).\end{aligned}$$

To show component fairness on $\mathcal{G}_N^{\mathcal{H}^{cf}}$, take any $(v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$ and $e \in H$. Let $K \in N/H$ be such that $e \in H|_K$ and let $|e| = m$. Note that $m \geq 2$. By Lemma 1, for every $r \in K$ there exists a unique $T_r \in \mathcal{T}_r^H(K)$. Since H is cycle-free, K has m components, say, $K^1, \ldots, K^m$, in $H \setminus \{e\}$. Take any $p \in \{1, \ldots, m\}$. For $r \in K$ it holds that

$$\sum_{h\in K^p} m_h^{T_r}(v,H) = v(K^p), \qquad \text{if } r \notin K^p,$$

and

$$\sum_{h\in K^p} m_h^{T_r}(v,H) = v(K) - \sum_{q\in\{1,\ldots,m\}\setminus\{p\}} v(K^q), \qquad \text{if } r \in K^p.$$

By Lemma 1, the number of admissible rooted spanning trees of $H|_K$ for which in the corresponding marginal contributions coalition K^p receives total payoff $v(K^p)$

is equal to $|K \setminus K^p|$ and the number of admissible rooted spanning trees of $H|_K$ for which in the corresponding marginal contributions coalition K^p receives total payoff $v(K) - \sum_{q \neq p} v(K^q)$ is equal to $|K^p|$. Therefore, we obtain

$$\sum_{h \in K^p} AT_h(v,H) = \frac{|K^p|\big(v(K) - \sum_{q \neq p} v(K^q)\big) + |K \setminus K^p| v(K^p)}{|K|}.$$

Rearranging yields

$$\frac{1}{|K^p|}\Big(\sum_{h \in K^p} AT_h(v,H) - v(K^p)\Big) = \frac{v(K) - \sum_{q=1}^{m} v(K^q)}{|K|}.$$

Since the average tree value satisfies component efficiency, it holds that

$$v(K^p) = \sum_{h \in K^p} AT_h(v, H \setminus \{e\}),$$

from which it follows that

$$\frac{1}{|K^p|}\Big(\sum_{h \in K^p} AT_h(v,H) - AT_h(v, H \setminus \{e\})\Big) = \frac{v(K) - \sum_{q=1}^{m} v(K^q)}{|K|},$$

which is independent of $p \in \{1, \ldots, m\}$. ∎

To prove uniqueness, we need the following property, see also Berge (1973).

Lemma 3 *For any component $K \in N/H$ of a cycle-free hypergraph H on N, with $|K| \geq 2$, it holds that $\sum_{e \in H|_K} (|e| - 1) = |K| - 1$.*

Proof. We prove the statement by induction on $|H|_K|$. If $|H|_K| = 1$, i.e., $H|_K = \{e\}$ for some $e \in H$, then $K = e$. So, we have $|e| - 1 = |K| - 1$.

Assume that the assertion is true for every component of a cycle-free hypergraph with less than ℓ hyperlinks for some $\ell \geq 2$. Let K be a component of a cycle-free hypergraph H with $|H|_K| = \ell$. Let $P = (i_1, e_1, i_2, e_2, \ldots, i_{k-1}, e_{k-1}, i_k)$ be a longest chain in $H|_K$. Then $k \geq 3$ and each player in e_{k-1}, except i_{k-1}, has degree 1, because if some $j \in e_{k-1}$, $j \neq i_{k-1}$, has degree more than 1, there exists $e \in H|_K$, $e \neq e_{k-1}$, such that $j \in e$. It contradicts that P is the longest chain in $H|_K$ if $e \neq e_h$ for every $h \in \{1, \ldots, k-2\}$, and it contradicts that H is cycle-free if $e = e_h$ for some $h \in \{1, \ldots, k-2\}$. Hence, $K' = (K \setminus e_{k-1}) \cup \{i_{k-1}\}$ is a component of N in the cycle-free hypergraph $H \setminus \{e_{k-1}\}$ satisfying $|H|_{K'}| = \ell - 1$. It holds that $H|_K = H|_{K'} \cup \{e_{k-1}\}$ and $|K| = |K'| + |e_{k-1}| - 1$. By the induction hypothesis, we have $\sum_{e \in H|_{K'}} (|e| - 1) = |K'| - 1$. Therefore,

$$\sum_{e \in H|_K} (|e| - 1) = |K'| - 1 + (|e_{k-1}| - 1) = |K| - 1.$$

∎

The lemma reveals the relation between the number of players and the number of hyperlinks in cycle-free hypergraph structures. In fact, there is a similar well-known result for cycle-free graphs (see Corollary 1.5.3 in Diestel (2000)) that in any component of a cycle-free graph the number of links is equal to the number of players minus one.

Lemma 4 *On the class of cycle-free hypergraph games, there is a unique value that satisfies component efficiency and component fairness.*

Proof. Suppose that on the class of cycle-free hypergraph games a value ξ satisfies component efficiency and component fairness. We show that for any cycle-free hypergraph game $(v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$ the two properties induce for every component $K \in N/H$ a system of $|K|$ linearly independent equations in $|K|$ unknowns, which uniquely determines $(\xi_i(v, H))_{i \in K}$.

Take any $K \in N/H$. If $|K| = 1$, let $K = \{i\}$, then component efficiency implies $\xi_i(v, H) = v(\{i\})$. Suppose $|K| \geq 2$, then $H|_K \neq \emptyset$ and take any $e \in H|_K$. Let $K_e^1, \ldots, K_e^{m_e}$ denote the $m_e = |e|$ components of K in $H \setminus \{e\}$. Component efficiency of ξ implies

$$\sum_{h \in K} \xi_h(v, H) = v(K) \tag{4}$$

and

$$\sum_{h \in K_e^p} \xi_h(v, H \setminus \{e\}) = v(K_e^p), \quad \text{for all } p \in \{1, \ldots, m_e\}. \tag{5}$$

Therefore, component fairness of ξ implies

$$\frac{1}{|K_e^p|}\Big(\sum_{h \in K_e^p} \xi_h(v, H) - v(K_e^p)\Big) = \frac{1}{|K_e^q|}\Big(\sum_{h \in K_e^q} \xi_h(v, H) - v(K_e^q)\Big),$$

for all $p, q \in \{1, \ldots, m_e\}$. Let $\alpha_e = (v(K) - \sum_{p=1}^{m_e} v(K_e^p))/|K|$, then by (4) for every $p \in \{1, \ldots, m_e\}$ it holds that

$$\frac{1}{|K_e^p|}\Big(\sum_{h \in K_e^p} \xi_h(v, H) - v(K_e^p)\Big) = \alpha_e$$

and therefore

$$\sum_{h \in K_e^p} \xi_h(v, H) = |K_e^p| \alpha_e + v(K_e^p). \tag{6}$$

Next, take any $r \in K$ and let T_r denote the unique admissible rooted spanning tree of $H|_K$ having player r as the root. Without loss of generality we assume $|\bar{S}_h^{T_r}| \leq |\bar{S}_k^{T_r}|$ whenever $h < k$. For $j \in K \setminus \{r\}$, let $j' \in K$ be such that $(j', j) \in T_r$, and let $e(j)$ be the unique hyperlink in $H|_K$ containing both j' and j. Since T_r is a rooted spanning tree of $H|_K$, such a hyperlink exists, and it is unique because H is cycle-free. Then there exists a unique $p \in \{1, \ldots, m_{e(j)}\}$ such that $\bar{S}_j^{T_r} = K_{e(j)}^p$. Note that $r \notin K_{e(j)}^p$. Conversely, for every $e \in H|_K$ and $p \in \{1, \ldots, m_e\}$ satisfying that K_e^p does not contain r, there exists a unique $j \in K \setminus \{r\}$ satisfying $\bar{S}_j^{T_r} = K_e^p$.

Hence, according to Lemma 3, for any $K \in N/H$ there are $|K| - 1 = \sum_{e \in H|_K}(|e| - 1)$ equations of type (6) for which $K_e^p = \bar{S}_j^{T_r}$ for some $j \in K \setminus \{r\}$. Combined with equation (4), they form the following system of $|K|$ linear equations with $|K|$ unknowns,

$$\sum_{h \in \bar{S}_j^{T_r}} \xi_h(v, H) = \begin{cases} |\bar{S}_j^{T_r}| \alpha_{e(j)} + v(\bar{S}_j^{T_r}), & \text{if } j \in K \setminus \{r\}, \\ v(K), & \text{if } j = r. \end{cases}$$

The coefficient matrix associated to this system is lower triangular with each diagonal element equal to 1. Therefore, the $|K|$ equations in the system are linearly independent and uniquely determine $\xi_h(v,H)$ for all $h \in K$. Since this holds for every $K \in N/H$, $\xi(v,H)$ is uniquely determined for any $(v,H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$. ∎

From Lemma 2 and Lemma 4 we obtain the following theorem.

Theorem 2 *On the class of cycle-free hypergraph games, the average tree value is the unique value that satisfies component efficiency and component fairness.*

4.2 Hypertree games

From Definition 2 we see that the payoffs in a component by using the average tree value do not affect the payoffs in other components, which implies that there is no loss of generality to consider a hypertree game instead of a cycle-free hypergraph game. Therefore, in this subsection we focus on hypertree games and provide another characterization of the average tree value. For simplicity we write in this and the following subsection $\mathcal{T}^H$ instead of $\mathcal{T}^H(N)$ to denote the set of admissible rooted spanning trees in a connected hypergraph H on N.

Before stating the characterization, we introduce several properties. The first two are well known properties in the theory of hypergraph games.

- A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}^c}$ is *efficient* if for every $(v,H) \in \mathcal{G}$ it holds that $\sum_{h\in N} \xi_h(v,H) = v(N)$.

Note that component efficiency reduces to efficiency on any subclass of connected hypergraph games.

- A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$ is *linear* if for every $(v,H),(w,H) \in \mathcal{G}$ and $a,b \in \mathbb{R}$, it holds that

$$\xi(av+bw,H) = a\xi(v,H) + b\xi(w,H).$$

The following three properties are adapted from the null-player property in Selçuk et al. (2013) and a symmetry and independence property in Mishra and Talman (2010) for graph games.

A player $i \in N$ is a *restricted null-player* in hypergraph game $(v,H) \in \mathcal{G}_N^{\mathcal{H}}$ if $v(S) = \sum_{K\in(S\setminus\{i\})/H} v(K)$ for all $S \in C^H(N)$ satisfying $i \in S$.

- A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$ satisfies the *restricted null-player property* if for every $(v,H) \in \mathcal{G}$ and restricted null-player i in (v,H) it holds that $\xi_i(v,H) = 0$.

The restricted null-player property states that if a player in a hypergraph game contributes nothing to any connected coalition, then this player gets zero payoff.

- A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}^c}$ satisfies *weak symmetry* if for every $(v,H) \in \mathcal{G}$ satisfying $v(S) = 0$ for all $S \in C^H(N)$, $S \neq N$, it holds that $\xi_i(v,H) = \xi_j(v,H)$ for all $i,j \in N$.

Weak symmetry states that if for a connected hypergraph game the worth of every connected proper coalition is zero, then all players get the same payoff.

• A value ξ on $\mathcal{G}_N^{\mathcal{H}^t}$ satisfies *independence in unanimity games* if for every $(u_Q, H) \in \mathcal{G}_N^{\mathcal{H}^t}$, $Q \in C^H(N)$, and $e \in H \setminus H|_Q$ satisfying $Q \cup e \in C^H(N)$, it holds that $\xi_i(u_Q, H) = \xi_i(u_{Q\cup e}, H)$ for all $i \in Q \setminus e$.

Independence in unanimity games for hypertree games states that when in a hypertree a hyperlink joins a connected coalition, then in the associated unanimity games every player in the coalition who is not a member of the hyperlink receives the same payoff.

From linearity and the restricted null-player property, we obtain the following property.

Lemma 5 *If a value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$ satisfies linearity and the restricted null-player property, then $\xi(v, H) = \xi(v^H, H)$ for all $(v, H) \in \mathcal{G}$.*

Proof. Consider the game $(w, H) \in \mathcal{G}$, where $w = v - v^H$. Every player is a restricted null-player in (w, H), since $w(S) = 0$ for all $S \in C^H(N)$. Therefore, $\xi_i(w, H) = 0$ for all $i \in N$. By linearity and since $v = w + v^H$, we obtain $\xi(v, H) = \xi(w, H) + \xi(v^H, H) = \xi(v^H, H)$. ∎

The following two lemmas show that on the class of connected hypergraph games the average tree value satisfies efficiency, linearity, and the restricted null-player property, and when the underlying structure is a hypertree, the average tree value satisfies weak symmetry.

Lemma 6 *On the class of connected hypergraph games, the average tree value satisfies efficiency, linearity, and the restricted null-player property.*

Proof. From Lemma 2 it follows that on the class of connected hypergraph games the average tree value satisfies efficiency.

Concerning linearity, for any (v, H), $(w, H) \in \mathcal{G}_N^{\mathcal{H}^c}$, $a, b \in \mathbb{R}$, and $T \in \mathcal{T}^H$, by (3), we have

$$\begin{aligned} m_i^T(av+bw, H) &= (av+bw)(\bar{S}_i^T) - \sum_{j\in\widehat{S}_i^T}(av+bw)(\bar{S}_j^T) \\ &= av(\bar{S}_i^T) + bw(\bar{S}_i^T) - \sum_{j\in\widehat{S}_i^T} av(\bar{S}_j^T) - \sum_{j\in\widehat{S}_i^T} bw(\bar{S}_j^T) \\ &= a\big(v(\bar{S}_i^T) - \sum_{j\in\widehat{S}_i^T} v(\bar{S}_j^T)\big) + b\big(w(\bar{S}_i^T) - \sum_{j\in\widehat{S}_i^T} w(\bar{S}_j^T)\big) \\ &= am_i^T(v, H) + bm_i^T(w, H), \quad \text{for all } i \in N. \end{aligned}$$

From Definition 2 it follows that the average tree value satisfies linearity.

If player $i \in N$ is a restricted null-player in $(v, H) \in \mathcal{G}_N^{\mathcal{H}^c}$, then for every $T \in \mathcal{T}^H$ it holds that

$$m_i^T(v, H) = v(\bar{S}_i^T) - \sum_{j\in\widehat{S}_i^T} v(\bar{S}_j^T) = 0.$$

Hence, again from Definition 2 it follows that $AT_i(v, H) = 0$. ∎

Lemma 7 *On the class of hypertree games the average tree value satisfies weak symmetry.*

Proof. Let $(v, H) \in \mathcal{G}_N^{\mathcal{H}^t}$ be such that $v(S) = 0$ for all $S \in C^H(N)$, $S \neq N$, then for every $r \in N$

$$m_i^{T_r}(v, H) = \begin{cases} v(N), & \text{if } i = r, \\ 0, & \text{otherwise,} \end{cases}$$

where T_r is the unique admissible rooted spanning tree of H having player r as the root. Therefore, $AT_i(v, H) = v(N)/n$ for all $i \in N$. ∎

Next, we examine the dividends on hypergraph-restricted games. The following lemma shows that only connected coalitions have non-zero dividends, which generalizes the same result for graph games in Owen (1986) and has a flavor similar to the result in Algaba et al. (2015) for games on union stable systems.

Lemma 8 *For any $(v, H) \in \mathcal{G}_N^{\mathcal{H}}$ it holds that $\Delta_{v^H}(S) = 0$ for all $S \notin C^H(N)$.*

Proof. We prove the property by induction on $|S|$. Since every singleton player is connected, the initial step is for $|S| = 2$. If $S = \{i, j\} \notin C^H(N)$, then $\Delta_{v^H}(S) = v^H(S) - v(\{i\}) - v(\{j\}) = 0$.

Assume that $S \notin C^H(N)$ for some $|S| \geq 3$ and $\Delta_{v^H}(Q) = 0$ for all $Q \notin C^H(N)$ with $|Q| < |S|$, then we show that $\Delta_{v^H}(S) = 0$. From (1) it follows that $v^H(S) = \sum_{Q \subseteq S} \Delta_{v^H}(Q)$. Hence,

$$\begin{aligned} \Delta_{v^H}(S) &= v^H(S) - \sum_{Q \subsetneq S} \Delta_{v^H}(Q) \\ &= \sum_{K \in S/H} v(K) - \sum_{Q \in C^H(S)} \Delta_{v^H}(Q) \\ &= \sum_{K \in S/H} \sum_{Q \in C^H(K)} \Delta_{v^H}(Q) - \sum_{Q \in C^H(S)} \Delta_{v^H}(Q) \\ &= 0, \end{aligned}$$

where the second equality follows from the definition of v^H and the hypothesis, the third equality follows from (1) and the fact that S is not connected, and the last equality holds because $\{Q \in C^H(K) : K \in S/H\} = C^H(S)$. ∎

From Lemma 8 and equation (1) it follows that for every hypergraph the unanimity games with respect to the set of connected coalitions form a linear basis for the set of hypergraph-restricted games. More specifically, for every $(v, H) \in \mathcal{G}_N^{\mathcal{H}}$ it holds that

$$v^H = \sum_{S \in C^H(N)} \Delta_{v^H}(S) u_S, \tag{7}$$

and therefore

$$v^H(Q) = \sum_{S \in C^H(Q)} \Delta_{v^H}(S), \text{ for all } Q \in 2^N \setminus \{\emptyset\}.$$

From Lemma 5 and (7) we obtain the following result.

Lemma 9 *If a value* ξ *on* $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$ *satisfies linearity and the restricted null-player property, then*

$$\xi(v,H) = \sum_{S \in C^H(N)} \Delta_{v^H}(S)\xi(u_S,H), \ \textit{for all } (v,H) \in \mathcal{G}. \tag{8}$$

Lemma 9 implies that if on a subclass of hypergraph games a value satisfies linearity and the restricted null-player property, then the value for any hypergraph game in this subclass is a linear combination of the value for the connected coalitions unanimity games with the same hypergraph structure.

Now, we rewrite the expression of the average tree value for hypertree unanimity games, which is similar to Herings et al. (2008) for the average tree solution for cycle-free graph games.

First, we need to introduce some notation. For a hypertree H, $S \in C^H(N)$, and $j \in S$, let $P_S^H(j)$ be the set of players outside S represented by player j, where player $j \in S$ represents a player $h \in N \setminus S$ if h is connected to j and the hyperlinks of the unique chain between j and h do not contain any player in S except player j. Specifically,

$$P_S^H(j) = \{k \in K : K \in (N \setminus S)/H, h \in K \text{ for some } \{j,h\} \subseteq e, e \in H\}.$$

Lemma 10 *For any* $(u_Q,H) \in \mathcal{G}_N^{\mathcal{H}^t}$, $Q \in C^H(N)$, *it holds that*

$$AT_i(u_Q,H) = \begin{cases} \frac{1+|P_Q^H(i)|}{n}, & \textit{if } i \in Q, \\ 0, & \textit{if } i \in N \setminus Q. \end{cases} \tag{9}$$

Proof. Note that $|Q|+\sum_{j\in Q}|P_Q^H(j)| = n$. For $r \in N$, let T_r be the unique admissible rooted spanning tree of H having player r as the root. From (3) it follows that for every $r \in N$

$$m_i^{T_r}(u_Q,H) = \begin{cases} 1, & \text{if } i = r \text{ or } r \in P_Q^H(i), \\ 0, & \text{otherwise.} \end{cases}$$

Then, by Definition 2, we obtain (9). ∎

From this lemma it follows that two hypertree unanimity games assign the same payoff to a player if the set of players represented by that player is the same in both games, that is, for any $(u_S,H),(u_Q,H) \in \mathcal{G}_N^{\mathcal{H}^t}$, with $Q, S \in C^H(N)$, it holds that $AT_i(u_S,H) = AT_i(u_Q,H)$ whenever $P_S^H(i) = P_Q^H(i)$. From this observation it immediately follows that the average tree value for hypertree games satisfies independence in unanimity games.

Theorem 3 *On the class of hypertree games, the average tree value is the unique value that satisfies efficiency, linearity, the restricted null-player property, weak symmetry, and independence in unanimity games.*

Proof. Let ξ be a value on the class of hypertree games that satisfies all five axioms. We show that $\xi = AT$. Because of Lemma 9, we only need to consider the hypertree unanimity games with respect to connected coalitions.

Take any hypertree H on N. By induction on $|S|$ we will show that

$$\xi(u_S, H) = AT(u_S, H), \text{ for all } S \in C^H(N).$$

In the initial step, let $S = N$, then $u_N(Q) = 1$ if $Q = N$ and 0 otherwise. Note that N is connected in H. By weak symmetry, $\xi_i(u_N, H) = \xi_j(u_N, H)$ for all $i, j \in N$. From efficiency we obtain

$$\xi_i(u_N, H) = \frac{1}{n}, \quad \text{for all } i \in N.$$

Since $P_N^H(i) = \emptyset$ for all $i \in N$, from (9) it follows that

$$AT_i(u_N, H) = \frac{1}{n}, \quad \text{for all } i \in N.$$

Therefore, $\xi(u_N, H) = AT(u_N, H)$.

Next, take any $1 \leq t < n$ and assume that $\xi(u_Q, H) = AT(u_Q, H)$ for all $Q \in C^H(N)$ with $|Q| > t$. We show that $\xi(u_S, H) = AT(u_S, H)$ for every $S \in C^H(N)$ with $|S| = t$. Since each $i \in N \setminus S$ is a restricted null-player in $(u_S, H) \in \mathcal{G}_N^{\mathcal{H}^t}$, it follows from the restricted null-player property that

$$\xi_i(u_S, H) = AT_i(u_S, H) = 0, \quad \text{for all } i \in N \setminus S. \tag{10}$$

Since N is connected in H and $t < n$, there exists $e \in H \backslash H|_S$ such that $S \cup e \in C^H(N)$. Let $Q = S \cup e$. Note that $|Q| > t$. By the induction hypothesis, we have

$$\xi_i(u_Q, H) = AT_i(u_Q, H) = \frac{1 + |P_Q^H(i)|}{n}, \quad \text{for all } i \in Q. \tag{11}$$

Moreover, because $Q = S \cup e$ and $Q \in C^H(N)$, from independence in unanimity games and (11) it follows that

$$\xi_i(u_S, H) = \xi_i(u_Q, H) = \frac{1 + |P_Q^H(i)|}{n}, \quad \text{for all } i \in S \setminus e. \tag{12}$$

Since $P_Q^H(i) = P_S^H(i)$ for every $i \in S \setminus e$, from (9) and (12) it follows that

$$\xi_i(u_S, H) = \frac{1 + |P_S^H(i)|}{n} = AT_i(u_S, H), \quad \text{for all } i \in S \setminus e. \tag{13}$$

Since H is a hypertree, it holds that $|S \cap e| = 1$. Therefore, by (10), (13), and efficiency, we obtain that

$$\xi_i(u_S, H) = AT_i(u_S, H), \quad \text{for all } i \in N.$$

By Lemma 9, the proof is completed. ∎

4.3 Cycle hypergraph games

In the previous subsections we provide characterizations of the average tree value for cycle-free hypergraph games. In this subsection we study the average tree value on a subclass of hypergraph games which allows for a cycle in the hypergraph.

Definition 3 A hypergraph H on N is a *cycle hypergraph* if it satisfies the following conditions:

(i) H is connected;

(ii) H is linear;

(iii) H contains a unique cycle and this cycle contains all hyperlinks in H.

A hypergraph game $(v, H) \in \mathcal{G}_N^{\mathcal{H}}$ is a *cycle hypergraph game* if the underlying hypergraph H is a cycle hypergraph. $\mathcal{G}_N^{\mathcal{C}}$ denotes the set of cycle hypergraph games with fixed player set N.

For a given connected coalition $Q \in C^H(N)$, with $|Q| \geq 2$, of a cycle hypergraph H, some players in Q may be adjacent to players not in Q. A player $i \in Q$ is an *extreme player* in H with respect to Q, if there exists a hyperlink $e \in H \setminus H|_Q$ such that $i \in e$. $E^H(Q)$ denotes the set of extreme players in H with respect to Q. A player in $I^H(Q) = Q \setminus E^H(Q)$ is an *inner player* in H with respect to Q. If $2 \leq |Q| < n$, then $|E^H(Q)| = 2$ because a cycle hypergraph is linear. Note that $E^H(N) = \emptyset$ and therefore all players in N are inner players in H with respect to N.

Example 4 Consider the cycle hypergraph H as depicted in Figure 7. For $Q = e_4$, $E^H(Q) = \{5, 7\}$ and $I^H(Q) = \{6, 8\}$. For $Q = e_1 \cup e_2$, $E^H(Q) = \{1, 7\}$ and $I^H(Q) = \{2, 3, 4\}$.

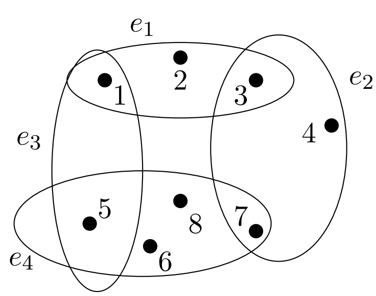


Figure 7: The cycle hypergraph H of Example 4.

In a cycle hypergraph that is not a graph, an admissible rooted spanning tree corresponds to a partial ordering of the players and not necessarily to a permutation on the set of players. This is an essential difference with the case of a cycle graph game as in Selçuk et al. (2013), because a cycle graph induces only admissible rooted spanning trees that correspond to permutations.

Based on the concepts introduced above, we have the following expression of the average tree value for cycle hypergraph unanimity games with respect to connected coalitions.

Lemma 11 *For any* $(u_Q, H) \in \mathcal{G}_N^{\mathcal{C}}$, $Q \in C^H(N)$, *it holds that*

$$AT_i(u_Q, H) = \begin{cases} 1, & \text{if } i \in Q \text{ and } |Q| = 1, \\ \frac{1}{n}, & \text{if } i \in I^H(Q) \text{ and } |Q| \geq 2, \\ \frac{n-|Q|+2}{2n}, & \text{if } i \in E^H(Q) \text{ and } |Q| \geq 2, \\ 0, & \text{if } i \in N \setminus Q. \end{cases} \tag{14}$$

Proof. Since H is a cycle hypergraph game, each $i \in N$ is exactly twice the root of an admissible rooted spanning tree of H. Indeed, for any $i \in N$ there is a unique component of $N \setminus \{i\}$ in H that consists of at least two players and it contains two distinct players, say, j_1 and j_2, such that $\{i, j_1\} \subseteq e$ and $\{i, j_2\} \subseteq e'$ for some $e, e' \in H$. Moreover, because the subhypergraph $H|_{N\setminus\{i\}}$ is cycle-free, from Lemma 1 it follows that each j_h, $h \in \{1, 2\}$, is only once the root of an admissible rooted spanning tree of this subhypergraph. Hence, it follows that $|\mathcal{T}^H| = 2n$.

If $|Q| = 1$, then for every $T \in \mathcal{T}^H$

$$m_i^T(u_Q, H) = u_Q(\bar{S}_i^T) - u_Q^H(S_i^T) = \begin{cases} 1, & \text{if } i \in Q, \\ 0, & \text{if } i \in N \setminus Q, \end{cases}$$

because $u_{\{i\}}(\bar{S}_i^T) = 1$ and $u_{\{i\}}^H(S_i^T) = 0$ for all $i \in N$. Therefore, if $Q = \{i\}$ for some $i \in N$, then $AT_i(u_Q, H) = 1$ and $AT_j(u_Q, H) = 0$ for all $j \neq i$.

Suppose $|Q| \geq 2$. Take any $r \in N$ and $T \in \mathcal{T}_r^H(N)$. Then we have one of the following two cases:

Case 1. If $r \in Q$, then

$$m_i^T(u_Q, H) = u_Q(\bar{S}_i^T) - u_Q^H(S_i^T) = \begin{cases} 1, & \text{if } i = r, \\ 0, & \text{if } i \neq r. \end{cases}$$

Case 2. If $r \in N \setminus Q$, then

$$m_i^T(u_Q, H) = u_Q(\bar{S}_i^T) - u_Q^H(S_i^T) = \begin{cases} 1, & \text{if } i \in E^H(Q) \text{ and } Q \subseteq \bar{S}_i^T, \\ 0, & \text{otherwise.} \end{cases}$$

A player $i \in N \setminus Q$ has in both Case 1 and Case 2 a zero marginal contribution corresponding to any admissible rooted spanning tree, and so $AT_i(u_Q, H) = 0$ if $i \in N \setminus Q$. A player $i \in I^H(Q)$ has only in Case 1 twice a non-zero marginal contribution of 1, and so $AT_i(u_Q, H) = \frac{2}{2n} = \frac{1}{n}$ if $i \in I^H(Q)$ and $|Q| \geq 2$. A player $i \in E^H(Q)$ has in Case 1 twice and in Case 2, $|N \setminus Q|$ times a non-zero marginal contribution of 1, and so $AT_i(u_Q, H) = \frac{n-|Q|+2}{2n}$ if $i \in E^H(Q)$ and $|Q| \geq 2$. ∎

Before stating a characterization of the average tree value for cycle hypergraph games, we need to introduce two other axioms.

- A value ξ on $\mathcal{G}_N^{\mathcal{C}}$ satisfies *symmetry in unanimity games* if for every $(u_Q, H) \in \mathcal{G}_N^{\mathcal{C}}$ with $Q \in C^H(N)$ and $|Q| \geq 2$, it holds that $\xi_i(u_Q, H) = \xi_j(u_Q, H)$ if either $i, j \in I^H(Q)$ or $i, j \in E^H(Q)$.

Symmetry in unanimity games for cycle hypergraph games features two kinds of symmetry, in a unanimity game both the inner players are symmetric and the extreme players are symmetric.

• A value ξ on $\mathcal{G}_N^{\mathcal{C}}$ satisfies *independence in unanimity games* if for every $(u_Q, H) \in \mathcal{G}_N^{\mathcal{C}}$, $Q \in C^H(N)$, and $e \in H \setminus H|_Q$ satisfying $Q \cup e \in C^H(N)$, it holds that $\xi_i(u_Q, H) = \xi_i(u_{Q\cup e}, H)$ for all $i \in I^H(Q)$.

Independence in unanimity games for cycle hypergraph games states that when in a cycle hypergraph a hyperlink joins a connected coalition, then in the associated unanimity games the inner players of the coalition receive the same payoff.

Theorem 4 *On the class of cycle hypergraph games, the average tree value is the unique value that satisfies efficiency, linearity, the restricted null-player property, symmetry in cycle unanimity games, and independence in unanimity games.*

Proof. Let ξ be a value satisfying all five axioms. We show that $\xi = AT$. By Lemma 9, we only have to consider cycle hypergraph unanimity games with respect to connected coalitions.

Take any cycle hypergraph H on N. By efficiency and symmetry in cycle unanimity games, it holds that

$$\xi_i(u_N, H) = \frac{1}{n}, \quad \text{for all } i \in N,$$

because all players in N are inner players in H with respect to N. By (14), we have

$$AT_i(u_N, H) = \frac{1}{n}, \quad \text{for all } i \in N.$$

Therefore, we obtain that $\xi(u_N, H) = AT(u_N, H)$.

If $|Q| = 1$, by efficiency and the restricted null-player property, we have $\xi_i(u_Q, H) = 1 = AT_i(u_Q, H)$ if $Q = \{i\}$ and $\xi_i(u_Q, H) = 0 = AT_i(u_Q, H)$ for all $i \in N \setminus Q$.

Next, take any $Q \in C^H(N)$ with $2 \leq |Q| < n$. Since $Q, N \in C^H(N)$ and $Q \neq N$, there exists a nonempty subset of hyperlinks $A = \{e \in H : e \not\subseteq Q\} \subsetneq H$ satisfying $Q \cup \{i \in e : e \in A\} = N$. Therefore, by applying $|A|$ times independence in unanimity games, we obtain

$$\xi_i(u_Q, H) = \xi_i(u_N, H) = \frac{1}{n} = AT_i(u_Q, H), \quad \text{for all } i \in I^H(Q).$$

Because each $i \in N \setminus Q$ is a restricted null-player in (u_Q, H), from the restricted null-player property it follows that $\xi_i(u_Q, H) = 0$ for all $i \in N \setminus Q$, as in the average tree value.

Since $2 \leq |Q| < n$, it holds that $|E^H(Q)| = 2$. Let $E^H(Q) = \{i_1, i_2\}$. From symmetry in unanimity games it follows that $\xi_{i_1}(u_Q, H) = \xi_{i_2}(u_Q, H)$. Together with efficiency, $\xi_i(u_Q, H) = \frac{1}{n}$ for all $i \in I^H(Q)$, and $\xi_i(u_Q, H) = 0$ for all $i \in N \setminus Q$, we obtain that

$$\xi_i(u_Q, H) = \frac{1}{2}\big(1 - (|Q| - 2)\frac{1}{n}\big) = \frac{n - |Q| + 2}{2n} = AT_i(u_Q, H), \quad \text{for all } i \in E^H(Q).$$

Therefore, we have $\xi(u_Q, H) = AT(u_Q, H)$ for all $Q \in C^H(N)$. By Lemma 9, this completes the proof. ∎

5 Logical independence

In this section we show that the axioms in Theorem 3 and Theorem 4, which are used to characterize the average tree value, are logical independent for each characterization. For the two axioms in Theorem 2, it is obvious that they are independent to each other.

The following five values on $\mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$ show the independence of the axioms in Theorem 3 and Theorem 4.

- Let ξ^1 be given by $\xi_i^1(v, H) = 0$ for all $i \in N$. This value satisfies all axioms of Theorem 3 and of Theorem 4, except efficiency.

- Let ξ^2 be given by $\xi_i^2(v, H) = \frac{v(N)}{n}$ for all $i \in N$. This value satisfies all axioms of Theorem 3 and of Theorem 4, except the restricted null-player property.

- Let ξ^3 be the Myerson value. This value satisfies all axioms of Theorem 3 and of Theorem 4, except independence in unanimity games. For efficiency, linearity, and the restricted null-player property, we refer to van den Nouweland et al. (1992). Take any $(v, H) \in \mathcal{G}_N^{\mathcal{H}^t}$ satisfying $v(S) = 0$ for all $S \in C^H(N)$, $S \neq N$, then $\xi_i^3(v, H) = \frac{v(N)}{n}$ for all $i \in N$, which shows weak symmetry of ξ^3 for hypertree games. Take any $(u_Q, H) \in \mathcal{G}_N^{\mathcal{C}}$, $Q \in C^H(N)$, then $\xi^3(u_Q, H) = \frac{1}{|Q|}$ for all $i \in Q$ and 0 otherwise, which shows symmetry in unanimity games of ξ^3 for cycle hypergraph games. Next, take any $(u_Q, H) \in \mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$ and $e \in H \setminus H(Q)$ satisfying $Q \in C^H(N)$ and $Q \cup e \in C^H(N)$, then $\xi_i^3(u_{Q\cup e}, H) = \frac{1}{|Q\cup e|}$ for all $i \in Q \cup e$ and 0 otherwise. Since $\frac{1}{|Q|} \neq \frac{1}{|Q\cup e|}$, ξ^3 fails independence in unanimity games for both hypertree games and cycle hypergraph games.

- Let ξ^4 be given by $\xi^4(v, H) = m^T(v, H)$ for some $T \in \mathcal{T}^H(N)$. This value satisfies all axioms of Theorem 3, except weak symmetry, and all axioms of Theorem 4, except symmetry in unanimity games. Let $(u_N, H) \in \mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$ and $T \in \mathcal{T}_r^H(N)$, $r \in N$, then $m_i^T(u_N, H) = 1$ if $i = r$ and 0 otherwise, which shows that ξ^4 fails weak symmetry for hypertree games and symmetry in unanimity games for cycle hypergraph games.

- Let ξ^5 be given by

$$\xi^5(v, H) = \begin{cases} AT(u_S, H), & \text{if } v = u_S \text{ for some } S \in C^H(N), \\ \xi^3(v, H), & \text{otherwise.} \end{cases}$$

  This value satisfies all axioms of Theorem 3 and of Theorem 4, except linearity. Take any $(v, H) \in \mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$, satisfying $v = u_{Q_1} + u_{Q_2}$ for some distinct $Q_1, Q_2 \in C^H(N)$ such that $Q_1 \cap Q_2 \neq \emptyset$, then for every $i \in Q_1 \cap Q_2$ with $|H_i| = 1$ it holds that $\xi_i^5(v, H) = \frac{1}{|Q_1|} + \frac{1}{|Q_2|} \neq \frac{2}{n} = \xi_i^5(u_{Q_1}, H) + \xi_i^5(u_{Q_2}, H)$, which shows that ξ^5 fails linearity for both hypertree games and cycle hypergraph games.

References

Algaba, E., Bilbao, J.M., Borm, P., López, J.J. (2001), The Myerson value for union stable systems, *Mathematical Methods of Operations Research*, 54(3), 359-371.

Algaba, E., Bilbao, J.M., van den Brink, R. (2015), Harsanyi power solutions for games on union stable systems, *Annals of Operations Research*, 225, 27–44.

Berge, C. (1973), *Graphs and Hypergraphs*, Amsterdam: North Holland.

Borm, P., Owen, G., Tijs, S. (1992), On the position value for communication situation, *SIAM Journal on Discrete Mathematics*, 5(3), 305–320.

Diestel, R. (2000), *Graph Theory*, Berlin: Springer-Verlag.

Herings, P.J.J., van der Laan, G., Talman, A.J.J. (2008) The average tree solution for cycle-free graph games, *Games and Economic Behavior*, 62, 77–92.

Herings, P.J.J., van der Laan, G., Talman, A.J.J., Yang, Z. (2010), The average tree solution for cooperative games with communication structure, *Games and Economic Behavior*, 68, 626–633.

Koshevoy, G., Talman, A.J.J. (2014), Solution concepts for games with general coalitional structure, *Mathematical Social Sciences*, 68(1), 19–30.

Meessen, R. (1988), Communication games, Master's thesis, Depart. of Mathematics, University of Nijmegen, the Netherlands (In Dutch).

Mishra, D., Talman, A.J.J. (2010), A characterization of the average tree solution for tree games, *International Journal of Game Theory*, 39, 105–111.

Myerson, R.B. (1977), Graphs and cooperation in games, *Mathematics of Operations Research*, 2, 225–229.

Myerson, R.B. (1980), Conference structures and fair allocation rules, *International Journal of Game Theory*, 9, 169–182.

van den Nouweland, A., Borm, P., Tijs, S. (1992), Allocation rules for hypergraph communication situations, *International Journal of Game Theory*, 20, 255–268.

Owen, G. (1986), Values of graph-restricted games, *SIAM J. Algebraic Discrete Methods*, 7, 210–220.

Selçuk, Ö., Suzuki, T., Talman, A.J.J. (2013), Equivalence and axiomatization of solutions for cooperative games with circular communication structure, *Economics Letters*, 121(3), 428–431.

Shapley, L.S. (1953), A value for n-person games, in: Kuhn, H., Tucker, A.W. (eds), *Contributions to the Theory of Games II*, Princeton: Princeton University Press, pp. 307–317.