Appendix to Temporal Analysis of Static Priority Preemptive Scheduled Cyclic Streaming Applications using CSDF Models

Philip S. Kurtin*
philip.kurtin@utwente.nl

Marco J.G. Bekooij*‡
marco.bekooij@nxp.com

*University of Twente, Enschede, The Netherlands
‡NXP Semiconductors, Eindhoven, The Netherlands

ABSTRACT

This is the appendix to the paper *Temporal Analysis of Static Priority Preemptive Scheduled Cyclic Streaming Applications using CSDF Models* [1].

The temporal analysis approach presented in [1] makes use of an iterative algorithm that computes so-called maximum busy periods over multiple task phases. The algorithm contains a stop criterion indicating after which iteration of the algorithm subsequent iterations do not need to be considered. The intuition behind that stop criterion is given in the paper and supplemented by a formal proof in this appendix.

A1. VALIDITY OF THE STOP CRITERION

Figure A1 recaps the algorithm presented in Figure 7 of [1]. In order to prove the validity of the stop criterion in line 14 we need to distinguish between the maximum busy periods and maximum finish times computed in different iterations. Consequently we introduce an index n that we use for $w_{ix}^{'\langle n\rangle}$, $w_{ix}^{\langle n\rangle}$, $\mathcal{Z}_{ix}^{\langle n\rangle}$ and $\hat{f}_{ix'}^{\langle n\rangle}$. We define the relation between x, $x' = x'_n$, $q = q_n$ and n as follows:

$$n = q_n \cdot \Theta_i + x'_n - x$$

As one can easily see this definition leads to n being initially zero and increasing by one in each iteration of the while-loop. Using this indexing and taking into account that $x' = x$ must hold for exiting the while-loop we can reformulate the stop criterion more explicitly as follows (with q^* the q for which the stop criterion is met):

$$w_{ix}^{'\langle q^*\Theta_i - 1\rangle} \leq q^* \cdot P_i \tag{A1}$$

Note that the term -1 appears as the increase of n to $q^*\Theta_i$ (and thus $x' = x$) occurs after the computation of the last maximum busy period. Moreover, the stop criterion for q^* would not be checked if it were already true for a q' with $0 < q' < q^*$. This implies:

$$\forall_{0<q'<q^*}:\ w_{ix}^{'\langle q'\cdot\Theta_i - 1\rangle} > q' \cdot P_i \tag{A2}$$

In the following we prove that given these two criteria we do not have to consider any $w_{ix}^{'\langle q^*\Theta_i+k\rangle}$ with $k \geq 0$ as the maximum finish times of task phases cannot become larger for any of these maximum busy periods. We conduct the proof by comparing interference characterizations, then extend these observations to maximum busy periods and finally maximum finish times. We begin with the period-and-jitter interference characterization:

Lemma A1. *It holds:*

$$\begin{aligned}\forall_{k\geq 0}:\ \eta_{jy}(w_{ix}^{'\langle q^*\Theta_i+k\rangle}) - \eta_{jy}(w_{ix}^{'\langle q^*\Theta_i-1\rangle}) &\leq \eta_{jy}(w_{ix}^{'\langle k\rangle})\\ w_{ix}^{'\langle q^*\Theta_i+k\rangle} - w_{ix}^{'\langle q^*\Theta_i-1\rangle} &\leq w_{ix}^{'\langle k\rangle}\end{aligned}$$

PROOF. With the subadditivity of the ceiling function $\lceil a+b\rceil \leq \lceil a\rceil + \lceil b\rceil$ it follows with $a = c-d$ and $b = d$ that $\lceil c\rceil - \lceil d\rceil \leq \lceil c-d\rceil$ and with $\hat{J}_{jy} \geq 0$:

$$\begin{aligned}\eta_{jy}(\Delta t_1) - \eta_{jy}(\Delta t_2) &= \left\lceil \frac{\hat{J}_{jy}+\Delta t_1}{P_i}\right\rceil - \left\lceil \frac{\hat{J}_{jy}+\Delta t_2}{P_i}\right\rceil\\ &\leq \left\lceil \frac{\hat{J}_{jy}+\Delta t_1-\Delta t_2}{P_i}\right\rceil = \eta_{jy}(\Delta t_1 - \Delta t_2)\end{aligned} \tag{A3}$$

By adding up extensions of maximum busy periods according to the algorithm in Figure A1 it follows that $w_{ix}^{'\langle q^*\Theta_i+k\rangle} - w_{ix}^{'\langle q^*\Theta_i-1\rangle}$ is the fixed point of a function $f(\Delta t)$ and $w_{ix}^{'\langle k\rangle}$ the fixed point of a function $g(\Delta t)$ with:

$$\begin{aligned}f(\Delta t) &= \sum_{k'=0}^{k} C_{ix'_{k'}} + \sum_{jy\in hp(i)} [\ \eta_{jy}(w_{ix}^{'\langle q^*\Theta_i-1\rangle} + \Delta t)\\ &\qquad - \eta_{jy}(w_{ix}^{'\langle q^*\Theta_i-1\rangle})] \cdot C_{jy}\\ g(\Delta t) &= \sum_{k'=0}^{k} C_{ix'_{k'}} + \sum_{jy\in hp(i)} \eta_{jy}(\Delta t)\cdot C_{jy}\end{aligned}$$

Note that in the definition of $f(\Delta t)$ we have used $\forall_{0\leq k'\leq k}$: $x'_{q\cdot\Theta_i+k'} = x'_{k'}$. From Equation A3 it immediately follows that $\forall_{\Delta t}:\ f(\Delta t) \leq g(\Delta t)$. With the monotonicity of both $f(\Delta t)$ and $g(\Delta t)$ and with $f(0) \geq 0$ one can further conclude that also the fixed point of $f(\Delta t)$ must be smaller or equal to the fixed point of $g(\Delta t)$, i.e.:

$$w_{ix}^{'\langle q^*\Theta_i+k\rangle} - w_{ix}^{'\langle q^*\Theta_i-1\rangle} \leq w_{ix}^{'\langle k\rangle}$$

And with Equation A3:

$$\begin{aligned}&\eta_{jy}(w_{ix}^{'\langle q^*\Theta_i+k\rangle}) - \eta_{jy}(w_{ix}^{'\langle q^*\Theta_i-1\rangle})\\ &\leq \eta_{jy}(w_{ix}^{'\langle q^*\Theta_i+k\rangle} - w_{ix}^{'\langle q^*\Theta_i-1\rangle}) \leq \eta_{jy}(w_{ix}^{'\langle k\rangle})\end{aligned}$$ □

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1  ∀_{0≤x<Θ_i}: f̂_ix = 0;
2  forall (x : e_jyix ∈ E^ext) {
3    x' = x; q = 0; w'_ix = w_ix = 0; Z_ix = ∅;
4    do {
5      w^⊕'_ix = C_ix' + Σ_{jy∈hp(i)} [η_jy(w'_ix + w^⊕'_ix) − η_jy(w'_ix)] · C_jy;
6      w^⊕_ix = C_ix' + Σ_{jy∈hp(i)} [ γ_jy(w'_ix + w^⊕'_ix, Z_ix ∪ {(v_ix', q)})
              −γ_jy(w'_ix, Z_ix)] · C_jy;
7      w'_ix = w'_ix + w^⊕'_ix;  w_ix = w_ix + w^⊕_ix;
8      Z_ix = Z_ix ∪ {(v_ix', q)};
9      f̂_ix' = max(f̂_ix', ŝ^ext_ix + w_ix − q · P_i);
10     x'++;
11     if (x' = Θ_i) {
12       q++; x' = 0;
13     }
14   } while (x' ≠ x || w'_ix > q · P_i);
15 }
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Figure A1: Algorithm to compute upper bounds on finish times of task phases (same as Figure 7 of [1]).

Lemma A1 allows us to prove similar inequalities for the combined interference characterizations, with a restriction that we relax in Lemma A4.

Lemma A2. *If it holds* $\forall_{jy \in hp(i)}: \ \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \geq q^*$ *it follows with Equations A1 and A2:*

$$\forall_{k \geq 0}: \ \gamma_{jy}(w_{ix}^{'\langle q^* \Theta_i + k \rangle}, \mathcal{Z}_{ix}^{\langle q^* \Theta_i + k \rangle}) - \gamma_{jy}(w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}, \mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \leq \gamma_{jy}(w_{ix}^{'\langle k \rangle}, \mathcal{Z}_{ix}^{\langle k \rangle}) \tag{A4}$$

$$w_{ix}^{\langle q^* \Theta_i + k \rangle} - w_{ix}^{\langle q^* \Theta_i - 1 \rangle} \leq w_{ix}^{\langle k \rangle} \tag{A5}$$

Proof. Recall that $\gamma_{jy}(\Delta t, \mathcal{Z}_i)$ is defined in Section 6.4 of [1] as:

$$\gamma_{jy}(\Delta t, \mathcal{Z}_i) = \min(\eta_{jy}(\Delta t), \zeta_{jy}(\mathcal{Z}_i))$$

If actors v_i and v_j are not connected via a cycle, for instance because the corresponding tasks belong to different task graphs, the interference characterizations considering cyclic data dependencies $\zeta_{jy}(\mathcal{Z}_i)$ all result in infinity and Equation A4 becomes the upper inequality of Lemma A1.

However, if both actors are connected via a cycle we have to differ between two cases. In the first case we assume that $\eta_{jy}(w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}) < \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle})$. Then the left-hand side of Equation A4 resolves to:

$$\begin{aligned}
&\gamma_{jy}(w_{ix}^{'\langle q^* \Theta_i + k \rangle}, \mathcal{Z}_{ix}^{\langle q^* \Theta_i + k \rangle}) - \gamma_{jy}(w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}, \mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \\
= \ &\min(\eta_{jy}(w_{ix}^{'\langle q^* \Theta_i + k \rangle}), \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i + k \rangle})) - \eta_{jy}(w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}) \\
\leq \ &\min(\eta_{jy}(w_{ix}^{'\langle k \rangle}), \delta(\mathcal{P}_{ix'_k jy}) + \delta(\mathcal{P}_{jyix}) + q^* + q_k - 1 - q^*) \\
= \ &\min(\eta_{jy}(w_{ix}^{'\langle k \rangle}), \zeta_{jy}(\mathcal{Z}_{ix}^{\langle k \rangle})) = \gamma_{jy}(w_{ix}^{'\langle k \rangle}, \mathcal{Z}_{ix}^{\langle k \rangle})
\end{aligned}$$

For the first argument of the minimum function we have used Lemma A1 and for the second argument we have substituted $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i + k \rangle})$ using Equation 3 of [1], which is:

$$\zeta_{jy}(\mathcal{Z}_i) = \delta(\mathcal{P}_{i\hat{x}jy}) + \hat{q} + \delta(\mathcal{P}_{jyi\check{x}}) - \check{q} - 1 \tag{A6}$$

Thereby we have taken into account that $\mathcal{Z}_{ix}^{\langle q^* \Theta_i + k \rangle}$ has the following suprema and infima with respect to the ordering relation defined in Section 6.4 of [1]:

$$(v_{i\hat{x}}, \hat{q}) = (v_{ix'_{q^* \Theta_i + k}}, q_{q^* \Theta_i + k}) = (v_{ix'_k}, q^* + q_k)$$

$$(v_{i\check{x}}, \check{q}) = (v_{ix}, 0)$$

And with Equation A2 and $P_i = P_j$ (which is implied by actors v_i and v_j being on a cycle, i.e. belonging to the same task graph) we have further concluded for the second argument:

$$\begin{aligned}
\eta_{jy}(w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}) &\geq \eta_{jy}((q^* - 1) \cdot P_i) \\
&= \left\lceil \frac{\hat{J}_{jy} + (q^* - 1) \cdot P_i}{P_j} \right\rceil = \left\lceil \frac{\hat{J}_{jy}}{P_j} \right\rceil + q^* - 1 \\
&\geq q^*
\end{aligned}$$

In the second case we assume that $\eta_{jy}(w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}) \geq \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle})$. Then the left-hand side of Equation A4 resolves to:

$$\begin{aligned}
&\gamma_{jy}(w_{ix}^{'\langle q^* \Theta_i + k \rangle}, \mathcal{Z}_{ix}^{\langle q^* \Theta_i + k \rangle}) - \gamma_{jy}(w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}, \mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \\
= \ &\min(\eta_{jy}(w_{ix}^{'\langle q^* \Theta_i + k \rangle}), \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i + k \rangle})) - \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \\
\leq &\min(q^* + \eta_{jy}(w_{ix}^{'\langle k \rangle}), \delta(\mathcal{P}_{ix'_k jy}) + \delta(\mathcal{P}_{jyix}) + q^* + q_k - 1) \\
&- q^* \\
= &\min(\eta_{jy}(w_{ix}^{'\langle k \rangle}), \zeta_{jy}(\mathcal{Z}_{ix}^{\langle k \rangle})) = \gamma_{jy}(w_{ix}^{'\langle k \rangle}, \mathcal{Z}_{ix}^{\langle k \rangle})
\end{aligned}$$

For the first argument of the minimum function we have used Lemma A1, the subadditivity of the ceiling function,

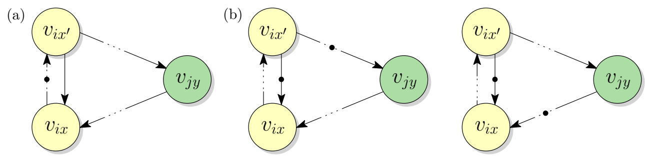

Figure A2: Special case $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) < q^*$.

Equation A1 and $P_i = P_j$ such that:

$$\begin{aligned}
\eta_{jy}(w_{ix}^{'\langle q^* \Theta_i + k \rangle}) &\leq \eta_{jy}(w_{ix}^{'\langle q^* \Theta_i - 1 \rangle} + w_{ix}^{'\langle k \rangle}) \\
&\leq \left\lceil \frac{w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}}{P_j} \right\rceil + \left\lceil \frac{\hat{J}_{jy} + \eta_{jy}(w_{ix}^{'\langle k \rangle})}{P_j} \right\rceil \\
&\leq \left\lceil \frac{q^* \cdot P_i}{P_j} \right\rceil + \eta_{jy}(w_{ix}^{'\langle k \rangle}) = q^* + \eta_{jy}(w_{ix}^{'\langle k \rangle})
\end{aligned} \tag{A7}$$

For the second argument we have again applied the substitution via Equation A6. Finally we have used that $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \geq q^*$.

This lets us conclude that Equation A4 holds for all $jy \in hp(i)$ if $\forall_{jy \in hp(i)}: \ \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \geq q^*$. In words this means that the differences between the interference characterizations of $w_{ix}^{'\langle q^* \Theta_i + k \rangle}$ and $w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}$ are smaller or equal to the interference characterizations of $w_{ix}^{\langle k \rangle}$ for all $jy \in hp(i)$.

Finally one can see from the algorithm in Figure A1 that if Equation A4 holds for all $jy \in hp(i)$ also the difference between the maximum busy periods $w_{ix}^{'\langle q^* \Theta_i + k \rangle}$ and $w_{ix}^{'\langle q^* \Theta_i - 1 \rangle}$ must be smaller than the maximum busy period $w_{ix}^{\langle k \rangle}$, i.e. Equation A5 holds as well. □

These lemmas allow us to establish a relation between the maximum finish time computations in algorithm iterations $q^* \Theta_i + k$ and k:

Lemma A3. *If it holds* $\forall_{jy \in hp(i)}: \ \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \geq q^*$ *it follows with Equations A1 and A2 for the arguments of the maximum function in line 9 of the algorithm that:*

$$\forall_{k \geq 0}: \ \hat{s}_{ix}^{ext} + w_{ix}^{\langle q^* \Theta_i + k \rangle} - q_{q^* \Theta_i + k} \cdot P_i \leq \hat{s}_{ix}^{ext} + w_{ix}^{\langle k \rangle} - q_k \cdot P_i$$

Proof. Using Lemma A2, $q_{q^* \Theta_i + k} = q^* + q_k$ and Equation A1 it follows:

$$\begin{aligned}
&\hat{s}_{ix}^{ext} + w_{ix}^{\langle q^* \Theta_i + k \rangle} - q_{q^* \Theta_i + k} \cdot P_i \\
\leq \ &\hat{s}_{ix}^{ext} + w_{ix}^{\langle q^* \Theta_i - 1 \rangle} + w_{ix}^{\langle k \rangle} - (q^* + q_k) \cdot P_i \\
\leq \ &\hat{s}_{ix}^{ext} + q^* \cdot P_i + w_{ix}^{\langle k \rangle} - (q^* + q_k) \cdot P_i \\
= \ &\hat{s}_{ix}^{ext} + w_{ix}^{\langle k \rangle} - q_k \cdot P_i
\end{aligned}$$

□

Now we show that Lemma A3 also holds if the restriction $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) \geq q^*$ is not true for all $jy \in hp(i)$:

Lemma A4. *Lemma A3 still holds if it holds for one or more* $jy \in hp(i)$ *that* $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) < q^*$.

Proof. First assume that it holds for only one $jy \in hp(i)$ that $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) < q^*$. From Equation A6 and the ordering relation in Section 6.4 of [1] it follows that $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle n \rangle})$ increases by one if n is increased by Θ_i. Thus it holds that:

$$\forall_{q' \geq 1}: \ \zeta_{jy}(\mathcal{Z}_{ix}^{\langle (q'+1) \Theta_i - 1 \rangle}) = \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q' \Theta_i - 1 \rangle}) + 1$$

From $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^* \Theta_i - 1 \rangle}) < q^*$ it therefore follows that also $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle \Theta_i - 1 \rangle}) < 1$ and thus $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle \Theta_i - 1 \rangle}) = 0$ must hold.

If v_{ix} is not the first phase of actor v_i (i.e. $0 < x < \Theta_i$) then q must have already become one before iteration $\Theta_i - 1$ of the algorithm in Figure A1. With $v_{ix'}$ the immediate

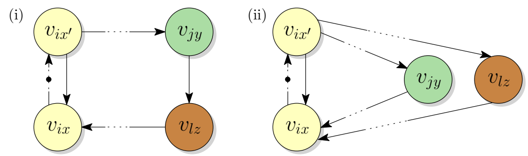

Figure A3: Case (a) for $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^*\Theta_i-1\rangle}) < q^*$ and $\zeta_{lz}(\mathcal{Z}_{ix}^{\langle q^*\Theta_i-1\rangle}) < q^*$.

predecessor of v_{ix} this implies that the supremum of $\mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle}$ is $(v_{i\hat{x}}, \hat{q}) = (v_{ix'}, 1)$, whereas the infimum of $\mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle}$ is $(v_{i\check{x}}, \check{q}) = (v_{ix}, 0)$. According to Equation A6 it then follows with $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle}) = 0$:

$$\zeta_{jy}(\mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle}) = \delta(\mathcal{P}_{ix'jy}) + \delta(\mathcal{P}_{jyix}) + 1 - 1 = 0$$

This equation can only be true if both $\delta(\mathcal{P}_{ix'jy})$ and $\delta(\mathcal{P}_{jyix})$ are zero, as depicted in Figure A2(a).

If v_{ix} is the first phase of actor v_i (i.e. $x = 0$) then q just becomes one at the end of algorithm iteration $\Theta_i - 1$. This implies that the supremum of $\mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle}$ is $(v_{i\hat{x}}, \hat{q}) = (v_{ix'}, 0)$ and it follows with $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle}) = 0$:

$$\zeta_{jy}(\mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle}) = \delta(\mathcal{P}_{ix'jy}) + \delta(\mathcal{P}_{jyix}) - 1 = 0$$

In this case the equation can only be true if either $\delta(\mathcal{P}_{ix'jy})$ or $\delta(\mathcal{P}_{jyix})$ is one and the other zero, as depicted in Figure A2(b).

In both cases (a) and (b) it holds that $v_{ix'}$ and v_{ix} can never be in consecutive execution as v_{jy} is always executed in between. This intermediate execution of v_{jy} is however already conservatively considered in the worst-case Linear Program (LP) presented in Section 7 of [1], which is:

$$\begin{aligned}
&\text{Minimize} \sum_{v_{ix}\in V} \hat{s}_{ix}^{ext} + \hat{s}_{ix} \\
&\text{Subject to } \hat{s}_{s0} = 0 \\
&\qquad \forall_{e_{ixjy}\in E^{ext}}: \quad \hat{s}_{jy}^{ext} - \hat{s}_{ix} \geq \hat{\rho}_{ix} - \delta(e_{ixjy}) \cdot P_j \\
&\qquad \forall_{e_{ixjy}\in E^{\langle 2,exp\rangle}}: \quad \hat{s}_{jy} - \hat{s}_{ix} \geq \hat{\rho}_{ix} - \delta(e_{ixjy}) \cdot P_j
\end{aligned}$$

From this follows for case (a) that $\hat{s}_{jy} \geq \hat{s}_{ix'} + \hat{\rho}_{ix'}$ and $\hat{s}_{ix}^{ext} \geq \hat{s}_{jy} + \hat{\rho}_{jy}$ and for case (b) with $P_i = P_j$ that $\hat{s}_{jy} \geq \hat{s}_{ix'} + \hat{\rho}_{ix'} - P_i$ and $\hat{s}_{ix}^{ext} \geq \hat{s}_{jy} + \hat{\rho}_{jy}$ or $\hat{s}_{jy} \geq \hat{s}_{ix'} + \hat{\rho}_{ix'}$ and $\hat{s}_{ix}^{ext} \geq \hat{s}_{jy} + \hat{\rho}_{jy} - P_i$. Thus it also holds that $\hat{s}_{ix}^{ext} \geq \hat{f}_{ix'} + C_{jy}$ in case (a) and $\hat{s}_{ix}^{ext} \geq \hat{f}_{ix'} + C_{jy} - P_i$ in case (b).

With $\hat{f}_{ix'} \geq \hat{s}_{ix}^{ext} + w_{ix}^{\langle\Theta_i-1\rangle} - P_i$ in case (a) and $\hat{f}_{ix'} \geq \hat{s}_{ix}^{ext} + w_{ix}^{\langle\Theta_i-1\rangle}$ in case (b) it further holds for both cases that $w_{ix}^{\langle\Theta_i-1\rangle} + C_{jy} \leq P_i$, which implies that the stop criterion is already met for $q^* = 1$.

Now assume that it holds for multiple $jy \in hp(i)$ that $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^*\Theta_i-1\rangle}) < q^*$. Figure A3 depicts case (a) for two such phases $jy, lz \in hp(i)$ (case (b) and more than two phases can be constructed analogously).

In case (i) the two phases are always firing one after the other. Thus it follows from the worst-case LP that $\hat{s}_{jy} \geq \hat{s}_{ix'} + \hat{\rho}_{ix'}$, $\hat{s}_{lz} \geq \hat{s}_{jy} + \hat{\rho}_{jy}$ and $\hat{s}_{ix}^{ext} \geq \hat{s}_{lz} + \hat{\rho}_{lz}$ from which we conclude with the same reasoning as above that $w_{ix}^{\langle\Theta_i-1\rangle} + C_{jy} + C_{lz} \leq P_i$.

In case (ii) the two phases are firing in parallel. Thus it follows from the worst-case LP that $\hat{s}_{jy} \geq \hat{s}_{ix'} + \hat{\rho}_{ix'}$, $\hat{s}_{lz} \geq \hat{s}_{ix'} + \hat{\rho}_{ix'}$ and $\hat{s}_{ix}^{ext} \geq \max(\hat{s}_{jy} + \hat{\rho}_{jy}, \hat{s}_{lz} + \hat{\rho}_{lz})$. From this we derive with above reasoning that $w_{ix}^{\langle\Theta_i-1\rangle} + \max(\hat{\rho}_{jy}, \hat{\rho}_{lz}) \leq P_i$. Additionally we know that the phases v_{jy} and v_{lz} cannot be on a cycle with one token (otherwise we would have case (i) again), which implies that the phases can interfere with each other. As one of the two phases must have a higher priority than the other it follows that either $\hat{\rho}_{jy} \geq C_{jy} + C_{lz}$ or $\hat{\rho}_{lz} \geq C_{jy} + C_{lz}$. This lets us conclude again that $w_{ix}^{\langle\Theta_i-1\rangle} + C_{jy} + C_{lz} \leq P_i$.

This observation can be generalized as follows. Let $hp'(i)$ be the set of all $jy \in hp(i)$ for which it holds that $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^*\Theta_i-1\rangle}) < q^*$. If the set is not empty it holds that the stop criterion is already met for $q^* = 1$ and it holds:

$$w_{ix}^{\langle\Theta_i-1\rangle} + \sum_{jy\in hp'(i)} C_{jy} \leq P_i \tag{A8}$$

Moreover it holds for all $jy \in hp'(i)$:

$$\begin{aligned}
&[\gamma_{jy}(w_{ix}^{'\langle\Theta_i+k\rangle}, \mathcal{Z}_{ix}^{\langle\Theta_i+k\rangle}) - \gamma_{jy}(w_{ix}^{'\langle\Theta_i-1\rangle}, \mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle})] \cdot C_{jy} \\
&= [\min(\eta_{jy}(w_{ix}^{'\langle\Theta_i+k\rangle}), \zeta_{jy}(\mathcal{Z}_{ix}^{\langle\Theta_i+k\rangle})) - 0] \cdot C_{jy} \\
&\leq [\min(1 + \eta_{jy}(w_{ix}^{'\langle k\rangle}), \zeta_{jy}(\mathcal{Z}_{ix}^{\langle k\rangle}) + 1)] \cdot C_{jy} \\
&= [\gamma_{jy}(w_{ix}^{'\langle k\rangle}, \mathcal{Z}_{ix}^{\langle k\rangle}) + 1] \cdot C_{jy}
\end{aligned}$$

For the first argument of the minimum function we have used Equation A7 from the proof of Lemma A2, for the second argument that $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle n\rangle})$ increases by one if n is increased by Θ_i and for $\gamma_{jy}(w_{ix}^{'\langle\Theta_i-1\rangle}, \mathcal{Z}_{ix}^{\langle\Theta_i\rangle})$ that $\zeta_{jy}(\mathcal{Z}_{ix}^{\langle\Theta_i-1\rangle}) = 0$.

From this and Lemma A2 for the other $jy \in hp(i) \setminus hp'(i)$ it follows:

$$w_{ix}^{\langle\Theta_i+k\rangle} - w_{ix}^{\langle\Theta_i-1\rangle} \leq w_{ix}^{\langle k\rangle} + \sum_{jy\in hp'(i)} C_{jy} \tag{A9}$$

If $hp'(i) \neq \emptyset$ (i.e. $\exists_{jy\in hp(i)}: \zeta_{jy}(\mathcal{Z}_{ix}^{\langle q^*\Theta_i-1\rangle}) < q^*$) we can therefore conclude with Equation A8 that $q^* = 1$ and with $q_{\Theta_i+q_k} = 1 + q_k$ and Equations A8 and A9 that for all $k \geq 0$:

$$\begin{aligned}
&\hat{s}_{ix}^{ext} + w_{ix}^{\langle\Theta_i+k\rangle} - q_{\Theta_i+q_k} \cdot P_i \\
&\leq \hat{s}_{ix}^{ext} + w_{ix}^{\langle\Theta_i-1\rangle} + w_{ix}^{\langle k\rangle} + \sum_{jy\in hp'(i)} C_{jy} - (1 + q_k) \cdot P_i \\
&\leq \hat{s}_{ix}^{ext} + P_i + w_{ix}^{\langle k\rangle} - (1 + q_k) \cdot P_i \\
&= \hat{s}_{ix}^{ext} + w_{ix}^{\langle k\rangle} - q_k \cdot P_i \qquad \square
\end{aligned}$$

Finally we can prove that if Equation A1 holds we do not need to consider any maximum finish times computed after the stop criterion is met:

Lemma A5. *It follows with Equation A1 for the maximum finish times computed with the algorithm that:*

$$\forall_{k\geq 0}: \hat{f}_{ix'}^{\langle q^*\Theta_i+k\rangle} = \hat{f}_{ix'}^{\langle q^*\Theta_i-1\rangle}$$

Proof. Based on line 9 of the algorithm and our indexing scheme we can write the maximum finish times computed in an iteration n as follows:

$$\hat{f}_{ix'}^{\langle n\rangle} = \max_{0\leq n'\leq n} (\hat{s}_{ix}^{ext} + w_{ix}^{\langle n'\rangle} - q_{n'} \cdot P_i)$$

Moreover we can rewrite any $k \geq 0$ as $k = l \cdot q^*\Theta_i + k'$, with $l \geq 0$ and $0 \leq k' \leq q^*\Theta_i - 1$. Considering that Lemma A4 holds we obtain by iteratively applying Lemma A3 for all $k \geq 0$:

$$\begin{aligned}
&\hat{s}_{ix}^{ext} + w_{ix}^{\langle q^*\Theta_i+k\rangle} - q_{q^*\Theta_i+k} \cdot P_i \\
&= \hat{s}_{ix}^{ext} + w_{ix}^{\langle(l+1)\cdot q^*\Theta_i+k'\rangle} - q_{(l+1)\cdot q^*\Theta_i+k'} \cdot P_i \\
&\leq \hat{s}_{ix}^{ext} + w_{ix}^{\langle l\cdot q^*\Theta_i+k'\rangle} - q_{l\cdot q^*\Theta_i+k'} \cdot P_i \\
&\leq \ldots \leq \hat{s}_{ix}^{ext} + w_{ix}^{\langle k'\rangle} - q_{k'} \cdot P_i
\end{aligned}$$

From this we finally get with $0 \leq k' \leq q^*\Theta_i - 1$:

$$\begin{aligned}\hat{f}_{ix'}^{\langle q^*\Theta_i+k\rangle} &= \max_{0\leq n\leq q^*\Theta_i+k}(\hat{s}_{ix}^{ext} + w_{ix}^{\langle n\rangle} - q_n \cdot P_i)\\ &= \max_{0\leq n\leq q^*\Theta_i-1}(\hat{s}_{ix}^{ext} + w_{ix}^{\langle n\rangle} - q_n \cdot P_i)\\ &= \hat{f}_{ix'}^{\langle q^*\Theta_i-1\rangle}\end{aligned}$$

□

Lemma A5 implies that as soon as the stop criterion is met we know that thereafter the maximum finish times cannot increase anymore, which proves the validity of the stop criterion.

A2. REFERENCES

[1] P. Kurtin et al. Temporal analysis of static priority preemptive scheduled cyclic streaming applications using CSDF models. In *ESTIMedia*, 2016.