Decay and Resonance of Coherent States

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Summary
We describe examples of two phenomena: the decay of a nonlinear coherent state under the influence of friction and the resonance of such a state due to a time periodic external forcing. In the latter case a period doubling transition to chaotic motion can be found as function of a parameter in the forcing. Essential is that in each case the state remains spatially coherent. The main features of these phenomena can be described with only two observables: say the amplitude and the phase of the coherent state. Here one method is given to choose these observables properly and to formulate their equations of motion in each case.

Decaying and resonant states
In this paper we consider systems with equations of motion

$$\dot{w} = F_o(w) + F_e(w,t) - D(w). \tag{1}$$

Here w is an element of the state space whose dimension may be finite or infinite. $F_o(w)$ is a Hamiltonian *autonomous* term, $F_e(w,t)$ is a *time periodic* external forcing and D(w) represents dissipation. The specific form of these terms will be specified later .
As examples for the autonomous Hamiltonian part one may think of a spherical pendulum, a chain of N particles on a ring with nonlinear interaction between nearest neighbours (such as the well-known Toda chain with periodic boundary conditions), or a continous system such as the Korteweg-de Vries equation, which describes one way running waves in many different physical systems. In the latter case the unperturbed part of (1) reads (subscripts denote partial differentiation)

$$u_t = \partial_x (u - 3u^2 - u_{xx}) \tag{2}$$

which is Hamiltonian indeed with Hamiltonian $\int dx(u^2/2 + u_x^2/2 - u^3)$, and ∂_x is the appropriate anti-symmetric operator.

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In all three examples given the autonomous part has special time periodic solutions with common properties, both in appearance and in their analytical 'background' as we will see: we mean the mode rotating with fixed angle to the vertical axis of the pendulum, a travelling 'one hump' density wave running around in ring of particles (cf. [1] for the integrable Toda chain and [2] for nonintegrable chains) and a travelling 'cnoidal' wave in the KdV case [3].

We describe the behaviour of such nonlinear excitations under influence of the extra terms F_e and D in (1) in two different cases. First consider the case of only dissipation: F_e is zero, and take the pendulum with uniform viscous friction as an example. If w = [p,q], p and q denoting momenta and coordinates respectively, the dissipation is given by

$$D(w) = \lambda\ [p,0]. \tag{3}$$

Choose as initial condition an exact rotating mode. Then obviously the motion decays and goes to equilibrium. Especially, for the friction coefficient λ below a certain treshold we observe numerically that the trajectory remains nearly circular: i.e. at each instant it is near a rotating mode, with slowly decaying amplitude and angular momentum. In fact, if L denotes the angular momentum and $\theta(L)$ the angle with the vertical axis of the rotating mode for a given L, the damped motion is described by $L(t) = L_0 \exp(-\lambda t)$ and $\theta \approx \theta\ (L(t))$ .This may not be very surprising, but a similar phenomenon is found in the Toda chain and in KdV [4,5]: In the former case, again with friction as in (3) and starting with an initial condition of an exact localized wave solution of the autonomous system, the wave decays but remains localized at each instant: if $r_n(t) = r\ (a;\ 2\pi n/N - \omega(a)t)$ describes the coordinate of the n^{th} particle of the density wave of the autonomous system (with 'amplitude parameter' a, particle number N, circular frequency ω), the decaying wave is *approximately* given by the same expression but with the parameter a decreasing as a function of time. In the KdV case the same behaviour is observed for two different types of damping, i.e. if eq. (2) is extended with uniform damping and with a term due to viscosity, given by respectively

$$-\lambda_1 u \quad \text{and} \quad -\lambda_2 u_{xx}. \tag{4}$$

Next consider a case in which all three terms in (1) are present. In a particle chain (actually with a nonintegrable interaction potential and with two fixed ends instead of an arrangement on a ring) with friction as above it appeared to be possible to choose a time periodic forcing $F_e(w,t\ )$ in such a way that there is an attracting

periodic state which shows a density wave bouncing back and forth beween the two fixed ends [6,7]. Increasing a strength parameter in the forcing a period doubling transition to chaotic motion appeared, whereas the 'one hump' character of the solution was conserved. This means that it remained a strongly localized wave going to and fro between the walls. Thus the spatial coherence was maintained and the chaotic behaviour was observed e.g. in the time intervals between the collisions of the wave with the wall. Note the similarity with a ball moving back and forth between two reflecting walls, the so called Fermi problem, [8] and with a ball dancing on a periodically vertically oscillating table [9].
Transitions to chaos via a period doubling sequence based on a coherent structure of the unperturbed system were observed in continuous systems as well, e.g.in the Nonlinear Schrödinger [10] and in the Sine-Gordon equation [11].

All these phenomena have in common that there seem to be only *a few relevant observables.* The aim of this paper is to describe briefly one way to describe these seemingly different behaviours and to derive equations of motion for the few dominant observables.

Analytical description

Fundamental for this description is that in all cases mentioned, the autonomous system has a *family of periodic solutions* depending on some parameter. In the integrable cases such families come about in the same way. Consider the pendulum. One readily verifies that the rotating mode represents at each time a minimum of the energy for given angular momentum, which is a constant of the motion because of rotational symmetry. Similarly, for KdV the integral $\int dx\, u^2$ is a constant of the motion related to translational symmetry of the system. It is called the wave action. The variational problem: minimize the energy for constant wave action yields the cnoidal solution [12]. In the case of the Toda chain the second constant of the motion is not known explicitly. However, defining the actions as in [1] a one hump wave going around in the chain represents a solution minimizing the energy for a given appropriate action. The period doubling transition discussed above was observed in a nonintegrable chain. Then no second constant of the motion is available. Nevertheless a family of bouncing waves is expected to exist. Using a variational formulation a family of solitary wave solutions can be proven to exist for particles on a ring [2].

In the integrable cases the minimization problem yields the value of the second constant of the motion, call it a, as a function of the energy, or vice versa. So naturally one can parametrize the solution with a. Then also the frequency of the motion depends on a and a solution of the autonomous system is given by

$$w(t) = v(a, \phi) \text{ with } a = a_0 \text{ and } \phi = \omega(a_0)\, t + \phi_0, \qquad (5)$$

a_0 and ϕ_0 being determined by the initial conditions, and $v(a, \phi)$ a 2π periodic function of ϕ.

Essential for the present analysis is that such a family spans *a twodimensional surface* in the phase space of the given system

$$V =\text{def } \{w \mid w = v(a, \phi),\ 0 \le \phi < 2\pi \text{ and a in some range } \}. \qquad (6)$$

This surface is composed of periodic orbits (5) of the unperturbed part of the equation for various values of a_0. Thus V is an invariant surface for this part. One can show [7] that $H_0(v(a, \phi))$ does not depend on ϕ and that for a proper scaling $dH_0/da = \omega(a)$. Thus one can say that *the restriction of the unperturbed part to* V *is a one degree of freedom integrable oscillator* with $[a, \phi]$ as action angle variables.

The experiments described above lead to the observation that in these cases the actual orbit of the full system remains *near the two-dimensional surface* V. This means that one can write

$$w(t) = v(a(t), \phi(t)) + z(t) \qquad (7)$$

with z small. Obviously any solution can be written like (7) but even for a solution near V, z is small only if a and ϕ have the right dependence on time. We now describe briefly how this can be achieved.

Let $T(a,\phi)$ denote the (two-dimensional) tangent space to V at the point $v(a,\phi)$ of V. Let $Y(a,\phi)$ be its complement in the phase space such that

$$(y, J x) = 0 \quad \text{for} \quad y \in Y,\ x \in T \quad \text{and with} \quad J = \begin{matrix} 0 & -Id \\ Id & 0 \end{matrix} . \qquad (8)$$

For a continuos system as KdV, J must be taken as $-\partial x^{-1}$. Now we require

$$(z(t), J x (a(t), \phi(t))) \equiv 0 \qquad (9)$$

i.e. $z \in Y$ *for all time.* This requirement serves two purposes: it makes the splitting $w = v(a,\phi) + z$ unique, and it decouples the dynamics of a and ϕ on the one hand and of z on the other in some optimal way

To obtain equations of motion for $[a,\phi]$ and z substitute (7) in the equation of motion (1) and split it in two parts by projecting on T and Y respectively. Using v_a and v_ϕ (subscripts denote partial differentiation) as the obvious basis for the tangent space T we obtain with (9), omitting the $O(z^2)$ terms

$$\dot{a}\ v_a + (\dot{\phi} - \omega(a))\ v_\phi = [\ F_e(v,t) - D(v)\ ]_T + \qquad (*)$$
$$[\ \{dF_e(v,t) - dD(v)\}\ z\ ]_T\ , \qquad (10a)$$

$$[\ \dot{z} - (dF_0(v) + dF_e(v,t) - dD(v))\ z\ ]_Y = [\ F_e(v,t) - D(v)\ ]_Y\ . \qquad (10b)$$

Before we draw conclusions, consider these coupled equations in more detail. The first line of (10a) (referred to as (*)) represents a one degree of freedom driven and damped oscillator with action angle variables a and ϕ. In fact it is precisely the restriction of the system to V (cf [5,6,7] for more details). The second equation (10b) is an inhomogeneous linear equation for z, the inhomogeneous term being the Y-component of the external forcing and the dissipation at the surface V. They are coupled in two ways: both sides of (10b) depend, through v, on a and ϕ, and there is the term linear in z in (10a).
Now we claim that the one degree of freedom (1 dof) oscillator (*) describes the dominant features of the decaying and resonating modes. This is supported by more detailed analysis of experimental data for Toda [4] and KdV [5]. The possible validity of this claim is also supported by the next paragraph.
One interesting question to answer is under what conditions the full system has a trajectory that remains near V and whose dominant features are given by (*). Two conditions seem to be sufficient: i) The zero solution of the homogeneous part of (10b), in which the solution of (*) is substituted, is attracting, and ii) The transversal component of the combined effect of forcing and damping (rhs of (10b)) is much smaller than its tangential component (rhs of (*)). i) causes that z is of the order of this transversal component so that ii) justifies that the linear term in z of the second line of (10a) can be neglected. One consequence is that carefully adjusting the driving term, as in [6], may cause behaviour described by (*). Indeed similar work on the Toda chain, but with a straight sinusoidal term [13], leads to different behaviour.

Comments

In conclusion, we claim that (10) and in particular (*) gives a useful description of the discussed phenomena, but obviously a detailed rigorous analysis, including the effect of the higher order terms, is necessary. Some final remarks are in order. Obviously the 1dof oscillator (*) can show time chaotic behaviour, depending on the chosen driving. In that case one cannot immediately conclude that the full system has a chaotic attractor, let alone that it is low-dimensional. Eq.(10a) is only an *approximate description* of the *projection* of the trajectory on V. The same

holds true if a complete period doubling sequence exists in (10a): One cannot simply conclude that the full system has a pd-sequence too [14].
The above described procedure is one way to introduce 'collective coordinates', that is coordinates which describe the 'bulk features' of the observed motion. Often special solutions are substituted in the equations of motion or in the Lagrangian of the system [3] and assuming several properties of the motion (e.g. slowly varying variables etc.) one ends up with a set ODE´s like here. Sometimes the derivation is purely based on the phenomenology. An interesting example is given in [15]. Our method seems to make optimal use of the Hamiltonian structure of the unperturbed system, via the projection (9) and the use of action-angle variables on V. It seems worthwile to investigate the relation between the several methods.
Finally, the present method seems to be extendable to higher dimensional dominant behaviour: The solution family of the unperturbed system is then for instance a family of two-tori and one can hope to be able to describe and understand 2-dof behaviour. Many interesting experiments in this area are done, as examples we just mention [16,17], presented at this Conference, and [18].

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