# Algebraic Aspects of Families of Fuzzy Languages

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**Abstract**

We study operations on fuzzy languages such as union, concatenation, Kleene $\star$, intersection with regular fuzzy languages, and several kinds of (iterated) fuzzy substitution. Then we consider families of fuzzy languages, closed under a fixed collection of these operations, which results in the concept of full Abstract Family of Fuzzy Languages or full AFFL. This algebraic structure is the fuzzy counterpart of full Abstract Family of Languages that has been encountered frequently in investigating families of crisp (i.e., non-fuzzy) languages.

In the second part of the paper we focus our attention to full AFFL's closed under iterated parallel fuzzy substitution, where the iterating process is prescribed by given crisp control languages. Proceeding inductively over the family of these control languages, yields an infinite sequence of full AFFL-structures with increasingly stronger closure properties.

**Keywords:** fuzzy languages, closure properties, full Abstract Family of Fuzzy Languages (full AFFL), controlled iterated fuzzy substitution, infinite hierarchies.

## 1 Introduction

Originally, a fuzzy formal language $L$ over an alphabet $\Sigma$ has been defined in [20] as a fuzzy subset of $\Sigma^\star$ with membership function $\mu_L : \Sigma^\star \to [0,1]$. Subsequently, the real closed interval $[0,1]$ has been replaced by a more general algebraic structure, viz. $\mu_L : \Sigma^\star \to \mathcal{L}$, where $\mathcal{L}$ is a (completely distributive) complete lattice; cf. [12]. Recently, the interest in fuzzy context-free grammars and their languages revived in an attempt to model grammatical errors and their rôle in robust parsing [3, 4, 8, 9]. However, in order to treat the accumulation of grammatical errors adequately ("Making an error twice is worse than making it once.") $\mathcal{L}$ ought to be augmented with an additional operation; so $\mathcal{L}$ became a commutative semigroup provided with a completely distributive complete lattice order [5, 6, 8]; cf. [15, 19].

In this framework we study fuzzy languages (§2), operations on fuzzy languages (§3), and the corresponding algebras, i.e., families of fuzzy languages closed under certain operations (§4). These operations are fuzzy analogues of well-known language-theoretic operations like union, concatenation, Kleene $\star$, homomorphism, inverse homomorphism, and intersection with regular languages. A nontrivial family of fuzzy languages closed under these six operations (extended to fuzzy languages) is called a full AFFL (full Abstract Family of Fuzzy Languages; §5), being the fuzzy counterpart of full AFL [13]. Next we briefly consider in §6 full AFFL's possessing stronger closure properties, such as full substitution-closed AFFL's [6], full super-AFFL's (full AFFL's closed under nested iterated fuzzy substitution [8]) and full hyper-AFFL's (full AFFL's closed under iterated fuzzy substitution [5]). Finally, we define an infinite sequence of such algebraic structures, each of which is "stronger" than its predecessor in the sequence, while all elements in the sequence are full hyper-AFFL's (§7). This sequence is obtained by (i) controlling the iteration of fuzzy substitutions by crisp languages that prescribe the order of the fuzzy substitutions, and (ii) proceeding inductively over the families of crisp control languages. The last section (§8) consists of a few concluding remarks.

## 2 Fuzzy Languages

We assume familiarity with basic definitions and results of formal language theory; cf. [17, 18, 21, 22] for basic texts and [13] for operations on languages. The rudiments of lattice theory can be found in many books on algebra; a summary of the relevant concepts is also included in [2].

Instead of the real interval $[0,1]$ as in [20], we take a more general structure as codomain of membership functions for fuzzy languages [5, 6, 8, 9]; cf. also [15, 19].

**Definition 2.1.** An algebraic structure $\mathcal{L}$ or $(\mathcal{L},\wedge,\vee,0,1,\star)$ is a *type-00 lattice* if

- $(\mathcal{L},\wedge,\vee,0,1)$ is a completely distributive complete lattice. So $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$ and $(\bigvee a_i) \wedge b = \bigvee_i (a_i \wedge b)$ hold for all $a_i$, $a$, $b_i$ and $b$ in $\mathcal{L}$. And 0 and 1 are the smallest and the greatest element of $\mathcal{L}$, respectively: $0 = \bigwedge \mathcal{L}$ and $1 = \bigvee \mathcal{L}$.
- $(\mathcal{L},\star)$ is a commutative semigroup.
- The following identities hold for all $a$ and $b$ in $\mathcal{L}$: $a \star \bigvee_i b_i = \bigvee_i (a \star b_i)$, $(\bigvee_i a_i) \star b = \bigvee_i (a_i \star b)$, $0 \wedge a = 0 \star a = a \star 0 = 0$, and $1 \wedge a = 1 \star a = a \star 1 = a$.

A type-00 lattice in which the operation $\star$ coincides with $\wedge$ is called a *type-01 lattice*: so it is a completely distributive complete lattice. A *type-10 lattice* is a type-00 lattice in which $(\mathcal{L},\wedge,\vee,0,1)$ is a totally ordered set or chain, i.e., for all $a$ and $b$ in $\mathcal{L}$, we have $a \wedge b = a$ or $a \wedge b = b$. In a type-10 lattice the operations $\vee$ and $\wedge$ are usually denoted by max and min, respectively. Finally, when $\mathcal{L}$ is both a type-01 lattice and a type-10 lattice, $\mathcal{L}$ is called a *type-11 lattice*. □

**Lemma 2.2.** [5, 6] *In each type-00 lattice $\mathcal{L}$, we have for all $a, b \in \mathcal{L}$, $a \star b \leq a \wedge b$.*

*Proof.* By the distributivity of $\star$ over $\vee$, $a \star (1 \vee b) = a \star 1 \vee a \star b$ holds. As $1 \vee b = 1$ and $a \star 1 = a$, this yields $a = a \vee a \star b$; so $a \star b \leq a$. Similarly, $a \star b \leq b$, and hence $a \star b \leq a \wedge b$. □

**Example 2.3.** Let $[0,1]$ be the interval of real numbers in between 0 and 1.
(1) Then $([0,1]\times[0,1],\wedge,\vee,(0,0),(1,1),\star)$ with $(x_1,y_1) \wedge (x_2,y_2) = (\min\{x_1,x_2\},\min\{y_1,y_2\})$, $(x_1,y_1) \vee (x_2,y_2) = (\max\{x_1,x_2\},\max\{y_1,y_2\})$ and $(x_1,y_1) \star (x_2,y_2) = (x_1x_2, y_1y_2)$ for all $x_1$, $x_2$, $y_1$ and $y_2$ in $[0,1]$ is a type-00 lattice.
(2) Let $\mathcal{L}$ be $(\{0,\xi,\eta,1\},\wedge,\vee,0,1,\wedge)$ with $0 < \xi < 1$, $0 < \eta < 1$, and $\xi$ and $\eta$ are incomparable. Then $\mathcal{L}$ is a type-01 lattice (and it is the 4-element distributive lattice that is not a chain).
(3) $([0,1],\min,\max,0,1,\star)$ with $x_1 \star x_2 = x_1x_2$ for all $x_1$ and $x_2$ in $[0,1]$ is a type-10 lattice.
(4) $([0,1],\min,\max,0,1,\min)$ is a type-11 lattice. □

In practical examples the real closed interval $[0,1]$ is usually restricted to (i.e., replaced by) the set of its computable or even its rational elements; cf. [9]. We refer to [12] for the impact of computability constraints in fuzzy formal languages.

**Definition 2.4.** Let $\mathcal{L}$ be a type-00 lattice and let $\Sigma$ be an alphabet. A *$\mathcal{L}$-fuzzy language* over $\Sigma$ is a $\mathcal{L}$-fuzzy subset of $\Sigma^\star$, i.e., it is a triple $(\Sigma,\mu_{L_0},L_0)$ where $\mu_{L_0}$ is a function $\mu_{L_0} : \Sigma^\star \to \mathcal{L}$, the *membership function* of $L_0$, and $L_0$ is the support of $\mu_{L_0}$; so $L_0 = \{w \in \Sigma^\star \mid \mu_{L_0}(w) > 0\}$. Usually, we write $L_0$ instead of $(\Sigma,\mu_{L_0},L_0)$.

When $\mathcal{L}$ is clear from the context, we use "fuzzy language" instead of "$\mathcal{L}$-fuzzy language". In the sequel we will also write $\mu(x;L_0)$ rather than $\mu_{L_0}(x)$.

For each fuzzy language $L_0$ over $\Sigma$, the *crisp language* $c(L_0)$ induced by $L_0$ —or the *crisp part* of $L_0$— is the subset $\{w \in \Sigma^\star \mid \mu(w;L_0) = 1\}$ of $\Sigma^\star$. Each ordinary (non-fuzzy) language $L_0$ coincides with its crisp part $c(L_0)$. So an ordinary language is also called a *crisp* language. □

**Example 2.5.** (1) Let $\mathcal{L}$ be the type-00 lattice of Example 2.3(1), and let the $\mathcal{L}$-fuzzy language $L_0$ over $\Sigma = \{a,b\}$ be defined by: $\mu(a^mb^na^m;L_0) = (m/(\max\{1,m,n\}), n/(\max\{1,m,n\}))$ if $m,n \geq 1$. (In definitions of this type, we always tacitly assume that $\mu(x;L_0) = (0,0)$, i.e., the zero-element of $\mathcal{L}$, in all other, unmentioned cases for $x$ in $\Sigma^\star$.) Then the crisp part of $L_0$ equals $c(L_0) = \{a^mb^ma^m \mid m \geq 1\}$: for each $x$ in $c(L_0)$, we have $\mu(x;L_0) = (1,1)$.
(2) Consider the type-01 lattice $\mathcal{L}$ of Example 2.3(2) and the $\mathcal{L}$-fuzzy languages $L_1$ and $L_2$ over $\{a,b\}$ defined by $\mu(a^mb^ma^n;L_1) = \xi$ and $\mu(a^mb^na^n;L_2) = \eta$ for $m,n \geq 1$. Then $c(L_1) = c(L_2) = \varnothing$ but both $L_1$ and $L_2$ are nonempty languages.

(3) Let again $\mathcal{L}$ be the type-01 lattice of Example 2.3(2). As a slight variation of the previous example, define the $\mathcal{L}$-fuzzy languages $L_3$ and $L_4$ over $\{a,b,c,d\}$ by $\mu(a^nb^nc^md^m;L_3)=\xi$ and $\mu(a^nb^mc^md^n : L_4)=\eta$ for $m,n\geq 1$. Of course, we have $c(L_3)=c(L_4)=\varnothing$, and both $L_3$ and $L_4$ are nonempty languages. □

Equality of fuzzy languages can be defined in several ways. Henceforth, we use full equality: two fuzzy languages $(\Sigma_1,\mu_{L_1},L_1)$ and $(\Sigma_2,\mu_{L_2},L_2)$ are *fully equal*, denoted by $L_1 \mathrel{\hat{=}} L_2$, if $\mu_{L_1}=\mu_{L_2}$, i.e., if for all $x\in(\Sigma_1\cup\Sigma_2)^\star$, $\mu(x;L_1)=\mu(x;L_2)$. Full equality implies equality of supports and of crisp parts, but not vice versa.

## 3 Operations on Fuzzy Languages

First, we recall the operations union, intersection, concatenation, Kleene $+$ and Kleene $\star$ for $\mathcal{L}$-fuzzy languages defined in [5, 6, 8]. We use $\lambda$ to denote the empty word.

Let $(\Sigma_1,\mu_{L_1},L_1)$ and $(\Sigma_2,\mu_{L_2},L_2)$ be fuzzy languages, then the *union*, the *intersection*, and the *concatenation* of the fuzzy languages $L_1$ and $L_2$, denoted by $(\Sigma_1\cup\Sigma_2,\mu_{L_1\cup L_2},L_1\cup L_2)$, $(\Sigma_1\cap\Sigma_2,\mu_{L_1\cap L_2},L_1\cap L_2)$ and $(\Sigma_1\cup\Sigma_2,\mu_{L_1L_2},L_1L_2)$ respectively (and abbreviated by $L_1\cup L_2$, $L_1\cap L_2$ and $L_1L_2$) are defined by: for all $x$ in $\Sigma^\star$,

$$\mu(x;L_1\cup L_2)=\mu(x;L_1)\vee\mu(x;L_2),$$

$$\mu(x;L_1\cap L_2)=\mu(x;L_1)\wedge\mu(x;L_2),\text{ and}$$

$$\mu(x;L_1L_2)=\bigvee\{\mu(y;L_1)\star\mu(z;L_2)\mid x=yz\}.$$

**Example 3.1.** (1) For the union and the intersection of the fuzzy languages $L_1$ and $L_2$ of Example 2.5(2), we have

$$\mu(x;L_1\cup L_2)=\begin{cases}1 & \text{if } x=a^mb^ma^m \text{ for some } m\geq 1,\\ \xi & \text{if } x=a^mb^ma^n \text{ and } m\neq n\ (m,n\geq 1),\\ \eta & \text{if } x=a^mb^na^n \text{ and } m\neq n\ (m,n\geq 1),\end{cases}$$

and $L_1\cap L_2=c(L_1\cap L_2)=\varnothing$, respectively. Note that $c(L_1\cup L_2)=c(L_0)\neq\varnothing$ ($L_0$ is the fuzzy language from Example 2.5(1)), whereas $c(L_1)=c(L_2)=\varnothing$.

(2) Similarly, for the union of $L_3$ and $L_4$ of Example 2.5.(3), we get

$$\mu(x;L_3\cup L_4)=\begin{cases}1 & \text{if } x=a^nb^nc^nd^n \text{ for some } n\geq 1,\\ \xi & \text{if } x=a^nb^nc^md^m \text{ and } m\neq n\ (m,n\geq 1),\\ \eta & \text{if } x=a^nb^mc^md^n \text{ and } m\neq n\ (m,n\geq 1),\end{cases}$$

We return to these unions in Example 6.4 and in §8 below. □

Next we consider the operations of *Kleene* $+$ and *Kleene* $\star$ for a fuzzy language $L$ defined by

$$L^+ \mathrel{\hat{=}} L\cup LL\cup LLL\cup\cdots \mathrel{\hat{=}} \bigcup\{L^i\mid i\geq 1\}\quad\text{and}\quad L^\star \mathrel{\hat{=}} \{\lambda\}\cup L\cup LL\cup\cdots \mathrel{\hat{=}} \bigcup\{L^i\mid i\geq 0\},$$

respectively, where $L^0 \mathrel{\hat{=}} \{\lambda\}$, and $L^{n+1} \mathrel{\hat{=}} L^nL$ with $n\geq 0$ [5, 6, 8]. Then, for $n\geq 0$ we have

$$\mu(x;L^n)=\bigvee\{\mu(x_1;L)\star\mu(x_2;L)\star\cdots\star\mu(x_n;L)\mid x_1x_2\cdots x_n=x\},\qquad\text{and}$$

$$\mu(x;L^\star)=\bigvee\{\mu(x_1;L)\star\mu(x_2;L)\star\cdots\star\mu(x_n;L)\mid n\geq 0,\ x_1x_2\cdots x_n=x\}.$$

Thus $\mu(\lambda;L^0)=1$, as $x_1x_2\cdots x_n=\lambda$ and $a_1\star a_2\star\cdots\star a_n=1$ ($a_1,\cdots,a_n\in\mathcal{L}$) in case $n=0$, and consequently $\mu(\lambda;L^\star)=1$. Hence, $L^\star \mathrel{\hat{=}} L^+\cup\{\lambda\}$ where the latter set in this union is a crisp set.

Other operations on fuzzy languages, like homomorphisms and substitutions, are defined as fuzzy functions on fuzzy languages. A fuzzy function is a special instance of a fuzzy relation. A *fuzzy relation* $R$ between crisp sets $X$ and $Y$ is a fuzzy subset of $X\times Y$. If $R\subseteq X\times Y$ and $S\subseteq Y\times Z$ are fuzzy relations, then their composition $R{\circ}S$ is defined by

$$\mu((x,z);R{\circ}S)=\bigvee\{\mu((x,y);R)\star\mu((y,z);S)\mid y\in Y\}. \tag{1}$$

Then a *fuzzy function* $f:X\to Y$ is a fuzzy relation $f\subseteq X\times Y$, satisfying the restriction that for all $x$ in $X$: if $\mu((x,y);f)>0$ and $\mu((x,z);f)>0$ hold, then $y=z$ and hence $\mu((x,y);f)=\mu((x,z);f)$. For fuzzy functions (1) holds as well, but we write the composition of two functions $f:X\to Y$ and $g:Y\to Z$ as $g{\circ}f:X\to Z$ rather than as $f{\circ}g$.

Let $\mathcal{P}(X)$ denote the power set of the crisp set $X$. In the sequel we will encounter functions $f : V^\star \to \mathcal{P}(V^\star)$ that will be extended to $f : \mathcal{P}(V^\star) \to \mathcal{P}(V^\star)$ by $f(L) = \bigcup\{f(x) \mid x \in L\}$ and for each subset $L$ of $V^\star$,

$$\mu(y; f(L)) = \bigvee\{\mu(x; L) \star \mu((x, y); f) \mid x \in V^\star\}. \tag{2}$$

Consequently, by (1) and (2) fuzzy functions like $f\circ f$, $f\circ f\circ f$, and so on, which are obtained by iterating the function $f$, are now defined. Clearly, each of these functions $f^{(k)}$ is of type $f^{(k)} : \mathcal{P}(V^\star) \to \mathcal{P}(V^\star)$. A finite set $\{f_1, \cdots, f_n\}$ of such functions can be iterated in the same way; cf. Definitions 6.1, 6.2, 6.3 and 7.1 below.

There are several ways to define the power set $\mathcal{P}(Z)$ of a fuzzy set $Z$. Based on the definition of inclusion —$Y \subseteq Z$ if and only if $\forall y : \mu(y; Y) \leq \mu(y; Z)$— we could define $\mathcal{P}(Z) = \{Y \mid Y \subseteq Z\}$. However, we will define the *power set* $\mathcal{P}_f(Z)$ of the fuzzy set $Z$ by

$$\mathcal{P}_f(Z) = \{Y \mid Y \doteq Y \cap Z\} = \{Y \mid \forall y : \mu(y; Y) = \mu(y; Y \cap Z)\}. \tag{3}$$

This latter definition implies that the power set $\mathcal{P}_f(Z)$ of a finite fuzzy set $Z$ is a crisp finite set of finite fuzzy sets.

## 4 Families of Fuzzy Languages

Let $\Sigma_\omega$ denote a countably infinite set of symbols. All families of languages that we will consider only use symbols from this set. And $\mathcal{L}$ is a type-00 lattice except when stated otherwise.

**Definition 4.1.** A *family of fuzzy languages* $K$ is a set of fuzzy languages $(\Sigma_L, \mu_L, L)$ such that each $\Sigma_L$ is a finite subset of $\Sigma_\omega$. We assume that for each fuzzy language $(\Sigma_L, \mu_L, L)$ in $K$, the alphabet $\Sigma_L$ is minimal, i.e., a symbol $\alpha$ belongs to $\Sigma_L$ if and only if there exists a word $w$ in which $\alpha$ occurs and for which $\mu(w : L) > 0$ or, equivalently, for which $w \in L$ holds.

A family $K$ of fuzzy languages is called *nontrivial* if $K$ contains a language $(\Sigma_L, \mu_L, L)$ with $c(L) \cap \Sigma_L^+ \neq \varnothing$, i.e., $(\Sigma_L, \mu_L, L)$ satisfies $\mu(x; L) = 1$ for some $x \in \Sigma_L^+$.

The *crisp part* $c(K)$ of a family $K$ of fuzzy languages is defined by $c(K) = \{c(L) \mid L \in K\}$. □

Henceforth, we assume that each family $K$ of fuzzy languages is closed under isomorphism ("renaming of symbols"), i.e., for each language $L$ in $K$ over some alphabet $\Sigma_L$ and for each bijective non-fuzzy mapping $i : \Sigma_L \to \Sigma_L'$ —extended to words and to languages in the usual way— we have that the language $i(L)$ also belongs to $K$. Remark that for all $x$ in $\Sigma_L^\star$, we have $\mu(x; L) = \mu(i(x); i(L))$.

Among the most simple nontrivial families of fuzzy languages, we have the family $\mathrm{FIN}_f$ of finite fuzzy languages $\mathrm{FIN}_f = \{\{w_1, w_2, \cdots, w_n\} \mid w_i \in \Sigma_\omega^\star,\ 1 \leq i \leq n;\ n \geq 0\}$, the family $\mathrm{ONE}_f$ of singleton fuzzy languages $\mathrm{ONE}_f = \{\{w\} \mid w \in \Sigma_\omega^\star\}$, the family $\mathrm{ALPHA}_f$ of fuzzy alphabets $\mathrm{ALPHA}_f = \{\Sigma \mid \Sigma \subset \Sigma_\omega,\ \ \Sigma \text{ is finite}\}$, and the family $\mathrm{SYMBOL}_f$ of singleton fuzzy alphabets $\mathrm{SYMBOL}_f = \{\{\alpha\} \mid \alpha \in \Sigma_\omega\}$. The crisp counterparts of these language families are denoted by FIN, ONE, ALPHA, and SYMBOL, respectively; cf. Lemma 4.3 below.

The family of regular fuzzy languages is denoted by $\mathrm{REG}_f$; it is defined in a way very similar to its crisp counterpart REG.

**Definition 4.2.** The *family of regular fuzzy languages* $\mathrm{REG}_f$ is the smallest set satisfying:

- The fuzzy subsets $\varnothing$ and $\{\lambda\}$ of $\varnothing^\star$ belong to $\mathrm{REG}_f$. Note that $\mu(\lambda; \varnothing^\star) = 1$.
- For each $\sigma$ in $\Sigma_\omega$, the fuzzy subset $\{\sigma\}$ of $\{\sigma\}^\star$ belongs to $\mathrm{REG}_f$.
- If $R_1$ and $R_2$ are in $\mathrm{REG}_f$, then so are $R_1 \cup R_2$, $R_1 R_2$, and $R_1^\star$. □

**Lemma 4.3.** (1) *$c(\mathrm{FIN}_f) = \mathrm{FIN}$, $c(\mathrm{ONE}_f) = \mathrm{ONE} \cup \{\varnothing\}$, $c(\mathrm{ALPHA}_f) = \mathrm{ALPHA}$, and $c(\mathrm{SYMBOL}_f) = \mathrm{SYMBOL} \cup \{\varnothing\}$.*
(2) *If $\mathcal{L}$ is a type-10 lattice, then for $\mathcal{L}$-fuzzy languages, $c(\mathrm{REG}_f) = \mathrm{REG}$.*

*Proof.* The inclusion $\mathrm{REG} \subseteq c(\mathrm{REG}_f)$ is obvious. The converse inclusion $c(\mathrm{REG}_f) \subseteq \mathrm{REG}$ can be easily established by a straightforward induction over the structure of a fuzzy regular language (Definition 4.2). The other equalities are trivial. □

Closely related to regular fuzzy languages is a kind of fuzzy finite automaton. The next definition and equivalence result (Proposition 4.6) is not surprising but useful.

**Definition 4.4.** A *nondeterministic fuzzy finite automaton with* $\lambda$*-moves* or *NFFA* $M$ is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where $Q$ is a finite fuzzy set of states, $\Sigma$ is an alphabet, $q_0$ is an element of $Q$ with $\mu(q_0; Q) > 0$, $F$ is a crisp subset of the crisp set $\{q \mid \mu(q; Q) > 0\}$, and $\delta$ is a fuzzy function of type $\delta : Q \times (\Sigma \cup \{\lambda\}) \to \mathcal{P}_f(Q)$. See (3) for $\mathcal{P}_f(Q)$.

The function $\delta$ is extended to $\hat{\delta} : Q \times \Sigma^\star \to \mathcal{P}_f(Q)$ by $\hat{\delta}(q, \lambda) \doteq \delta(q, \lambda)$ and $\hat{\delta}(q, \sigma\omega) \doteq \bigcup\{\hat{\delta}(q', \omega) \mid q' \in \delta(q, \sigma)\}$ for all $q$ in $Q$.

The language $L(M)$ accepted by $M$ is defined by $\mu(x; L(M)) = \bigvee\{\mu(q; \hat{\delta}(q_0, x)) \mid q \in F\}$. □

We use expressions like $X = \{\cdots, x/_{\mu(x;X)}, \cdots\}$ to denote finite fuzzy sets (including the degrees of membership) concisely.

**Lemma 4.5.** *Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFFA. Then there is an equivalent NFFA $M' = (Q', \Sigma, \delta', q_0', \{f\})$ such that $Q' \doteq Q \cup \{q_0'/_1, f/_1\}$, the in-degree of $q_0$ is zero and the out-degree of $f$ is zero, i.e., $\delta'$ is a fuzzy function of type $\delta' : (Q \cup \{q_0', f\}) \times (\Sigma \cup \{\lambda\}) \to \mathcal{P}_f(Q \cup \{f\})$ with $\delta'(f, \lambda) \doteq \{f/_1\}$ and $\forall \alpha \in \Sigma : \delta'(f, \alpha) \doteq \varnothing$.*

*Proof.* In order to obtain $\delta'$ we extend the fuzzy function $\delta$, viewed as fuzzy relation, by $\delta' \doteq \delta \cup \{((q_0', \lambda), q_0)/_{\mu(q_0;Q)}\} \cup \{((q, \lambda), f)/_1 \mid q \in F\}$. □

**Proposition 4.6.** *A fuzzy language $L$ is regular if and only if $L$ is accepted by a nondeterministic fuzzy finite automaton.*

*Proof.* Suppose $R$ is a regular fuzzy language. If $R$ equals $\varnothing$, $\{\lambda\}$ or $\{\sigma\}$ (Definition 4.2) we define $M = (Q, \Sigma, \delta, q_0, F)$ by

$\varnothing$: $Q \doteq \{q_0/_1, q_1/_1\}$, $F = \{q_1\}$, $\delta \doteq \varnothing$,
$\{\lambda\}$: $Q \doteq \{q_0/_1, q_1/_1\}$, $F = \{q_1\}$, $\delta(q_0, \lambda) \doteq \{q_1/_1\}$,
$\{\sigma\}$: $Q \doteq \{q_0/_1, q_1/_1\}$, $F = \{q_1\}$, $\delta(q_0, \sigma) \doteq \{q_1\}$, $\mu(q_1; \delta(q_0, \sigma)) = \mu(\sigma; \{\sigma\})$.

Then for each of these three cases we have $R \doteq L(M)$.

Next, let $R$ be equal to $R_1 \cup R_2$, $R_1R_2$ or $R_1^\star$ (Definition 4.2). Suppose $R_i \doteq L(M_i)$ for $i = 1, 2$ with $M_i = (Q_i, \Sigma_i, \delta_i, q_{i0}, \{f_i\})$ satisfying the properties of Lemma 4.5, and $Q_1 \cap Q_2 \doteq \varnothing$. We construct $M_j = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma_j, \delta_j, q_{j0}, F_j)$ for $j = 3, 4, 5$ by

$R_1 \cup R_2$: $F_3 = \{f_1, f_2\}$, $\delta_3 \doteq \delta_1 \cup \delta_2 \cup \{((q_0, \lambda), q_{10})/_1, ((q_0, \lambda), q_{20})/_1\}$,
$R_1R_2$: $F_4 = \{f_2\}$, $\delta_4 \doteq \delta_1 \cup \delta_2 \cup \{((q_0, \lambda), q_{10})/_1, ((f_1, \lambda), q_{20})/_1\}$,
$R_1^\star$: $F_5 = \{f_1\}$, $\delta_5 \doteq \delta_1 \cup \{((q_0, \lambda), f_1)/_1, ((q_0, \lambda), q_{10})/_1, ((f_1, \lambda), q_{10})/_1\}$.

Then $L(M_3) \doteq R_1 \cup R_2$, $L(M_4) \doteq R_1R_2$, and $L(M_5) \doteq R_1^\star$.

The converse implication can easily be established by adapting the standard construction (cf. e.g., [22] pp. 200–203). From §3 it will be clear how to apply the operations $\vee$ and $\star$ in updating the degree of membership when we meet a union, a concatenation or a Kleene $\star$ operation in that construction □

Other families of fuzzy languages are obtained by applying the operation of fuzzy substitution or some of its generalizations (Definitions 4.7, 5.5, 6.6 and 7.5 below). Fuzzy substitution plays the principal rôle in our approach: it is a straightforward extension of the notion of substitution for crisp languages.

**Definition 4.7.** Let $K$ be a family of fuzzy languages and let $V$ be an alphabet. A mapping $\tau : V \to K$ is called a *fuzzy $K$-substitution* on $V$; it is extended to words over $V$ by $\tau(\lambda) \doteq \{\lambda/_1\}$ and $\tau(\alpha_1 \cdots \alpha_n) \doteq \tau(\alpha_1) \cdots \tau(\alpha_n)$ with $\alpha_i \in V$ $(1 \le i \le n)$, and to languages by $\tau(L) \doteq \bigcup\{\tau(w) \mid w \in L\}$.

If for each $\alpha \in V$, $\tau(\alpha) \subseteq V^\star$, then $\tau : V \to K$ is called a *fuzzy $K$-substitution over $V$*.

If for each $\alpha \in V$, we have $\alpha \in \tau(\alpha)$, then $\tau : V \to K$ is called a *nested fuzzy $K$-substitution*.

If $K$ equals $\mathrm{FIN}_f$ or $\mathrm{REG}_f$, $\tau$ is called a *fuzzy finite* or a *fuzzy regular substitution*, respectively.

Given families $K$ and $K'$ of fuzzy languages, let $\hat{\text{Sûb}}(K, K') = \{\tau(L) \mid L \in K;\ \tau$ is a fuzzy $K'$-substitution$\}$. A family $K$ is *closed* under fuzzy $K'$-substitution if $\text{Sûb}(K, K') \subseteq K$, and $K$ is *closed under fuzzy substitution*, if $K$ is closed under fuzzy $K$-substitution. □

Since we assumed that each family of fuzzy languages is closed under isomorphism, the Sûb-operator is associative; cf. [14, 13].

Taking $K$ and $K'$ equal to families of crisp languages in Definition 4.7 yields the well-known notion of (ordinary, non-fuzzy, crisp) substitution. Then a ONE-substitution is just a homomorphism and an isomorphism ("renaming of symbols") is a one-to-one SYMBOL-substitution.

Similarly, we define an $\mathcal{L}$-fuzzy homomorphism $h : \Sigma_1^\star \to \Sigma_2^\star$ as an $\mathcal{L}$-fuzzy $\text{ONE}_f$-substitution. The inverse $h^{-1} : \mathcal{P}(\Sigma_2^\star) \to \mathcal{P}(\Sigma_1^\star)$ of such an $\mathcal{L}$-homomorphism is defined by $h^{-1}(L) = \{w \in \Sigma_1^\star \mid \mu(h(w); L) > 0\}$ with

$$\mu(x; h^{-1}(L)) = \bigvee\{\mu((x,y); h) \star \mu(y; L) \mid y \in \Sigma_2^\star\}. \tag{4}$$

Clearly, $h$ is viewed as a fuzzy relation of which we take the converse to obtain $h^{-1}$; cf. (2).

Note that in general for a fuzzy function $f : X \to Y$ and a fuzzy subset $S$ of $Y$, we have

$$\begin{aligned}\mu(y; ff^{-1}(S)) &= \bigvee\{\mu(x; f^{-1}(S)) \star \mu((x,y); f) \mid x \in X\} = \\ &= \bigvee\{(\bigvee\{\mu(z; S) \star \mu((x,z); f) \mid z \in Y\}) \star \mu((x,y); f) \mid x \in X\}.\end{aligned}$$

Since $f$ is a function, $\mu((x,z); f) > 0$ and $\mu((x,y); f) > 0$ imply $z = y$. Hence

$$\mu(y; ff^{-1}(S)) = \bigvee\{\mu(y; S) \star \mu((x,y); f) \star \mu((x,y); f) \mid x \in X\} \leq \mu(y; S).$$

So $ff^{-1}(S) \subseteq S$ and the equality $ff^{-1}(S) \cap S \doteq ff^{-1}(S)$ holds when $f$ is a crisp function. This latter fact we will use in the case of a crisp homomorphism $h : \Sigma^\star \to \text{ONE}$ for which we have $hh^{-1}(S) \cap S \doteq hh^{-1}(S)$; cf. the proofs of Lemma 5.6, Theorem 5.7 and Lemma 5.8.

## 5 Simple Algebraic Structures

We start with a very simple algebraic structure —viz. the fuzzy prequasoid— from which we arrive at more complicated ones such as full AFFL, full substitution-closed AFFL's, etc.; cf. Theorems 5.7, 6.7 and 7.6 below.

**Definition 5.1.** A nontrivial family $K$ of fuzzy languages is a *fuzzy prequasoid* if $K$ is closed under fuzzy finite substitution (i.e., $\text{Sûb}(K, \text{FIN}_f) \subseteq K$) and under intersection with fuzzy regular languages. A *fuzzy quasoid* is a fuzzy prequasoid that contains a fuzzy language $L_0$ such that $c(L_0)$ is infinite. □

**Lemma 5.2.** (1) *If $K$ is a fuzzy [pre]quasoid, then $K \supseteq \text{REG}_f$ [$K \supseteq \text{FIN}_f$, respectively].*

(2) *$\text{REG}_f$ [$\text{FIN}_f$, respectively] is the smallest fuzzy [pre]quasoid.*

(3) *Let $K$ be a fuzzy prequasoid. If $L \in K$ with $L \subseteq \Sigma^\star$ and $c \notin \Sigma$, then $\{c/_1\}L \in K$.*

*Proof.* (1) Let $K$ be a fuzzy prequasoid. Since $K$ is nontrivial, there is a fuzzy language $L$ over $\Sigma$ in $K$ that contains a nonempty word $x$ with $\mu(x; L) = 1$. Let $a$ be a symbol occurring in $x$, and define the fuzzy finite substitutions $\tau : \sigma \mapsto \{\lambda/_1, a/_1\}$ for each $\sigma \in \Sigma$, and $\varphi : a \mapsto L_F$ where $L_F$ is an arbitrary finite fuzzy language. Then $L_F \doteq \varphi(\tau(L) \cap \{a/_1\})$, and hence $L_F \in K$.

If $K$ is a fuzzy quasoid, then $K$ contains an $L_0$ over $\Sigma_0$ such that $c(L_0)$ is infinite. Let $R$ be an arbitrary regular fuzzy language over $\Sigma$. Define the fuzzy finite substitution $\tau$ by $\tau(\sigma) = \{\lambda/_1\} \cup \{\alpha/_1 \mid \alpha \in \Sigma\}$ for each $\sigma \in \Sigma_0$. Then $\tau(L_0) \cap R \doteq \{w/_1 \mid w \in \Sigma^\star\} \cap R \doteq R$, and so $R$ belongs to $K$.

(2) follows from (1) and the fact that $\text{REG}_f$ [$\text{FIN}_f$, respectively] is a [pre]quasoid.

(3) Define the crisp finite substitution $\tau : \Sigma^\star \to \text{FIN}$ by $\tau(a) = \{a, ca\}$ and the crisp regular set $R$ by $R = \{c\}\Sigma^\star$. Then $\{c/_1\}L \doteq \tau(L) \cap R$; hence $\{c/_1\}L \in K$. □

Lemma 5.2 implies that $\text{FIN}_f$ is the only fuzzy prequasoid that is not a fuzzy quasoid.

For each family $K$ of fuzzy languages, let $\Phi_f(K) = \text{Sûb}(K, \text{FIN}_f)$, $\Theta_f(K) = \text{Sûb}(K, \text{ONE}_f)$, and $\Delta_f(K) = \{L \cap R \mid L \in K,\ R \in \text{REG}_f\}$. Since $\text{REG}_f$ is closed under intersection, and both $\text{FIN}_f$ and $\text{ONE}_f$ are closed under fuzzy substitution, we have that for $X \in \{\Theta_f, \Delta_f, \Phi_f\}$, $X$ is a closure operator, i.e., (i) $X$ is *extensive*: $K \subseteq X(K)$, (ii) $X$ is *monotonic*: $K_1 \subseteq K_2$ implies

$X(K_1) \subseteq X(K_2)$, and (iii) $X$ is *idempotent*: $XX(K) \subseteq X(K)$. Of course, $K$, $K_1$ and $K_2$ are families of fuzzy languages.

Similarly, let for each family $K$ of fuzzy languages, $\Pi_f(K)$ denote the smallest fuzzy prequasoid that includes $K$. Clearly, $\Pi_f$ is a closure operator too.

For each family $K$ of fuzzy languages, we have $\Pi_f(K) = \{\Phi_f, \Delta_f, \Theta_f\}^\star(K)$ or even $\Pi_f(K) = \{\Phi_f, \Delta_f\}^\star(K)$. But instead of this infinite set of strings over $\{\Phi_f, \Delta_f, \Theta_f\}$ or over $\{\Phi_f, \Delta_f\}$ respectively, a single string suffices; see Proposition 5.3 and Corollary 5.4, respectively.

**Proposition 5.3.** *For each family $K$ of fuzzy languages,* $\Pi_f(K) = \Theta_f\Delta_f\Phi_f(K)$.

*Proof.* As $\Pi_f$, $\Theta_f$, $\Delta_f$ and $\Phi_f$ are closure operators, we have $\Pi_f(K) = ((\Delta_f\Phi_f)^\star \cup (\Phi_f\Delta_f)^\star)(K)$ for each $K$. Since $\Delta_f\Phi_f(K) \subseteq \Theta_f\Delta_f\Phi_f(K)$ and $\Phi_f\Delta_f(K) \subseteq (\Delta_f\Phi_f)^2(K) \subseteq (\Theta_f\Delta_f\Phi_f)^2(K)$, it remains to show that $(\Theta_f\Delta_f\Phi_f)^2(K) \subseteq \Theta_f\Delta_f\Phi_f(K)$ or, equivalently, that $\Theta_f\Delta_f\Phi_f\Delta_f\Phi_f(K) \subseteq \Theta_f\Delta_f\Phi_f(K)$ for each $K$.

Suppose $L \in \Theta_f\Delta_f\Phi_f\Delta_f\Phi_f(K)$, i.e., these exist an $L_0$ in $K$ with $L_0 \subseteq \Sigma_0^\star$, fuzzy finite substitutions $\tau_1 : \Sigma_0^\star \to \Sigma_1^\star$ and $\tau_2 : \Sigma_1^\star \to \Sigma_2^\star$, regular fuzzy languages $R_1 \subseteq \Sigma_1^\star$ and $R_2 \subseteq \Sigma_2^\star$, and a fuzzy homomorphism $h_1 : \Sigma_2^\star \to \Sigma_3^\star$ such that $L \doteq h_1(\tau_2(\tau_1(L_0) \cap R_1) \cap R_2)$.

We will define a fuzzy finite substitution $\tau : \Sigma_0^\star \to \Sigma_4^\star$, a regular fuzzy language $R \subseteq \Sigma_4^\star$, and a fuzzy homomorphism $h : \Sigma_4^\star \to \Sigma_3^\star$ such that $L \doteq h(\tau(L_0) \cap R)$. We assume that $\Sigma_1 \cap \Sigma_2 = \varnothing$. Then we define $\Sigma_4$ by $\Sigma_4 = \Sigma_1 \cup \Sigma_2$.

Define crisp homomorphisms $\varphi_i : (\Sigma_1 \cup \Sigma_2) \to \Sigma_i^\star$ by $\varphi_i(\alpha) = \alpha$ for each $\alpha \in \Sigma_i$ and $\varphi_i(\alpha) = \lambda$ otherwise. Let $\tau_2' : \Sigma_1 \to (\Sigma_1 \cup \Sigma_2)^\star$ be the fuzzy finite substitution defined by $\tau_2'(\alpha) \doteq \{\alpha/_1\}\tau_2(\alpha)$ for each $\alpha$ in $\Sigma_1$, let $R \doteq \varphi_1^{-1}(R_1) \cap \varphi_2^{-1}(R_2)$, and $\tau(\sigma) \doteq \tau_2' \circ \tau_1(\sigma)$ for each $\sigma \in \Sigma_0$, and $h(\alpha) = h_1(\alpha)$ for each $\alpha \in \Sigma_2$ and $h(\alpha) = \lambda$ for each $\alpha \in \Sigma_1$. Then $L \doteq h(\tau(L_0) \cap R)$. □

**Corollary 5.4.** *For each family $K$ of fuzzy languages,* $\Pi_f(K) = \Phi_f\Delta_f\Phi_f(K)$. □

The following algebraic structure is the fuzzy counterpart of the full Abstract Family of Languages or full AFL; cf. [13]. Full substitution-closed AFL's have been investigated in [14].

**Definition 5.5.** A *full Abstract Family of Fuzzy Languages* or *full AFFL* is a nontrivial family of fuzzy languages closed under union, concatenation, Kleene $\star$, (possibly erasing) fuzzy homomorphism, inverse fuzzy homomorphism, and intersection with regular fuzzy languages.

A *full substitution-closed AFFL* is a full AFFL closed under fuzzy substitution. □

The remaining part of this section consists of some elementary results which are straightforward generalizations of their crisp originals (see [13, 14, 2]). First, we consider a characterization of full AFFL in Theorem 5.7 for which we need the following Lemma.

**Lemma 5.6.** (1) *A fuzzy prequasoid $K$ is closed under union, concatenation and Kleene $\star$ if and only if $K$ is closed under fuzzy substitution in the fuzzy regular languages (*$\hat{\text{Sub}}(\text{REG}_f, K) \subseteq K$*).*

(2) *If a family $K$ of fuzzy languages is closed under fuzzy regular substitution, intersection with regular fuzzy languages and union with regular fuzzy languages, then $K$ is closed under inverse fuzzy homomorphisms.*

*Proof.* (1) Let $K$ be a fuzzy prequasoid closed under union, concatenation and Kleene $\star$, and let $L_0$ be a fuzzy language over $\Sigma_0$ from $\hat{\text{Sub}}(\text{REG}_f, K)$. Then there is a fuzzy $K$-substitution $\tau : \Sigma_0 \to K$ and a regular fuzzy language $R \subseteq \Sigma_0^\star$ such that $L_0 \doteq \tau(R)$. By induction on the structure of $R$ we show that $L_0 \in K$.
Basis: If $R$ equals $\varnothing$, $\{\lambda\}$ or $\{\sigma\}$ ($\sigma \in \Sigma_0$), then clearly $\tau(R) \in K$.
Induction step: Assume that for fuzzy regular languages $R_1$ and $R_2$ over $\Sigma_0$, we have that both $\tau(R_1)$ and $\tau(R_2)$ are in $K$.

If $R \doteq R_1 \cup R_2$, $R \doteq R_1R_2$ or $R \doteq R_1^\star$, we conclude from the induction hypothesis, the closure properties of $K$ and the equalities $\tau(R_1 \cup R_2) \doteq \tau(R_1) \cup \tau(R_2)$, $\tau(R_1R_2) \doteq \tau(R_1)\tau(R_2)$ and $\tau(R_1^\star) \doteq (\tau(R_1))^\star$ that $\tau(R) \in K$, which completes the induction.

The converse implication easily follows from substituting fuzzy $K$-languages into the crisp regular sets $\{a, b\}$, $\{ab\}$ and $a^\star$.

(2) Let $(L, \Sigma_L, \mu_L)$ be an arbitrary fuzzy language in $K$ where $\Sigma_L$ is the minimal alphabet of $L$. Let $h : \Sigma^\star \to \Sigma_L^\star$ be a fuzzy homomorphism with $\Sigma = \{\sigma_1, \cdots, \sigma_k\}$ and $h(\sigma_i) = w_i$ ($w_i \in \Sigma_L^\star$,

$1 \leq i \leq k$). We will show that $h^{-1}(L)$ is in $K$.

First, we assume that $L$ is $\lambda$-free. Then we take a new alphabet $\Sigma_0 = \{\sigma'_1, \cdots, \sigma'_k\}$ and a crisp $\lambda$-free regular substitution $\tau$ defined by $\tau(\sigma) = \Sigma_0^\star \sigma \Sigma_0^\star$ for each $\sigma$ in $\Sigma_L$.

Define $L_1$ as the finite fuzzy language $L_1 \doteq \{\sigma'_i w_i/_{\mu((\sigma_i, w_i);h)} \mid 1 \leq i \leq k\}$ and the fuzzy language $L_2$ by $L_2 \doteq \tau(L) \cap L_1^\star$. Let $h_1$ be the crisp homomorphism defined by $h_1(\sigma'_i) = \sigma_i$ ($\sigma'_i \in \Sigma_0$, $\sigma_i \in \Sigma$, and $1 \leq i \leq k$) and $h(\alpha) = \lambda$ for each $\alpha$ in $\Sigma_L$. Then from the closure properties of $K$ we obtain $L_2 \in K$ and $h_1(L_2) \in K$. It is left to the reader to verify that $h_1(L_2) \doteq h^{-1}(L)$.

When $L$ contains $\lambda$, we have $L \doteq (L - \{\lambda\}) \cup \{\lambda\}$ and $h^{-1}(L) \doteq h^{-1}(L - \{\lambda\}) \cup h^{-1}(\lambda)$. Now by the first part of this proof we have $h^{-1}(L - \{\lambda\}) \in K$. If $h$ is $\lambda$-free, then $h^{-1}(\lambda) \doteq \{\lambda/_1\}$. Otherwise $h^{-1}(\lambda) \doteq \{a/_{\mu((a,\lambda);h)} \mid h(a) = \lambda\}^\star$. In either case $h^{-1}(\lambda) \in \mathrm{REG}_f$, and hence $h^{-1}(L) \in K$. □

**Theorem 5.7.** *A family $K$ of fuzzy languages is a full AFFL if and only if $K$ is a fuzzy prequasoid closed under fuzzy regular substitution (i.e., $\hat{\mathrm{Sub}}(K, \mathrm{REG}_f) \subseteq K$ ), and under substitution in the regular fuzzy languages (i.e., $\hat{\mathrm{Sub}}(\mathrm{REG}_f, K) \subseteq K$ ).*

*Proof.* In view of Lemma 5.6 it is sufficient to show that $\hat{\mathrm{Sub}}(K, \mathrm{REG}_f) \subseteq K$ when $K$ is closed under fuzzy homomorphism, inverse fuzzy homomorphism and intersection with fuzzy regular languages. Note that $\hat{\mathrm{Sub}}(K, \mathrm{REG}_f) \subseteq K$ implies closure under finite fuzzy substitution as well.

Let $L$ be a fuzzy $K$-language over $\Sigma$, and let $\tau : \Sigma \to \mathrm{REG}_f$ be a fuzzy regular substitution with $\tau(\alpha) \subseteq \Sigma_\alpha^\star$ for each $\alpha$ in $\Sigma$. Define alphabets $\Sigma_0 = \bigcup\{\Sigma_\alpha \mid \alpha \in \Sigma\}$ and $\Sigma_1 = \{\alpha' \mid \alpha \in \Sigma\}$, crisp homomorphisms $h_i : (\Sigma_0 \cup \Sigma_1) \to \mathrm{ONE}$ ($i = 1, 2$) by $h_1(\alpha') = \alpha$ ($\alpha \in \Sigma_1$), $h_1(\beta) = \lambda$ ($\beta \in \Sigma_0$), $h_2(\alpha') = \lambda$ ($\alpha \in \Sigma_1$), $h_2(\beta) = \beta$ ($\beta \in \Sigma_0$), and the regular fuzzy language $R \doteq (\bigcup\{\alpha'\tau(\alpha) \mid \alpha \in \Sigma\})^\star$ with $\mu(\alpha' x; R) = \mu(x; \tau(\alpha))$ and $\alpha \in \Sigma$. Then $\tau(L) \doteq h_2(h_1^{-1}(L) \cap R)$ and hence $\tau(L) \in K$. □

**Lemma 5.8.** *If $K_1$ and $K_2$ are fuzzy prequasoids, then so is $\hat{\mathrm{Sub}}(K_1, K_2)$.*

*Proof.* It is sufficient to show that $\hat{\mathrm{Sub}}(K_1, K_2)$ is closed under $\Phi_f$ and $\Delta_f$.

First, we have $\Phi_f(\hat{\mathrm{Sub}}(K_1, K_2)) = \hat{\mathrm{Sub}}(\hat{\mathrm{Sub}}(K_1, K_2), \mathrm{FIN}_f) = \hat{\mathrm{Sub}}(K_1, \hat{\mathrm{Sub}}(K_2, \mathrm{FIN}_f)) = \hat{\mathrm{Sub}}(K_1, \Phi_f(K_2)) = \hat{\mathrm{Sub}}(K_1, K_2)$ by the associativity of the $\hat{\mathrm{Sub}}$-operation.

Next we prove that $\Delta_f(\hat{\mathrm{Sub}}(K_1, K_2)) \subseteq \Theta_f(\hat{\mathrm{Sub}}(\Delta_f \Phi_f(K_1), \Delta_f(K_2))) = \Theta_f(\hat{\mathrm{Sub}}(K_1, K_2)) = \hat{\mathrm{Sub}}(\hat{\mathrm{Sub}}(K_1, K_2), \mathrm{ONE}_f) = \hat{\mathrm{Sub}}(K_1, \hat{\mathrm{Sub}}(K_2, \mathrm{ONE}_f)) = \hat{\mathrm{Sub}}(K_1, \Theta_f(K_2)) = \hat{\mathrm{Sub}}(K_1, K_2)$. In order to establish this inclusion, let $L$ be a fuzzy language over $\Sigma$ from $K_1$, let $\tau : \Sigma^\star \to K_2$ be a fuzzy $K_2$-substitution such that $\tau(L) \subseteq \Sigma_1^\star$ with $\Sigma_1 \cap \Sigma = \varnothing$, and let $R$ be a regular fuzzy language over $\Sigma_1$. We will prove that $\tau(L) \cap R$ belongs to $\Theta(\hat{\mathrm{Sub}}(\Delta_f \Phi_f(K_1), \Delta_f(K_2)))$.

We first define the fuzzy substitution $\tau_2$ on $\Sigma^\star$ by $\tau_2(a) \doteq \{a/_1\}\tau(a)$ for each $a$ in $\Sigma$. Note that by Lemma 5.2(3), $\tau_2$ is a fuzzy $K_2$-substitution. Next we define the crisp homomorphism $h : (\Sigma \cup \Sigma_1)^\star \to \mathrm{ONE}$ by $h(a) = \lambda$ for each $a$ in $\Sigma$ and $h(a) = a$ for each $a$ in $\Sigma_1$. Then $\tau = h \circ \tau_2$ and $\tau(L) \cap R \doteq h\tau_2(L) \cap R \doteq h(\tau_2(L) \cap h^{-1}(R))$ since $h$ is crisp.

Since both $R$ and $h^{-1}(R)$ are regular fuzzy languages (Lemma 5.6), there is according to Proposition 4.6 and Lemma 4.5 an NFFA $M = (Q, \Sigma \cup \Sigma_1, \delta, q_0, \{f\})$ with $\mu(q_0; Q) = 1$ that accepts $h^{-1}(R)$. Let $R_0$ be defined by

$$R_0 = (L(M) \cap \{\lambda\}) \cup \{(q_0, a_1, q_1) \cdots (q_{m-1}, a_m, q_m) \mid a_i \in \Sigma,\ q_i \in Q,\ 1 \leq i \leq m,\ q_m = f\}.$$

Then $R_0$ is a crisp regular set (Theorem 2.1, p. 130 in [21]). Now define for each $a$ in $\Sigma$ and each $p$ and $q$ in $Q$ the fuzzy language $R(a, p, q)$ by $R(a, p, q) = \{w \mid w \in \Sigma_1^\star,\ q \in \delta(p, aw)\}$ with $\mu(w; R(a, p, q)) = \mu(q; \delta(p, aw))$. Clearly, $R(a, p, q)$ is a regular fuzzy language by Proposition 4.6, since $R(a, p, q) \doteq L(M(a, p, q))$ where $M(a, p, q)$ is the NFFA defined by $M(a, p, q) = (Q, \Sigma_1, \delta, \delta(p, a), \{q\})$.

Let $\tau_3$ be the regular fuzzy substitution on $(\Sigma \times Q \times Q)^\star$ defined by $\tau_3((a, p, q)) \doteq \{a/_1\}R(a, p, q)$. Then $\tau_3(R_0)$ consists of all words of $h^{-1}(R)$ that do not start with a symbol of $\Sigma_1$. Because $\tau_2(L)$ does not contain words starting with a symbol of $\Sigma_1$, we have $\tau_2(L) \cap h^{-1}(R) \doteq \tau_2(L) \cap \tau_3(R_0)$.

Define the finite crisp substitution $\tau'$ on $\Sigma^\star$ by $\tau'(a) = \{a\} \times Q \times Q$ for each $a$ in $\Sigma$, and the fuzzy $K_2$-substitution $\tau''$ on $(\Sigma \times Q \times Q)^\star$ by $\tau''((a, p, q)) \doteq \tau_2(a)$ for each $(a, p, q)$ in $\Sigma \times Q \times Q$.

Then $\tau_2 = \tau'' \circ \tau'$, and $\tau_2(L) \cap h^{-1}(R) \doteq \tau''\tau'(L) \cap \tau_3(R_0)$.

Finally, let $\tau_3''$ be the fuzzy $K_1$-substitution on $(\Sigma \times Q \times Q)^\star$ defined by $\tau_3''((a,p,q)) \doteq \tau''((a,p,q)) \cap \tau_3((a,p,q)) \doteq \{a/_1\}\tau(a) \cap \{a/_1\}R(a,p,q)$ for each $(a,p,q)$ in $\Sigma \times Q \times Q$.

Then we have $\tau_2(L) \cap h^{-1}(R) \doteq \tau''\tau'(L) \cap \tau_3(R_0) \doteq \tau_3''(\tau'(L) \cap R_0)$. (The actual proof of these two equalities is left as an exercise to the reader.) Consequently, $\tau(L) \cap R \doteq h(\tau_3''(\tau'(L) \cap R_0))$ and hence $\tau(L) \cap R$ belongs to $\Theta_f(\hat{\mathrm{Sub}}(\Delta_f \Phi_f(K_1), \Delta_f(K_2)))$. □

For each family $K$ of fuzzy languages, let $\hat{\mathcal{F}}_f(K)$ denote the smallest full AFFL that includes $K$. So $\hat{\mathcal{F}}_f$ is a closure operator.

**Theorem 5.9.** *Let $K$ be a family of fuzzy languages.*
(1) $\hat{\mathrm{Sub}}(\hat{\mathrm{Sub}}(\mathrm{REG}_f, \Pi_f(K)), \mathrm{REG}_f) = \hat{\mathrm{Sub}}(\mathrm{REG}_f, \hat{\mathrm{Sub}}(\Pi_f(K), \mathrm{REG}_f))$. *This family of fuzzy languages is a full AFFL that includes $K$.*
(2) $\hat{\mathcal{F}}_f(K) = \hat{\mathrm{Sub}}(\hat{\mathrm{Sub}}(\mathrm{REG}_f, \Pi_f(K)), \mathrm{REG}_f) = \hat{\mathrm{Sub}}(\mathrm{REG}_f, \hat{\mathrm{Sub}}(\Pi_f(K), \mathrm{REG}_f))$.

*Proof.* (1) The equality follows from the associativity of the Sûb-operator. Next we show that $\hat{\mathrm{Sub}}(\hat{\mathrm{Sub}}(\mathrm{REG}_f, \Pi(K)), \mathrm{REG}_f)$, abbreviated by $Z(K)$, is a full AFFL that includes $K$.

By the monotonicity of $\Pi_f$, $\hat{\mathrm{Sub}}(\mathrm{REG}_f, \cdot)$ and of $\hat{\mathrm{Sub}}(\cdot, \mathrm{REG}_f)$, we have $K \subseteq Z(K)$. So it remains to prove that $Z(K)$ is a full AFFL. By the equality of 5.9(1) and the idempotency of $\hat{\mathrm{Sub}}(\mathrm{REG}_f, \cdot)$ and of $\hat{\mathrm{Sub}}(\cdot, \mathrm{REG}_f)$, it remains to show that $Z(K)$ is a fuzzy prequasoid. However, this follows from Lemmas 5.2 and 5.8.

(2) The inclusion $K \subseteq \hat{\mathcal{F}}_f(K)$, the monotonicity of $Z$ and Theorem 5.7, imply that $Z(K) \subseteq Z\hat{\mathcal{F}}_f(K) = \hat{\mathcal{F}}_f(K)$. As $Z(K)$ is a full AFFL that includes $K$, we obtain $\hat{\mathcal{F}}_f(K) = Z(K)$. □

Finally, we turn to full substitution-closed AFFL. Let $K_\infty$ denote the smallest family of fuzzy languages that includes a given family $K$ of fuzzy languages and that is closed under fuzzy substitution.

**Theorem 5.10.** (1) *If* $\mathrm{SYMBOL} \subseteq K$, *then*

$$K_\infty = \bigcup_{n=0}^{\infty} \mathrm{SUB}^n(K), \qquad \textit{with}$$

$$\mathrm{SUB}^0(K) = K\ , \qquad \textit{and}$$

$$\mathrm{SUB}^{n+1}(K) = \hat{\mathrm{Sub}}(\bigcup_{i=0}^{n} \mathrm{SUB}^i(K), K)\ , \qquad \textit{for each } n \geq 0.$$

(2) *If $K$ is a fuzzy quasoid, then $K_\infty$ is a full substitution-closed AFFL.*

*Proof.* (1) Let $K_1$ denote $\bigcup_{n=0}^{\infty} \mathrm{SUB}^n(K)$ for short. Since $\mathrm{SYMBOL} \subseteq K$, we have $\mathrm{SUB}^n(K) \subseteq \mathrm{SUB}^{n+1}(K)$ for each $n \geq 0$. Consequently, $\mathrm{SUB}^{n+1}(K) = \hat{\mathrm{Sub}}(\mathrm{SUB}^n(K), K)$ for each $n \geq 0$, and $K = \mathrm{SUB}^0(K) \subseteq K_1$.

Obviously, $K_1$ is closed under fuzzy $K$-substitution: viz. let $L$ be a fuzzy language from $K_1$ —i.e., there is an $i \geq 0$ such that $L \in \mathrm{SUB}^i(K)$— and let $\tau$ be a fuzzy $K$-substitution. Then $\tau(L) \in \mathrm{SUB}^{i+1}(K)$ and therefore $\tau(L) \in K_1$. Hence $K_\infty \subseteq K_1$.

In order to prove the converse inclusion we show by induction that $\mathrm{SUB}^n(K) \subseteq K_\infty$ $(n \geq 0)$.
*Basis:* $(n = 0)$ $\mathrm{SUB}^0(K) = K \subseteq K_\infty$.
*Induction hypothesis:* $\mathrm{SUB}^i(K) \subseteq K_\infty$
*Induction step:* $\mathrm{SUB}^{i+1}(K) = \hat{\mathrm{Sub}}(\mathrm{SUB}^i(K), K) \subseteq \hat{\mathrm{Sub}}(K_\infty, K) \subseteq K_\infty$ by the monotonicity of the $\hat{\mathrm{Sub}}(\cdot, K)$-operation, the induction hypothesis and the definition of $K_\infty$.

Now the inclusions $\mathrm{SUB}^n(K) \subseteq K_\infty$ $(n \geq 0)$ imply that $K_1 \subseteq K_\infty$.

(2) By Lemma 5.2 we have $\mathrm{REG}_f \subseteq K \subseteq K_\infty$. Thus $K_\infty$ is closed under $\hat{\mathrm{Sub}}(\mathrm{REG}_f, \cdot)$ and under $\hat{\mathrm{Sub}}(\cdot, \mathrm{REG}_f)$. According to Theorem 5.7, it suffices to show that $K_\infty$ is a fuzzy prequasoid. However, this can be done using the equality $K_\infty = K_1$ and a straightforward induction in which we use Lemma 5.8. □

## 6 More Complicated Algebraic Structures

We first recall the definitions of some generalized fuzzy grammars; they are generalized in the sense that they possess a countably infinite number of rules rather than a finite number. This countable number of rules is restricted in the following way: for each symbol $\alpha$, the set containing all right-hand sides of rules with left-hand side equal to $\alpha$ forms a fuzzy language that belongs to a given family $K$ of fuzzy languages. This restriction allows us to formulate these grammars in terms of fuzzy $K$-substitutions. The grammars that have been generalized in this way are: ETOL-system (Definition 6.1), context-free grammar (Definition 6.2), and non-self-embedding context-free grammar (Definition 6.3).

In each case such a family of fuzzy generalized grammars give rise to an algebraic closure operator —viz. $H_f$, $A_f$ and $R_f$, respectively— acting on (a slightly restricted class of) families $K$ of fuzzy languages.

**Definition 6.1.** [5] Let $K$ be a family of fuzzy languages. A *fuzzy $K$-iteration grammar* $G = (V, \Sigma, U, S)$ consists of an alphabet $V$, a terminal alphabet $\Sigma$ ($\Sigma \subseteq V$), an initial symbol $S$ ($S \in V$), and a finite set $U$ of fuzzy $K$-substitutions over $V$. The fuzzy language $L(G)$ generated by $G$ is defined by $L(G) \doteq U^\star(S) \cap \Sigma^\star \doteq \bigcup\{\tau_p(\cdots(\tau_1(S))\cdots) \mid p \geq 0;\ \tau_i \in U,\ 1 \leq i \leq p\} \cap \Sigma^\star$. The family of fuzzy languages generated by fuzzy $K$-iteration grammars is denoted by $H_f(K)$. □

**Definition 6.2.** [8] Let $K$ be a family of fuzzy languages. A *fuzzy context-free $K$-grammar* $G$ is a fuzzy $K$-iteration grammar $G = (V, \Sigma, U, S)$ of which each substitution $\tau$ from $U$ is a nested fuzzy $K$-substitution over $V$; so $\alpha \in \tau(\alpha)$ for each $\alpha \in V$ and each $\tau \in U$. The family of fuzzy languages generated by fuzzy context-free $K$-grammars is denoted by $A_f(K)$. □

**Definition 6.3.** [6] Let $K$ be a family of fuzzy languages and let $U$ be a finite set of nested fuzzy $K$-substitutions over an alphabet $V$. Then $U$ is called *not self-embedding* if for all $u \in U^\star$ and for all $\alpha$ in $V$, the implication $w_1\alpha w_2 \in u(\alpha) \ \Rightarrow\ (w_1 = \lambda \text{ or } w_2 = \lambda)$ holds for all $w_1, w_2 \in V^\star$.

A *fuzzy regular $K$-grammar* $G = (V, \Sigma, U, S)$ is a fuzzy context-free $K$-grammar where $U$ is a non-self-embedding set of nested fuzzy $K$-substitutions over $V$. The family of fuzzy languages generated by fuzzy regular $K$-grammars is denoted by $R_f(K)$. □

**Example 6.4.** When we take $K$ equal to $\mathrm{FIN}_f$, we have $H_f(\mathrm{FIN}_f) = \mathrm{ETOL}_f$ (the family of fuzzy ETOL-languages), $A_f(\mathrm{FIN}_f) = \mathrm{CF}_f$ (the family of fuzzy context-free languages; [20]), and $R_f(\mathrm{FIN}_f) = \mathrm{REG}_f$ (Definition 4.2).

Clearly, we have $\mathrm{CF} \subseteq c(\mathrm{CF}_f)$ where CF is the family of (ordinary, crisp) context-free languages. The converse inclusion does not hold in general; cf. Examples 2.5(2) and 3.1(1), and §8 below. However, when we restrict ourselves to type-10 lattices $\mathcal{L}$, then $c(\mathrm{CF}_f) = \mathrm{CF}$. □

Next we turn to some elementary properties of the families $H_f(K)$, $A_f(K)$ and $R_f(K)$.

**Proposition 6.5.** [5, 6, 8] (1) *Let $K$ be a family of fuzzy languages closed under union with* SYMBOL*-languages. If $K \supseteq$ SYMBOL, then $K \subseteq H_f(K)$, $K \subseteq A_f(K)$, and $K \subseteq R_f(K)$.*
(2) *If the family $K$ is a fuzzy prequasoid, then so are the families $R_f(K)$, $A_f(K)$, and $H_f(K)$.* □

Now we are ready to consider some algebraic structures that are special cases of full AFFL (Definitions 6.6 and 6.9) and to relate them to these generalized fuzzy grammars (Theorems 6.7, 6.8, 6.10 and 6.11).

**Definition 6.6.** A family $K$ of fuzzy languages is closed under *iterated fuzzy substitution* if for each fuzzy language $L$ from $K$ with $L \subseteq V^\star$ for some alphabet $V$, and for each finite set $U$ of fuzzy $K$-substitutions over $V$, the fuzzy language $U^\star(L)$, defined by

$$U^\star(L) \doteq \bigcup\{\tau_p \cdots \tau_1(L) \mid p \geq 0,\ \tau_i \in U\ (1 \leq i \leq p)\},$$

belongs to $K$. In case each fuzzy substitution in $U$ is nested, then $K$ is called closed under *nested iterated fuzzy substitution.*

A *full hyper-AFFL* [5] is a full AFFL closed under iterated fuzzy substitution; a *full super-AFFL* [8] is a full AFFL closed under nested iterated fuzzy substitution. □

For the crisp originals of full substitution-closed AFFL, full super-AFFL and full hyper-AFFL we refer to [14, 13], [16] and [1], respectively. See also [2] for an overview including other algebraic structures weaker than full AFL.

In establishing the following few results Proposition 6.5 played a principal part; cf. [5, 6, 8] for details.

**Theorem 6.7.** [5, 6, 8] *Let $K$ be a family of fuzzy languages. Then*
(1) *$K$ is a full substitution-closed AFFL, if and only if $K$ is a fuzzy prequasoid and $R_f(K) = K$.*
(2) *$K$ is a full super-AFFL, if and only if $K$ is a fuzzy prequasoid and $A_f(K) = K$.*
(3) *$K$ is a full hyper-AFFL, if and only if $K$ is a fuzzy prequasoid and $H_f(K) = K$.* □

Theorems 6.7 and 6.8 play the same rôle as Theorems 5.7 and 5.9(1) do with respect to full AFFL's. The proof of Theorem 6.7(1) in [6] heavily relies on Theorem 5.10 above.

**Theorem 6.8.** [5, 6, 8] *Let $K$ be a family of fuzzy languages. Then*
(1) *$R_f\Pi_f(K)$ is a full substitution-closed AFFL that includes $K$.*
(2) *$A_f\Pi_f(K)$ is a full super-AFFL that includes $K$.*
(3) *$H_f\Pi_f(K)$ is a full hyper-AFFL that includes $K$.* □

**Definition 6.9.** Let $K$ be a family of fuzzy languages . By $\hat{\mathcal{R}}_f(K)$ [$\hat{\mathcal{A}}_f(K)$, and $\hat{\mathcal{H}}_f(K)$] we denote the smallest full substitution-closed AFFL, [full super-AFFL, and full hyper-AFFL, respectively] that includes $K$. □

Theorem 6.7(3) says that $K$ is a full hyper-AFFL if and only if it is a prequasoid —i.e., $\Pi_f(K) = K$— and $H_f(K) = K$. Consequently, the smallest full hyper-AFFL $\hat{\mathcal{H}}_f(K)$, that includes a family $K$, equals $\hat{\mathcal{H}}_f(K) = \bigcup\{w(K) \mid w \in \{\Pi_f, H_f\}^\star\}$ or, equivalently, $\hat{\mathcal{H}}_f(K) = \{\Pi_f, H_f\}^\star(K)$. According Theorem 6.10(3) below, this infinite set of strings over $\{\Pi_f, H_f\}$ can be reduced to the single string $H_f\Pi_f$. Obviously, an analogous remark applies to the other full AFFL-structures in Theorems 6.7 and 6.10.

**Theorem 6.10.** [5, 6, 8] *Let $K$ be a family of fuzzy languages. Then*
(1) $\hat{\mathcal{R}}_f(K) = R_f\Pi_f(K) = R_f\Theta_f\Delta_f\Phi_f(K)$,
(2) $\hat{\mathcal{A}}_f(K) = A_f\Pi_f(K) = A_f\Theta_f\Delta_f\Phi_f(K)$, *and*
(3) $\hat{\mathcal{H}}_f(K) = H_f\Pi_f(K) = H_f\Theta_f\Delta_f\Phi_f(K)$. □

Clearly, the latter equalities in Theorem 6.10 have been obtained using Proposition 5.3.

**Theorem 6.11.** $\mathrm{REG}_f$ [$\mathrm{CF}_f$, $\mathrm{ETOL}_f$, *respectively*] *is the smallest full substitution-closed AFFL* [*full super-AFFL, full hyper-AFFL*]. □

Each full hyper-AFFL is a full super-AFFL, and each full super-AFFL is a full substitution-closed AFFL. But none of the converse implications hold; cf. Theorem 6.11.

## 7 An Infinite Sequence of Algebraic Structures

Definition 6.1 is a special instance of a more general fuzzy $K$-iteration grammar in which the application order of fuzzy $K$-substitutions is prescribed by a crisp control language over $U$; viz.

**Definition 7.1.** [5] Let $\Gamma$ be a family of crisp languages, and let $K$ be a families of fuzzy languages. A *$\Gamma$-controlled fuzzy $K$-iteration grammar* or *fuzzy $(\Gamma, K)$-iteration grammar* is a pair $(G, M)$ that consists of a fuzzy $K$-iteration grammar $G = (V, \Sigma, U, S)$ and a crisp *control language* $M$, i.e., $M$ is a language over $U$, and $M \in \Gamma$. The fuzzy language $L(G, M)$ generated by $(G, M)$ is defined by

$$L(G, M) \mathrel{\hat=} M(S) \cap \Sigma^\star \mathrel{\hat=} \bigcup\{\tau_p(\cdots(\tau_1(S))\cdots) \mid p \geq 0;\ \tau_i \in U,\ \tau_1\cdots\tau_p \in M\} \cap \Sigma^\star.$$

The family of fuzzy languages generated by fuzzy $(\Gamma, K)$-iteration grammars is denoted by both $H_f(\Gamma, K)$ and by $H_{f,\Gamma}(K)$. □

In comparing Definition 7.1 with Definition 6.1 it is useful to mention the fact that regular control does not extend the generating power of fuzzy $K$-iteration grammars.

**Theorem 7.2.** [5] *Let $K$ be a family of fuzzy languages. Then $H_f(\mathrm{REG}, K) = H_f(K)$ holds, provided that $K \supseteq \mathrm{ONE}$.* □

The number of fuzzy $K$-substitutions in a ($\Gamma$-controlled) fuzzy $K$-iteration grammar can be reduced to two in case the parameters $\Gamma$ and $K$ satisfy some very simple conditions [5]. In case of a [non-self-embedding] fuzzy context-free $K$-grammars a reduction to a single, equivalent [non-self-embedding] fuzzy $K$-substitution is possible [8, 6]. Therefore, providing fuzzy regular or fuzzy context-free $K$-grammars with a control language, that prescribes the application order of the [non-self-embedding] fuzzy $K$-substitutions, will probably not results into an interesting topic.

In order to give some elementary properties of $H_{f,\Gamma}(K)$ we need the following concepts.

**Definition 7.3.** A family $\Gamma$ is closed under *left marking* [*right marking*] if for each language $L$ in $\Gamma$ with $L \subseteq \Sigma^\star$ for some $\Sigma$, and for each $c$ not in $\Sigma$, the language $\{c\}L$ [$L\{c\}$, respectively] belongs to $\Gamma$. And $\Gamma$ is closed under *full marking* if $\Gamma$ is closed under both left and right marking. □

**Proposition 7.4.** [5] (1) *Let $\Gamma$ be a family closed under right marking, and let $K$ be a family of fuzzy languages with $K \supseteq$ ONE. Then $\Gamma \subseteq H_{f,\Gamma}(K)$ and $K \subseteq H_{f,\Gamma}(K)$.*
(2) *Let $\Gamma$ be a family closed under (i) left or right marking, (ii) union or concatenation, and (iii) Kleene $\star$. If $K$ is a family of fuzzy languages with $K \supseteq$ SYMBOL, then $H_f(K) \subseteq H_{f,\Gamma}(K)$.*
(3) *Let $\Gamma$ be a family closed under full marking. If $K$ is a fuzzy prequasoid, then so is $H_{f,\Gamma}(K)$.* □

It is useful to compare Proposition 7.4(1) and (3) with the corresponding statements in Proposition 6.5(1) and (2), respectively.

Next we generalize the notion of iterated fuzzy substitution to $\Gamma$-controlled iterated fuzzy substitution where $\Gamma$ is a family of crisp languages.

**Definition 7.5.** Let $\Gamma$ be a family of crisp languages. A family $K$ of fuzzy languages is closed under $\Gamma$-*controlled iterated fuzzy substitution*, if for each fuzzy language $L$ from $K$ with $L \subseteq V^\star$ for some alphabet $V$, for each finite set $U$ of fuzzy $K$-substitutions over $V$, and for each crisp language $M$ over $U$ from the family $\Gamma$, the fuzzy language $M(L)$, defined by

$$M(L) \doteq \bigcup\{\tau_p \cdots \tau_1(L) \mid p \geq 0,\ \tau_i \in U\ (1 \leq i \leq p),\ \tau_1 \cdots \tau_p \in M\},$$

belongs to $K$; cf. Definition 6.6. A *full* $\Gamma$-*hyper-AFFL* is a full AFFL closed under $\Gamma$-controlled iterated fuzzy substitution.

For each family $K$, let $\hat{\mathcal{H}}_{f,\Gamma}(K)$ be the smallest full $\Gamma$-hyper-AFFL that includes $K$. □

**Theorem 7.6.** *Let the crisp family $\Gamma$ be a full substitution-closed AFL. Then a family $K$ of fuzzy languages is a full $\Gamma$-hyper-AFFL if and only if $K$ is a fuzzy prequasoid and $H_{f,\Gamma}(K) = K$.*

*Proof.* Suppose $K$ is a full $\Gamma$-hyper-AFFL. By Theorem 5.7, $K$ is a fuzzy prequasoid; so it remains to show that $H_{f,\Gamma}(K) \subseteq K$ since the converse inclusion follows from Proposition 7.4(1).

Let $(G, M)$ be an arbitrary $\Gamma$-controlled fuzzy $K$-iteration grammar. So $M \in \Gamma$ and $G = (V, \Sigma, U, S)$. Because $K$ is a full $\Gamma$-hyper-AFFL, the fuzzy languages $\{S/_1\}$, $M(\{S/_1\})$ and $M(\{S/_1\}) \cap \Sigma^\star$ all belong to the family $K$. But the latter fuzzy language equals $L(G, M)$. Hence $L(G, M) \in K$ and $H_{f,\Gamma}(K) \subseteq K$.

Conversely, let $K$ be a fuzzy prequasoid that satisfies $H_{f,\Gamma}(K) = K$. First, we show that $K$ is closed under $\Gamma$-controlled iterated fuzzy substitution.

Let $L_0$ be an arbitrary fuzzy language in $K$ with $L_0 \subseteq V^\star$ for some alphabet $V$, and let $U$ be a finite set of fuzzy $K$-substitutions over $V$ and let $M \subseteq U^\star$ be a crisp language from $\Gamma$. Consider the $\Gamma$-controlled fuzzy $K$-iteration grammar $(G, M)$ with $G = (V \cup \{S\}, V, U \cup \{\tau\}, S)$, $S \notin V$, $\tau \notin U$ and $\tau(S) \doteq L_0 \cup \{S/_1\}$ and $\tau(\alpha) \doteq \{\alpha/_1\}$ for each $\alpha$ in $V$.

Then $L(G, M) \doteq M^\star(L_0)$, $L(G, M) \in H_{f,\Gamma}(K) = K$, and hence $M^\star(L_0) \in K$, i.e., $K$ is closed under $\Gamma$-controlled iterated fuzzy substitution.

As $K$ is a fuzzy prequasoid, we have $\mathrm{FIN}_f \subseteq K$ and thus $\mathrm{REG}_f \subseteq \mathrm{ETOL}_f = H_f(\mathrm{FIN}_f) = H_{f,\mathrm{REG}}(\mathrm{FIN}_f) \subseteq H_{f,\Gamma}(K) = K$ by Example 6.4 and Theorem 7.2. But $K \subseteq R_f(K) \subseteq H_f(K) = H_{f,\mathrm{REG}}(K) \subseteq H_{f,\Gamma}(K) = K$ according to Definitions 6.1 and 6.3, Theorem 7.2 and the fact that $\Gamma \supseteq$ REG. So $R_f(K) = K$ and by Theorem 5.7 or 6.7(1) $K$ is a full AFFL. □

Theorem 7.6 is the analogue of Theorem 6.7 as Theorem 7.8(2), (3) and (4) is of Theorems 6.8, 6.10 and 6.11, respectively. However, to establish Theorem 7.8 we need the main result from [5], viz.

**Theorem 7.7.** (1) *Let* $\Gamma_1$ *and* $\Gamma_2$ *be families of crisp languages and let* $\Gamma_2$ *be closed under full marking, union or concatenation, and Kleene* $\star$*. If* $K$ *is a family of fuzzy languages with* $K \supseteq \text{ALPHA}$*, then* $H_f(\Gamma_1, H_f(\Gamma_2, K)) \subseteq H_f(\hat{\text{Sub}}(\Gamma_1, \Gamma_2), K)$*.*
(2) *Let* $\Gamma$ *be a family of crisp languages closed under full marking and under substitution that satisfies* $\Gamma \supseteq \text{REG}$*. If* $K$ *is a family of fuzzy languages with* $K \supseteq \text{ALPHA} \cup \text{ONE}$*, then* $H_f(\Gamma, H_f(\Gamma, K)) = H_f(\Gamma, K)$*.*
(3) *Let* $\Gamma$ *be a family of crisp languages closed under full marking, union, concatenation, and Kleene* $\star$*. If* $K$ *is a family of fuzzy languages with* $K \supseteq \text{ALPHA} \cup \text{ONE}$*, then* $H_f(H_f(\Gamma, K)) = H_f(\Gamma, K)$*.* □

**Theorem 7.8.** *Let the crisp family* $\Gamma$ *be a full substitution-closed AFL, and let* $K$ *be a family of fuzzy languages.*
(1) *Each full* $\Gamma$*-hyper-AFFL is a full hyper-AFFL.*
(2) $H_{f,\Gamma}\Pi_f(K)$ *is a full* $\Gamma$*-hyper-AFFL that includes* $K$*.*
(3) $\hat{\mathcal{H}}_{f,\Gamma}(K) = H_{f,\Gamma}\Pi_f(K) = \Theta_f\Delta_f\Phi_f H_{f,\Gamma}(K)$*.*
(4) $H_{f,\Gamma}(\text{FIN}_f)$ *is the smallest full* $\Gamma$*-hyper-AFFL.*

*Proof.* (1) Clearly, by Theorems 6.7(3) and 7.6 it is sufficient to show that $H_{f,\Gamma}(K) = K$ implies $H_f(K) = K$. Since $\Gamma \supseteq \text{REG}$, we have by Propositions 6.5(1) and 7.4(2): $K \subseteq H_f(K) \subseteq H_{f,\Gamma}(K) = K$. Hence $H_f(K) = K$.

(2) This result follows from Proposition 7.4(3), Theorems 7.6 and 7.7(2).

(3) By the inclusion $K \subseteq \hat{\mathcal{H}}_{f,\Gamma}(K)$ and the monotonicity of both $H_{f,\Gamma}$ and $\Pi_f$, we have $H_{f,\Gamma}\Pi_f(K) \subseteq H_{f,\Gamma}\Pi_f\hat{\mathcal{H}}_{f,\Gamma}(K)$. According to Theorem 7.6, this yields $H_{f,\Gamma}\Pi_f(K) \subseteq \hat{\mathcal{H}}_{f,\Gamma}(K)$. Now Theorem 7.8(2) implies that $H_{f,\Gamma}\Pi_f(K)$ is a full $\Gamma$-hyper-AFFL that includes $K$. Hence we obtain that $\hat{\mathcal{H}}_{f,\Gamma}(K) = H_{f,\Gamma}\Pi_f(K)$.

(4) This statement follows from Theorem 7.8(3) and Lemma 5.2(2) ($\text{FIN}_f$ is the smallest fuzzy prequasoid). □

Comparing Theorem 7.8(3) and the obvious equality $\hat{\mathcal{H}}_{f,\Gamma}(K) = \{H_{f,\Gamma}, \Pi_f\}^\star(K)$, shows that the single string $H_{f,\Gamma}\Pi_f$ suffices rather than the countably infinite set $\{H_{f,\Gamma}, \Pi_f\}^\star$.

The free parameter $\Gamma$ allows us to proceed inductively over the crisp family of control languages, yielding an infinite sequence of signatures (types/classes of algebras).

**Theorem 7.9.** *Let* $K$ *be a* $\mathcal{L}$*-fuzzy prequasoid, and let* $Q_0 = \text{REG}$ *and* $Q_{i+1} = H_f(c(Q_i), K)$ *for each* $i \geq 0$*. Then for each* $i \geq 0$*,* $Q_j$ *is a full* $c(Q_i)$*-hyper-AFFL provided that* $j > i$*.*

*Proof.* A straightforward inductive argument on $i$ —applying Theorem 7.2, Propositions 7.4(1) and 7.4(3), and Theorem 7.7(3)— yields the following facts:

- (7.9-i) $Q_i$ is a full hyper-AFFL for each $i \geq 1$, and
- (7.9-ii) $Q_i \subseteq Q_j$ provided $j \geq i$.

Using these facts we prove by induction on $i$ that $Q_j$ is a full $c(Q_i)$-hyper-AFFL ($0 \leq i < j$).
*Basis:* ($i = 0$). We have to show that for each $j \geq 1$, $Q_j$ is a full $c(Q_0)$-hyper-AFFL. Since $Q_0 = \text{REG}$ and each $Q_j$ is a full $c(\text{REG})$-hyper-AFFL if and only if $Q_j$ is a full hyper-AFFL (Theorem 7.8(1)), the statement follows from (7.9-i).
*Induction hypothesis:* Assume that for each $j > i$, $Q_j$ is a full $c(Q_i)$-hyper-AFFL.
*Induction step:* We have to show that each family $Q_j$ with $j > i+1$ is a full $c(Q_{i+1})$-hyper-AFFL.

Consider an arbitrary $Q_j$ with $j > i + 1$; then $Q_j = H_f(c(Q_{j-1}), K)$. As $j - 1 > i$, the induction hypothesis implies that $Q_{j-1}$ is a full $c(Q_i)$-hyper-AFFL. Now by Theorem 7.8(1) and Proposition 7.4(3), $Q_j$ is a fuzzy prequasoid.

So it remains to show that $H_f(c(Q_{i+1}), Q_j) \subseteq Q_j$, since the converse inclusion follows from Proposition 7.4(2) and (7.9-i).

From the definition of $Q_j$ and Theorem 7.7(1) respectively, we obtain

$$H_f(c(Q_{i+1}), Q_j) = H_f(c(Q_{i+1}), H_f(c(Q_{j-1}), K)) \subseteq H_f(c(\hat{\text{Sub}}(Q_{i+1}, Q_{j-1})), K).$$

We already remarked that the induction hypothesis implies that $Q_{j-1}$ is a full $c(Q_i)$-hyper-AFFL. By Theorem 7.8(1), $Q_{j-1}$ is a full hyper-AFFL and so $Q_{j-1}$ is closed under fuzzy substitution. Consequently, $c(Q_{j-1})$ is closed under (ordinary, crisp) substitution. As $j-1 \geq i+1$, we have $Q_{i+1} \subseteq Q_{j-1}$ by (7.9-ii), and consequently, $c(\hat{\mathrm{Sub}}(Q_{i+1}, Q_{j-1})) \subseteq c(Q_{j-1})$. Hence we have $H_f(c(Q_{i+1}), Q_j) \subseteq H_f(c(\hat{\mathrm{Sub}}(Q_{i+1}, Q_{j-1})), K) \subseteq H_f(c(Q_{j-1}), K) = Q_j$, which completes the induction. □

Note that the statement of Theorem 7.9 still contains two free parameters, viz. (i) the fuzzy prequasoid $K$, and (ii) the type-00 lattice $\mathcal{L}$. To make the latter dependency explicit, we wrote "$\mathcal{L}$-fuzzy" rather than "fuzzy" in Theorem 7.9.

By fixing $K$ and restricting $\mathcal{L}$ we are able to establish the existence of a countably infinite sequence of full AFFL-structures: see Theorem 7.12, the proof of which relies on Theorem 7.9 and the following two results.

In Theorem 7.10 $H(\Gamma, K)$ denotes the family of languages $L(G, M)$ generated by (ordinary, crisp) $(\Gamma, K)$-iteration grammars $(G, M)$, i.e., all substitutions involved in $G$ are crisp $K$-substitutions; cf. [1].

**Theorem 7.10.** [10, 11] *Let $K_0 = \mathrm{REG}$ and $K_{i+1} = H(K_i, \mathrm{FIN})$ for each $i \geq 0$. Then $\{K_i\}_{i\geq 1}$ is an infinite hierarchy of full hyper-AFL's, i.e.,*

- *for each $i \geq 1$, $K_i$ is a full hyper-AFL, and*
- *for each $i \geq 1$, $K_i$ is properly included in $K_{i+1}$: $K_i \subset K_{i+1}$.* □

**Corollary 7.11.** *Let $\mathcal{L}$ be an arbitrary type-10 lattice, and let $\{F_i\}_{i\geq 1}$ be the sequence of families of $\mathcal{L}$-fuzzy languages defined by $F_0 = \mathrm{REG}_f$ and $F_{i+1} = H_f(c(F_i), \mathrm{FIN}_f)$ for each $i \geq 0$. Then $\{F_i\}_{i\geq 1}$ is an infinite hierarchy of full hyper-AFFL's, i.e.,*

- *for each $i \geq 1$, $F_i$ is a full hyper-AFFL, and*
- *for each $i \geq 1$, $F_i$ is properly included in $F_{i+1}$: $F_i \subset F_{i+1}$.*

*Proof.* First, we show by induction that for each $i \geq 0$, $c(F_i) = K_i$ where $\{K_i\}_{i\geq 1}$ is as in Theorem 7.10.
*Basis:* $(i = 0)$. $c(F_0) = K_0$ follows from Lemma 4.3(2).
*Induction hypothesis:* Assume $c(F_m) = K_m$.
*Induction step:* In order to prove $c(F_{m+1}) = K_{m+1}$, we first remark that $K_{m+1} \subseteq c(F_{m+1})$. Hence it remains to show that $c(F_{m+1}) \subseteq K_{m+1}$.

So let $L_0$ be an arbitrary element of $c(F_{m+1})$, i.e., $L_0 = c(L_f)$ for some $L_f \in F_{m+1}$. Thus there exists a fuzzy $(c(F_m), \mathrm{FIN}_f)$-iteration grammar $(G, M)$ with $G = (V, \Sigma, U, S)$ such that $L(G, M) \doteq L_f$. By the induction hypothesis, $(G, M)$ is a fuzzy $(K_m, \mathrm{FIN}_f)$-iteration grammar. Next we will construct an equivalent $(K_m, \mathrm{FIN})$-iteration grammar $(G', M)$ by $G' = (V, \Sigma, U', S)$, $U' = \{\tau' \mid \tau \in U\}$ and for each $\alpha$ in $V$ and each $\tau$ in $U$, we define $\tau'(\alpha) = c(\tau(\alpha))$.

Since $\mathcal{L}$ is linearly ordered, the max-operation applies and as $a \leq \max\{a, b\} = \max\{b, a\}$ for all $a, b \in \mathcal{L}$, we have that the crisp language $L(G', M)$ equals $c(L_f)$. Consequently, $L_0 \in H(K_m, \mathrm{FIN})$ or, equivalently, $L_0 \in K_{m+1}$, which completes the induction.

Now the statement follows from Theorems 7.7(3) and 7.10. □

Finally, we are ready for the main result.

**Theorem 7.12.** *Let $\mathcal{L}$ be an arbitrary type-10 lattice and consider the following families of $\mathcal{L}$-fuzzy languages: $F_0 = \mathrm{REG}_f$ and $F_{m+1} = H_f(c(F_m), \mathrm{FIN}_f)$ for $m \geq 0$. Let $\mathcal{C}_m$ be the class of all full $c(F_m)$-hyper-AFFL's. Then for each $m \geq 1$,*

(1) *the class $\mathcal{C}_m$ is a proper superset of $\mathcal{C}_{m+1}$: $\mathcal{C}_m \supset \mathcal{C}_{m+1}$,*

(2) *the class $\mathcal{C}_m$ contains an infinite hierarchy of full $c(F_m)$-AFFL's, i.e., a countably infinite chain of families of fuzzy language $F_{m,n}$ $(n \geq 1)$ such that*

(i) *each $F_{m,n}$ is a full $c(F_m)$-AFFL, and*

(ii) *for each $n \geq 1$, $F_{m,n}$ is properly included in the next one: $F_{m,n} \subset F_{m,n+1}$.*

*Proof.* (1) The statement follows from Corollary 7.11 and Theorems 7.9 (with $K = \mathrm{FIN}_f$) and 7.8(4).

(2) For fixed $m$ ($m \geq 1$), we define $\{F_{m,n}\}_{n\geq 1}$ by $F_{m,n} = F_{m+n}$ for each $n \geq 1$. By Corollary 7.11 and Theorem 7.9 this is an infinite hierarchy of full $c(F_m)$-hyper-AFFL's. □

## 8 Concluding Remarks

In §§4–5 we showed that some basic results for crisp language families (like prequasoid and full AFL) can be generalized to their fuzzy analogues(fuzzy prequasoid and full AFFL, respectively), provided the operations on fuzzy languages have been defined appropriately (§3). Then in §6 we surveyed some results on full substitution-closed AFFL's, full super-AFFL's and full hyper-AFFL's from [5, 6, 8]. In §7 we extended this finite chain of algebraic structures to a countably infinite sequence of full AFFL-structures, each of which possesses properties (Theorem 7.8) similar to those of the members of the initial, finite sequence (Theorems 6.8, 6.10 and 6.11). And each new class of full AFFL-structures in this sequence is nontrivial in the sense that it contains a countably infinite hierarchy (Theorem 7.12).

Note that this latter conclusion has only been proved for fuzzy languages of which the codomain $\mathcal{L}$ of the membership function is linearly ordered (a type-10 lattice; §2). Whether this result can be generalized to arbitrary type-00 lattices is an open question, but its answer is probably negative. The approach in §7, i.e., deriving Corollary 7.11 from Theorem 7.10, will not work as we will show. More precisely: if $\mathcal{L}$ is a type-01 lattice, $K_f$ is a family of $\mathcal{L}$-fuzzy languages and $K$ is its crisp counterpart, then —apart from a few trivial exceptions (viz. $K_f$ equals $\mathrm{FIN}_f$ or $\mathrm{ALPHA}_f$; cf. Lemma 4.3)— in general $K$ seems to be a proper subset of $c(K_f)$: $K \subset c(K_f)$.

Our evidence is based on (i) the inherent ambiguity of some context-free languages (like, e.g., $\{a^m b^m a^n \mid m, n \geq 1\} \cup \{a^m b^n a^n \mid m, n \geq 1\}$ or $\{a^n b^n c^m d^m \mid m, n \geq 1\} \cup \{a^n b^m c^m d^n \mid m, n \geq 1\}$; cf. Example 2.5(2-3)), and (ii) the structure of the simplest type-00 lattice that is not linearly ordered (cf. Example 2.3(2) in which we have $\xi \vee \eta = 1$).

Consider the type-01 lattice $\mathcal{L}$ of Example 2.3(2) and the $\mathcal{L}$-fuzzy context-free $\mathrm{FIN}_f$-grammars $G_1 = (V, \Sigma, \{\tau_1\}, S)$ and $G_2 = (V, \Sigma, \{\tau_2\}, S)$ with $V = \{S, A\}$, $\Sigma = \{a, b\}$ and

$$\tau_1(\alpha) \mathrel{\hat=} \tau_2(\alpha) \mathrel{\hat=} \{\alpha/_1\}, \qquad \text{for each } \alpha \text{ in } \Sigma,$$

$$\tau_1(S) \mathrel{\hat=} \{S/_1, Sa/_1, Aa/_\xi\},$$

$$\tau_2(S) \mathrel{\hat=} \{S/_1, aS/_1, aA/_\eta\},$$

$$\tau_1(A) \mathrel{\hat=} \tau_2(A) \mathrel{\hat=} \{A/_1, aAb/_1, ab/_1\}.$$

Then $L(G_1) \mathrel{\hat=} L_1$, $L(G_2) \mathrel{\hat=} L_2$ and $L(G_1) \cup L(G_2) \mathrel{\hat=} L_0$; for $L_0$, $L_1$ and $L_2$ we refer to Example 2.5. Note that both $G_1$ and $G_2$ are linear context-free and that the support of $L(G_1) \cup L(G_2)$ is an inherently ambiguous, linear context-free language. Since $L_0$ is not (linear) context-free, we have $\mathrm{CF} \subset c(\mathrm{CF}_f)$ and $\mathrm{LCF} \subset c(\mathrm{LCF}_f)$, where LCF [$\mathrm{LCF}_f$, respectively] is the family of [fuzzy] linear context-free languages. Proper inclusions of this kind prevent us to apply an argument as in the proof of Corollary 7.11 in case $\mathcal{L}$ is a type-00 lattice that is not linearly ordered.

Note that the question whether $c(\mathrm{REG}_f) = \mathrm{REG}$ in case $\mathcal{L}$ is a type-00 lattice, is still open; cf. Lemma 4.3(2).

A "crisp version" of Theorem 7.12 has been established in [7]: in that case the smallest elements (Theorem 7.8(4)) are subfamilies of the family of context-sensitive languages CS; see [7] for details.

Another topic for further investigation is the limit family of fuzzy languages $F_\omega$, defined by $F_\omega = \bigcup_{n\geq 0} F_n$ (cf. Theorem 7.12). As its crisp counterpart $K_\omega = \bigcup_{n\geq 0} K_n$ (Theorem 7.10), it possesses closure properties, even stronger than those of full $c(F_n)$-hyper-AFFL, viz. $F_\omega = H_f(c(F_\omega), F_\omega)$. With respect to $K_\omega$ we know that $K_\omega \subset \mathrm{CS}$ [7, 10, 11], but where the position of $F_\omega$ is in the extended "fuzzified" Chomsky hierarchy, is still open.

## References


1. P.R.J. Asveld, Controlled iteration grammars and full hyper-AFL's, *Inform. Contr.* **34** (1977) 248–269.
2. P.R.J. Asveld, An algebraic approach to incomparable families of formal languages, pp. 455–475 in: G. Rozenberg & A. Salomaa (eds.), *Lindenmayer Systems — Impacts on Theoretical Computer Science, Computer Graphics, and Developmental Biology* (1992), Springer, Berlin, etc.

3. P.R.J. Asveld, Towards robustness in parsing — Fuzzifying context-free language recognition, pp. 443–453 in: J. Dassow, G. Rozenberg & A. Salomaa (eds.), *Developments in Language Theory II — At the Crossroads of Mathematics, Computer Science and Biology*, World Scientific (1996), Singapore.
4. P.R.J. Asveld, A fuzzy approach to erroneous inputs in context-free language recognition, pp. 14–25 in: *Proc. 4th Internat. Workshop on Parsing Technologies IWPT'95* (1995), Prague / Karlovy Vary, Czech Republic.
5. P.R.J. Asveld, Controlled fuzzy parallel rewriting, pp. 49–70 in: Gh. Păun & A. Salomaa (eds.), *New Trends in Formal Languages — Control, Cooperation, and Combinatorics* (1997), Lect. Notes. in Comp. Sci. **1218**, Springer, Berlin, etc.
6. P.R.J. Asveld, The non-self-embedding property for generalized fuzzy context-free grammars, *Publ. Math. Debrecen* **54** *Suppl.* (1999) 553–573.
7. P.R.J. Asveld, An infinite sequence of full AFL-structures, each of which possesses an infinite hierarchy (1999), CTIT Tech. Rep. No. 99-06, Enschede, the Netherlands.
8. P.R.J. Asveld, Fuzzy context-free languages — Part I: Generalized fuzzy context-free grammars (2000), CTIT Tech. Rep. No. 00-03, Enschede, the Netherlands.
9. P.R.J. Asveld, Fuzzy context-free languages — Part II: Recognition and parsing algorithms (2000), CTIT Techn. Rep. No. 00-??, Enschede, the Netherlands (in preparation).
10. J. Engelfriet: Three hierarchies of transducers, *Math. Syst. Th.* **15** (1982) 95-125.
11. J. Engelfriet: Hierarchies of hyper-AFL's, *J. Comp. Syst. Sci.* **30** (1985) 86–115.
12. G. Gerla, Fuzzy grammars and recursively enumerable fuzzy languages, *Inform. Sci.* **60** (1992) 137–143.
13. S. Ginsburg, *Algebraic and Automata-Theoretic Properties of Formal Languages* (1975). North-Holland, Amsterdam.
14. S. Ginsburg & E.H. Spanier, Substitution in families of languages, *Inform. Sci.* **2** (1970) 83–110.
15. J.A. Goguen, *L*-fuzzy sets, *J. Math. Analysis Appl.* **18** (1967) 145–174.
16. S.A. Greibach, Full AFL's and nested iterated substitution. *Inform. Contr.* **16** (1970) 7–35.
17. M.A. Harrison, *Introduction to Formal Language Theory* (1978). Addison-Wesley, Reading, Mass.
18. J.E. Hopcroft & J.D. Ullman, *Introduction to Automata Theory, Languages, and Computation* (1979). Addison-Wesley, Reading, Mass.
19. H.H. Kim, M. Mizumoto, J. Toyoda & K. Tanaka, *L*-fuzzy grammars, *Inform. Sci.* **8** (1975) 123–140.
20. E.T. Lee & L.A. Zadeh, Note on fuzzy languages, *Inform. Sci.* **1** (1969) 421–434.
21. A. Salomaa, *Formal Languages* (1973), Academic Press, New York.
22. T.A. Sudkamp, *Languages and Machines — An Introduction to the Theory of Computer Science*, Second edition (1997), Addison-Wesley, Reading, Mass.