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Necessary conditions for the compensation approach for a random walk in the quarter-plane

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Abstract
We consider the invariant measure of homogeneous random walks in the quarter-plane. In particular, we consider measures that can be expressed as a countably infinite sum of geometric terms which individually satisfy the interior balance equations. We demonstrate that the compensation approach is the only method that may lead to such a type of invariant measure. In particular, we show that if a countably infinite sum of geometric terms is an invariant measure, then the geometric terms in an invariant measure must be the union of at most six pairwise-coupled sets of countably infinite cardinality each. We further show that for such invariant measure to be a countably infinite sum of geometric terms, the random walk cannot have transitions to the north, northeast or east. Finally, we show that for a countably infinite weighted sum of geometric terms to be an invariant measure at least one of the weights must be negative.

Keywords Compensation approach · Random walk · Quarter-plane · Invariant measure · Geometric term · Algebraic curve · Pairwise-coupled

Mathematics Subject Classification 60J10 · 60G50

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1 Introduction

Random walks for which the invariant measure is a geometric product form are often used to model practical systems. The most prominent is the Jackson network of single-server queues. The advantage of such models is that their performance can be analyzed using tractable closed-form expressions. However, the class of random walks with a product-form invariant measure is rather limited. Therefore, it is of interest to find larger classes of tractable measures that can be the invariant measure for random walks in the quarter-plane. One such a class is formed by measures that can be expressed as a sum of geometric terms. In [9] we have characterized random walks with invariant measure that is a sum of finitely many geometric terms. The motivation for the work in [9] is that it provides a means of establishing error bounds on the performance of random walks for which the invariant measure is not known explicitly; see, for instance, [16,24]. Indeed, in [10] we have shown that using sums of geometric terms instead of a single term product-form invariant measure may yield better error bounds.

One of the main results in [9] is that a finite sum of geometric terms can only be an invariant measure if the terms have a structure that we refer to as pairwise-coupled. In this paper we consider the case of countably infinitely many terms. We start from the observation that the compensation approach [1–4] enables us to express the invariant measure as a sum of countably infinitely many geometric terms that are pairwise-coupled. In particular, the compensation approach constructs this invariant measure by consecutively adding terms to the invariant measure, with each new term being coupled to the previous term. It is shown in [3] that a sufficient condition under which the compensation approach can be applied is that the random walk does not have any transitions to the north, northeast or east. This implies that the invariant measure for any random walk without these transitions can be expressed as a sum of countably infinitely many pairwise-coupled geometric terms. The question addressed in this paper is whether the pairwise-coupled structure and the absence of transitions to the north, northeast or east are also necessary. We will answer this question affirmatively by providing necessary conditions for the invariant measure of a random walks to be a sum of countably infinitely many geometric terms.

We consider discrete-time nearest-neighbor random walks in the quarter-plane, i.e., on $\{0, 1, 2, \dots\}^2$, that are homogeneous in the sense that the transition probabilities in the interior and along the axes are translation invariant. For geometric terms that individually satisfy the interior balance equations, we consider finite measures $m(i, j)$ that can be expressed as a sum of these geometric terms:

$$m(i, j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho, \sigma)\rho^i \sigma^j, \tag{1}$$

where the set Γ containing the parameters ρ and σ of the individual geometric terms has countably infinite cardinality. We will show that if such a measure m is the invariant measure, it can be obtained via the compensation approach. More precisely, this paper gives the following result: If the invariant measure $m(i, j)$ is of the form (1), then the following conditions *must* hold:

1. The set Γ is the union of at most six sets, each of which have infinite cardinality and have a structure that we refer to as pairwise-coupled. (Theorem 19)
2. In the interior of the state space, the random walk has no transitions to the north, northeast or east. (Theorem 20)
3. At least one of the coefficients $\alpha(\rho, \sigma)$ in (1) is negative (Theorem 21).

Observe that the first and the third property deal with the structure of the invariant measure. The second property deals with the structure of the random walk itself. Therefore, our results demonstrate that the structure of the random walk determines if its invariant measure can be expressed as a countably infinite sum of geometric terms. If this is possible (only if there are no transitions to north, northeast or east), the terms in this sum will necessarily be pairwise-coupled and involve at least one negative coefficient. The compensation approach [3] then provides a means of constructing this invariant measure.

We want to emphasize that contrary to much other work, for instance [11,13,22,23], our interest is not in finding the invariant measure for specific random walks. Instead, our interest is in characterizing the fundamental properties of random walks, sets Γ and coefficients α that allow an invariant measure to be expressed in the form (1). A related study for the reflected Brownian motion in a wedge is [12], where it is shown that for the invariant measure of this process to be a linear combination of finitely many exponential measures, there must be an odd number of terms that have a pairwise-coupled structure. The methods developed in [12] for the continuous state space Brownian motion do not carry over to the discrete state space random walk.

There exists work that aims at expressing the invariant measure of a random walk in the quarter-plane as a sum of geometric terms. For the symmetric shortest queue problem, Kingman [19], as well as Flatto and Mckean [14], have shown that the equilibrium probabilities can be written as an infinite sum of geometric terms. They obtain these results by analyzing the generating function of the invariant measure. However, this approach does not seem to provide a tractable means of finding explicit expressions for the geometric terms that are involved. A similar approach has been used by Hofri [17] for a multiprogramming queueing problem with two queues. There is also a numerical oriented method for multi-dimensional exponential queueing systems in the form of a sum of geometric terms method; see Hooghiemstra et al. [18]. Blanc [5] reports numerically satisfactory results for the shortest queue problem. However, this method lacks a theoretical foundation and no error bounds are provided.

This paper is structured as follows: In Sect. 2 we present the model. The geometric properties of the algebraic curve arising from the balance equations in the interior of the state space are studied in Sect. 3. A closely related curve arises as the kernel of the boundary value problems studied in [11,13] and related work. Some of its basic properties were derived in [13]. An important part of the present paper consists of studying this algebraic curve in more detail. Section 3 presents new results on the geometric properties of this curve. The necessary conditions for the invariant measure of a random walk to be a countably infinite sum of geometric terms are derived in Sect. 4. Section 5 gives concluding remarks.

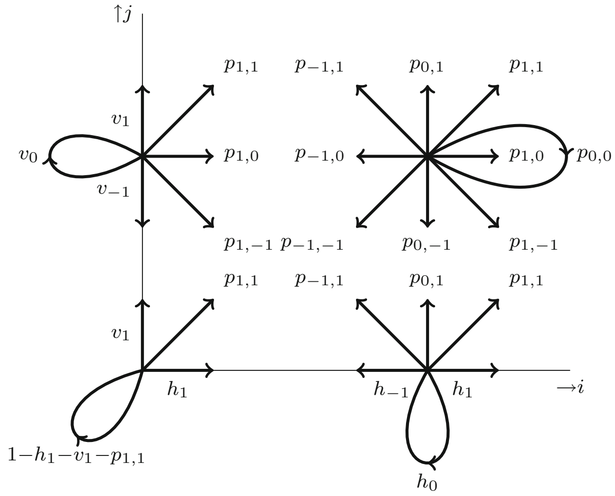

Fig. 1 Random walk R in the quarter-plane

2 Model

Consider a two-dimensional random walk R on the pairs $S = \{(i, j)|i, j \in \mathbb{N}_0\}$ of non-negative integers. We refer to $\{(i, j)|i > 0, j > 0\}$, $\{(i, j)|i > 0, j = 0\}$, $\{(i, j)|i = 0, j > 0\}$ and $(0, 0)$ as the interior, the horizontal axis, the vertical axis and the origin of the state space, respectively. Let $\mathbb{R}^2_+ = \{(x, y)|x \geq 0, y \geq 0\}$, $R_B = \{(x, y) \in \mathbb{R}^2_+|xy = 0\}$, $U = (0, 1)^2$ and $\bar{U} = [0, 1)^2$. Let $p_{s,t}(i, j)$ denote the transition probability from state (i, j) to state $(i + s, j + t)$. Transitions are restricted to the neighboring states (horizontally, vertically and diagonally), i.e., $p_{s,t}(k, l) = 0$ if $|s| > 1$ or $|t| > 1$. The process is homogeneous in the sense that, for each pair (i, j), (k, l) in the interior (respectively, on the horizontal axis and on the vertical axis) of the state space

$$p_{s,t}(i, j) = p_{s,t}(k, l) \quad \text{and} \quad p_{s,t}(i - s, j - t) = p_{s,t}(k - s, l - t), \tag{2}$$

for all $-1 \leq s \leq 1$ and $-1 \leq t \leq 1$. Note that the first equality of (2) implies that the transition probabilities for each part of the state space are translation invariant. The second equality ensures that also the transition probabilities entering the same part of the state space are translation invariant. For $i > 0$, $j > 0$, let $p_{s,t} = p_{s,t}(i, j)$, $h_s = p_{s,0}(i, 0)$ and $v_t = p_{0,t}(0, j)$, so that $p_{1,0}(0, 0) = h_1$ and $p_{0,1}(0, 0) = v_1$. The model and notation are illustrated in Fig. 1.

We assume that the random walk has invariant measure m, i.e., for $i, j > 0$,

$$m(i, j) = \sum_{s=-1}^{1} \sum_{t=-1}^{1} m(i - s, j - t) p_{s,t}, \tag{3}$$

$$m(i,0) = \sum_{s=-1}^{1} m(i-s,1)p_{s,-1} + \sum_{s=-1}^{1} m(i-s,0)p_{s,0}, \tag{4}$$

$$m(0,j) = \sum_{t=-1}^{1} m(1,j-t)p_{-1,t} + \sum_{t=-1}^{1} m(0,j-t)p_{0,t}. \tag{5}$$

We will refer to the above equations as the balance equations in the interior, the horizontal axis and the vertical axis of the state space, respectively. The balance equation at the origin is implied by the balance equations for the other states.

We are interested in measures that are linear combinations of geometric measures. We first classify the geometric measures.

Definition 1 (*Geometric measure*) The measure $m(i,j) = \rho^i\sigma^j$ is called a geometric measure. It is called horizontally degenerate if $\sigma = 0$, vertically degenerate if $\rho = 0$ and non-degenerate if $\rho > 0$ and $\sigma > 0$. We define $0^0 \equiv 1$.

A horizontally (vertically) degenerate measure is supported on the horizontal (vertical) axis. Degenerate measures have been studied in [9, Theorem 2] for the case of a finite number of terms. It can be easily verified that these results also hold for countably infinitely many geometric terms. Hence, we focus on the case when $\rho > 0$ and $\sigma > 0$ from now on.

We represent a geometric measure $\rho^i\sigma^j$ by its pair of parameters (ρ,σ) in $(0,\infty)^2$. As a consequence, a set $\Gamma \subset (0,\infty)^2$ characterizes a set of geometric measures. To identify the geometric measures that individually satisfy the balance equations in the interior of the state space (3), we introduce the polynomial

$$Q(x,y) = xy\left(\sum_{s=-1}^{1}\sum_{t=-1}^{1} x^{-s}y^{-t}p_{s,t} - 1\right). \tag{6}$$

Then $Q(\rho,\sigma) = 0$, $\rho,\sigma > 0$, implies that $m(i,j) = \rho^i\sigma^j$, $(i,j) \in S$, satisfies (3). Similarly, we introduce the polynomials $H(x,y)$, $V(x,y)$ to identify the geometric measures that satisfy (4) and (5), respectively:

$$H(x,y) = x\left(1 - \sum_{s=-1}^{1} x^{-s}yp_{s,-1} - \sum_{s=-1}^{1} x^{-s}h_s\right),$$

$$V(x,y) = y\left(1 - \sum_{t=-1}^{1} xy^{-t}p_{-1,t} - \sum_{t=-1}^{1} y^{-t}v_t\right).$$

Let Q, H, V denote the set of real (x,y) satisfying $Q(x,y) = 0$, $H(x,y) = 0$, $V(x,y) = 0$, i.e.,

$$Q = \left\{(x,y) \in \mathbb{R}^2 \mid Q(x,y) = 0\right\}, \tag{7}$$

$$H = \left\{(x,y) \in \mathbb{R}^2 \mid H(x,y) = 0\right\}, \tag{8}$$

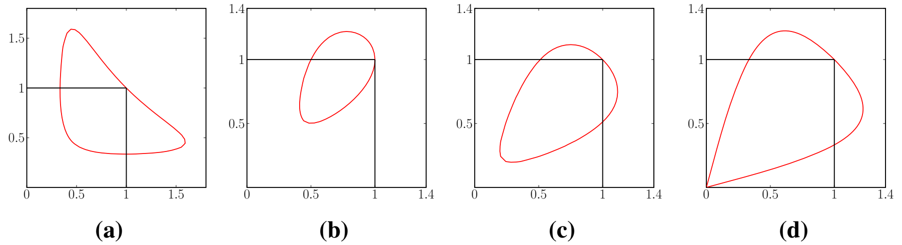

Fig. 2 Examples of Q_0^+, the origin is also part of Q_0^+ in **a**, **b** and **d**. **a** $p_{1,0} = p_{0,1} = \frac{1}{5}$, $p_{-1,-1} = \frac{3}{5}$. **b** $p_{1,0} = \frac{1}{5}$, $p_{0,-1} = p_{-1,1} = \frac{2}{5}$. **c** $p_{1,1} = \frac{1}{62}$, $p_{-1,1} = p_{1,-1} = \frac{10}{31}$, $p_{-1,-1} = \frac{21}{62}$. **d** $p_{-1,1} = p_{1,-1} = \frac{1}{4}$, $p_{-1,-1} = \frac{1}{2}$

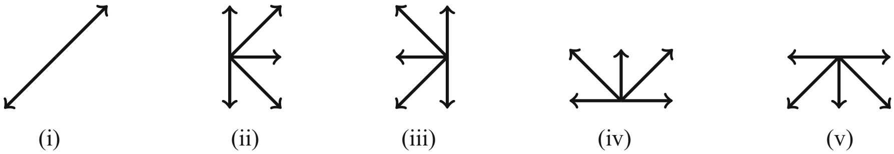

Fig. 3 Singular random walks: nonzero transitions in the interior of the state space

$$V = \Big\{(x, y) \in \mathbb{R}^2 \mid V(x, y) = 0\Big\}. \tag{9}$$

We are mainly interested in the properties of the algebraic curve Q in $\mathbb{R}_+^2$, which we denote by Q_0^+. Note that U contains those values of (ρ, σ) that yield a finite measure. Several examples of Q_0^+ are displayed in Fig. 2. The origin is on Q_0^+ iff $p_{1,1} = 0$ because $Q(0, 0) = p_{1,1}$.

Definition 2 (*Singular random walk* [13]) The random walk R is called singular if the associated polynomial $Q(x, y)$ is either reducible or of degree 1 in at least one of the variables.

It was shown in [13] that a random walk is singular if and only if it has a transition structure that corresponds to one of the cases depicted in Fig. 3. Singular random walks were analyzed in [9] for the case that $|\Gamma| < \infty$. It is readily verified that the proofs in [9] extend to the case that Γ has countably infinite cardinality. In the current paper we, therefore, restrict our attention to non-singular random walks.

Let

$$M_x = \sum_{t=-1}^{1} p_{1,t} - \sum_{t=-1}^{1} p_{-1,t}$$

and

$$M_y = \sum_{s=-1}^{1} p_{s,1} - \sum_{s=-1}^{1} p_{s,-1}$$

denote the drift in the horizontal and vertical direction, respectively. In Sect. 3 we will use the condition that in a positive recurrent random walk at least one of the conditions $M_x < 0$ or $M_y < 0$ holds [13, Theorem 1.2.1]. We restrict our attention to random walks that are non-singular, irreducible, aperiodic and positive recurrent.

Definition 3 (*Induced measure*) The measure m is called induced by a countable set $\Gamma \subset \mathbb{R}_+^2$ if

$$m(i, j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho, \sigma)\rho^i \sigma^j,$$

with $\alpha(\rho, \sigma) \in \mathbb{R}\backslash\{0\}$ for all $(\rho, \sigma) \in \Gamma$.

We make the following assumptions on the sets Γ and coefficients α that we will consider.

Assumptions 4 1. Measures are positive, finite and absolute convergent in the sense that

$$\sum_{(\rho,\sigma)\in\Gamma} |\alpha(\rho, \sigma)| \frac{1}{1-\rho} \frac{1}{1-\sigma} < \infty. \tag{10}$$

2. The geometric measures that correspond to the elements of Γ are finite measures that individually satisfy the balance equations in the interior of the state space, i.e., $\Gamma \subset U \cap Q_0^+$.

Since the compensation approach deals with geometric measures that individually satisfy the balance equation in the interior, it is natural to also impose this condition here. Furthermore, it is not clear how this condition can be relaxed in a meaningful way. The reason is that countably infinite sums of geometric measures may, in general, satisfy an algebraic equation; see [7]. In particular, such sums may satisfy the balance equations at the boundaries of the state space without satisfying balance individually. This fact is used in the compensation approach and in Lemma 18 of the current paper.

The assumption that $\Gamma \subset U \cap Q_0^+$ implies that all geometric terms individually satisfy the balance equation in the interior and that they are non-degenerate, i.e., neither ρ nor σ from (ρ, σ) is zero. Moreover, this assumption guarantees that each (ρ, σ) from the set Γ is on the algebraic curve Q_0^+. Finally, we are interested in finite measures (corresponding to positive recurrence) in which the sum does not depend on the ordering of the terms. Therefore, we assume absolute convergence.

3 Algebraic curve Q in $\mathbb{R}^2$

This section considers the algebraic curve Q in $\mathbb{R}^2$. The algebraic curve that arises from $xy(\sum_{s=-1}^{1} \sum_{t=-1}^{1} x^s y^t \tilde{p}_{s,t} - 1) = 0$, with $\tilde{p}_{i,j} = p_{-i,-j}$, is extensively studied in [13]. For convenience of notation, we consider the algebraic curve defined by (6). Observe that $\tilde{p}_{i,j}$ might induce a non-ergodic random walk. This will not introduce complications in our analysis, since the results from [13] that will be used in this paper do not require the random walk to be ergodic. We will present some of the results from [13] that will be used in the sequel as well as a number of new results. The results from [13] are mostly algebraic in nature. The new results that we present deal with the geometry of Q.

The algebraic results that we use from [13] are expressed in terms of the branch points of the multi-valued algebraic functions $X(y)$ and $Y(x)$ which are defined through

$$Q(X(y), y) = Q(x, Y(x)) = 0.$$

These functions are most naturally treated as complex valued functions for complex variables $x, y \in \mathbb{C}$. In particular, the embedding in $\mathbb{C}$ allows us to define the branch points of $X(y)$ and $Y(x)$.

As a first step toward analysis of the branch points of $X(y)$ and $Y(x)$, observe that reordering the terms in $Q(x, y) = 0$ gives

$$\left(\sum_{s=-1}^{1} y^{-s+1} p_{-1,s}\right) x^2 + \left(\sum_{s=-1}^{1} y^{-s+1} p_{0,s} - y\right) x + \left(\sum_{s=-1}^{1} y^{-s+1} p_{1,s}\right) = 0. \quad (11)$$

Therefore, the branch points of $X(y)$ are the roots of $\Delta_x(y) = 0$, where

$$\Delta_x(y) = \left(\sum_{s=-1}^{1} y^{-s+1} p_{0,s} - y\right)^2 - 4\left(\sum_{s=-1}^{1} y^{-s+1} p_{-1,s}\right)\left(\sum_{s=-1}^{1} y^{-s+1} p_{1,s}\right). \quad (12)$$

In similar fashion, by rewriting $Q(x, y) = 0$ into

$$\left(\sum_{t=-1}^{1} x^{-t+1} p_{t,-1}\right) y^2 + \left(\sum_{t=-1}^{1} x^{-t+1} p_{t,0} - x\right) y + \left(\sum_{t=-1}^{1} x^{-t+1} p_{t,1}\right) = 0, \quad (13)$$

it follows that the branch points of $Y(x)$ are the roots of $\Delta_y(x) = 0$, where

$$\Delta_y(x) = \left(\sum_{t=-1}^{1} x^{-t+1} p_{t,0} - x\right)^2 - 4\left(\sum_{t=-1}^{1} x^{-t+1} p_{t,-1}\right)\left(\sum_{t=-1}^{1} x^{-t+1} p_{t,1}\right). \quad (14)$$

The following two lemmas, adopted from [13], fully characterize the location of the branch points of $Y(x)$ and $X(y)$ in terms of the transition probabilities of the random walk. These results enable us to connect the geometry of Q with the interior transition probabilities. Lemma 5 readily follows from Lemmas 2.3.8–2.3.10 of [13], taking into account that we consider only ergodic random walks, whereas [13] also allows for non-ergodic random walks. Lemma 6 considers multiplicity of the branch points.

Lemma 5 ([13, Lemmas 2.3.8–2.3.10]) *For all non-singular random walks such that $M_y \neq 0$, $Y(x)$ has four real branch points. Moreover, $Y(x)$ has two branch points x_1 and x_2 (respectively, x_3 and x_4) inside (respectively, outside) the unit circle.*

For the pair (x_3, x_4), the following classification holds:

1. *if $p_{-1,0} > 2\sqrt{p_{-1,-1}p_{-1,1}}$, then x_3 and x_4 are positive;*

2. *if* $p_{-1,0} = 2\sqrt{p_{-1,-1}p_{-1,1}}$, *then one point is infinite and the other is positive, possibly infinite;*
3. *if* $p_{-1,0} < 2\sqrt{p_{-1,-1}p_{-1,1}}$, *then one point is positive and the other is negative.*

Similarly, for the pair (x_1, x_2)*,*

1. *if* $p_{1,0} > 2\sqrt{p_{1,-1}p_{1,1}}$, *then* x_1 *and* x_2 *are positive;*
2. *if* $p_{1,0} = 2\sqrt{p_{1,-1}p_{1,1}}$, *then one point is* 0 *and the second is non-negative;*
3. *if* $p_{1,0} < 2\sqrt{p_{1,-1}p_{1,1}}$, *then one point is positive and the other is negative.*

For all non-singular random walks for which $M_y = 0$*, one of the branch points of* $Y(x)$ *is equal to* 1*. In addition,*

1. *if* $M_x < 0$*, then two other branch points have a modulus larger than* 1 *and the remaining one has a modulus less than* 1*;*
2. *if* $M_x > 0$*, then two branch points are less than* 1 *and the modulus of the remaining one is larger than* 1.

Furthermore, the positivity conditions are the same as the case when $M_y \neq 0$*. This lemma holds also for* $X(y)$*, up to a proper symmetric change of the parameters.*

Lemma 6 ([13, Lemma 2.3.10]) *The branch points of* $X(y)$ *and* $Y(x)$ *with multiplicity* 2 *occur only at* 0*,* 1 *and* ∞*.*

The remainder of this section provides new results on the geometry of Q. First, we consider the intersection of Q and R_B, where R_B is the boundary of the first quadrant.

Lemma 7 *Consider the random walk* R*. If* $(x, y) \in Q \cap \mathbb{R}^2_+$*, then either* $x > 0$ *and* $y > 0$ *or* $x = y = 0$*,* i.e.*,* Q *cannot cross* R_B *except at the origin.*

Proof If (x, y) is the intersection of Q and $x = 0$, then y must be the root of the following quadratic equation:

$$Q(0, y) = p_{1,-1}y^2 + p_{1,0}y + p_{1,1} = 0. \tag{15}$$

We now show that the roots of (15) are non-positive by considering all possible choices of $p_{1,-1}$, $p_{1,0}$ and $p_{1,1}$. If $p_{1,-1} \neq 0$, then (15) has either no root or two non-positive roots by investigating the relations of the roots using Vieta's formulas. If $p_{1,-1} = 0$ and $p_{1,0} \neq 0$, then (15) has one non-positive root. If $p_{1,-1} = p_{1,0} = 0$ and $p_{1,1} \neq 0$, then (15) has no root. The random walk with $p_{1,-1} = p_{1,0} = p_{1,1} = 0$ is excluded because of the assumption of non-singularity. In similar fashion, it follows that Q can only intersect $y = 0$ when $x \leq 0$. Therefore, the only possible intersection of Q and R_B is the origin. □

Now we characterize the number of connected components in the first quadrant.

Lemma 8 *Consider random walk* R*. The algebraic curve* Q_0^+ *consists of either (i) exactly one closed connected component, or (ii) a closed connected component plus an isolated point at* $(0, 0)$*. Moreover, this connected component has a non-empty intersection with the unit square* U*.*

Proof Observe that $Q(e^x, e^y) < 0$, where $(x, y) \in \mathbb{R}^2$, forms a convex set; see [20,21]. Moreover, the boundary of this convex set is a closed curve [20, Section 3.3]. By Lemma 7, Q_0^+ can only intersect the axes at (0, 0). This immediately implies that Q_0^+ has a single closed connected component. The origin is on Q_0^+ iff $p_{1,1} = 0$ because $Q(0, 0) = p_{1,1}$. When $p_{1,1}$ is 0, if the origin is already on the single closed connected component, then Q_0^+ consists of exactly one closed connected component. If the origin is not on the single closed connected component, then Q_0^+ consists of a closed connected component plus an isolated point at (0, 0). When $p_{1,1}$ is not 0, it is obvious that Q_0^+ consists of exactly one closed connected component.

Notice that Q intersects $\{(x, y)|x = 1, y > 0\}$ at

$$\left(1, \frac{\sum_{s=-1}^{1} p_{s,1}}{\sum_{s=-1}^{1} p_{s,-1}}\right), \text{ and } \quad (1, 1). \tag{16}$$

Also, Q intersects $\{(x, y)|x > 0, y = 1\}$ at

$$\left(\frac{\sum_{t=-1}^{1} p_{1,t}}{\sum_{t=-1}^{1} p_{-1,t}}, 1\right), \text{ and } \quad (1, 1). \tag{17}$$

For R to be ergodic it must be that at least one of the conditions $M_x < 0$ or $M_y < 0$ holds. Therefore, at least one of the following restrictions must be satisfied:

$$0 < \frac{\sum_{s=-1}^{1} p_{s,1}}{\sum_{s=-1}^{1} p_{s,-1}} < 1, \quad 0 < \frac{\sum_{t=-1}^{1} p_{1,t}}{\sum_{t=-1}^{1} p_{-1,t}} < 1. \tag{18}$$

Hence, we conclude that the component of Q in the first quadrant has a non-empty intersection with U. □

From the results above we obtain that Q_0^+, the intersection of Q and $\mathbb{R}_+^2$, contains a connected component. We denote this connected component by Q^+. Moreover, denote the branch points of $Y(x)$ and $X(y)$ on Q^+ by x_l, x_r, with $x_l < x_r$, and y_b, y_t, with $y_b < y_t$, respectively. Then y_l, y_r, x_b, x_t satisfy $(x_l, y_l), (x_r, y_r), (x_t, y_t), (x_b, y_b) \in Q$; see Fig. 4a. We will refer to $(x_l, y_l), (x_r, y_r), (x_t, y_t), (x_b, y_b)$ as branch points of Q^+. From Lemma 5, we obtain that $0 \le x_l \le 1 \le x_r$, $0 \le y_b \le 1 \le y_t$. Since we are only interested in finite measures, we only consider Q^+ in $\bar{U} = [0, 1)^2$. Lemma 8 states that $Q_U^+ = Q^+ \cap U$ is a non-empty set for an ergodic random walk with nonzero drift.

We proceed with the analysis of Q^+.

Definition 9 (*Partition of* Q^+) The partition $\{Q_{00}, Q_{01}, Q_{10}, Q_{11}\}$ of Q^+ is defined as follows: Q_{00} is the part of Q connecting (x_l, y_l) and (x_b, y_b); Q_{10} is the part of Q connecting (x_b, y_b) and (x_r, y_r); Q_{01} is the part of Q connecting (x_l, y_l) and (x_t, y_t); Q_{11} is the part of Q connecting (x_r, y_r) and (x_t, y_t).

Figure 4b illustrates the partition of Q^+.

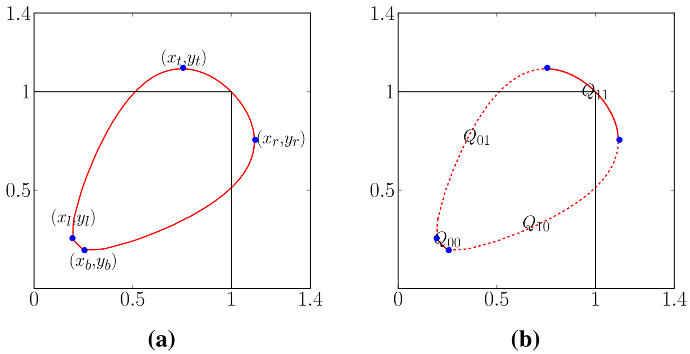

Fig. 4 Q^+ for the random walk from Fig. 2c. **a** Branch points of Q^+. **b** Partition of Q^+

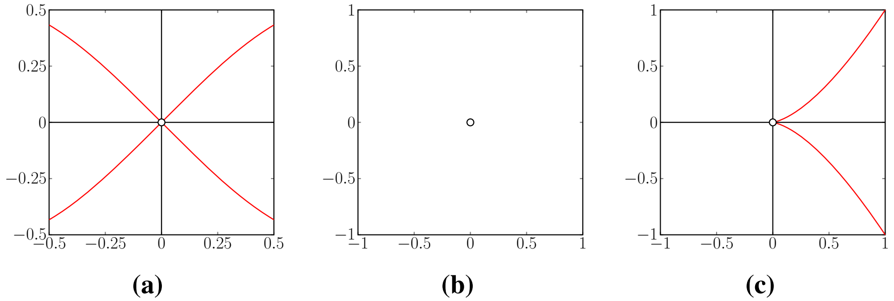

Fig. 5 Singularities of order 2: **a** crunode, **b** acnode, **c** ordinary cusp

Monotonicity of $X(y)$ and $Y(x)$ will play a crucial role in analyzing the structure of Γ. The following result immediately follows from the fact that $\{(x, y) \in \mathbb{R}^2 \mid Q(e^x, e^y) < 0\}$ is convex.

Lemma 10 *1. Let $(x, y) \in Q_{i,1-i}$ and $(\tilde{x}, \tilde{y}) \in Q_{i,1-i}$, with $i \in \{0, 1\}$ and $\tilde{x} > x$. Then $\tilde{y} > y$.*
2. Let $(x, y) \in Q_{i,i}$ and $(\tilde{x}, \tilde{y}) \in Q_{i,i}$, with $i \in \{0, 1\}$ and $\tilde{x} > x$. Then $\tilde{y} < y$.

We now consider the singularities of Q. We will see below that if one set from the partition of Q^+ is empty, then the curve Q^+ will have a singularity. Singularity plays an important role in the analysis later.

Definition 11 (*Singularity of Q*) The point $(x, y) \in Q$ is a singularity of multiplicity m, $m > 1$, iff at (x, y) all partial derivatives of $Q(x, y)$ of order less than m vanish and at least one partial derivative of order m is nonzero.

Lemma 12 *For all random walks with nonzero drift, (x, y) is a singularity of Q^+ iff it is a crunode of order 2, and x and y are branch points of multiplicity 2 of $Y(x)$ and $X(y)$, respectively.*

Proof We prove by contradiction that it is not possible to have a singularity of order larger than 2. Suppose that $(\tilde{x}, \tilde{y}) \in Q^+$ is a singularity of order larger than 2. From

Lemma 7 it follows that we need to consider the cases (i) $\tilde{x} > 0$ and $\tilde{y} > 0$, and (ii) $(\tilde{x}, \tilde{y}) = (0, 0)$. If $\tilde{x} > 0$ and $\tilde{y} > 0$, it follows from

$$\frac{\partial^2 Q(x, y)}{\partial^2 x} = \sum_{t=-1}^{1} p_{-1,t} y^{-t+1} = 0, \quad \text{and} \quad \frac{\partial^2 Q(x, y)}{\partial^2 y} = \sum_{s=-1}^{1} p_{s,-1} x^{-s+1} = 0,$$

that $p_{-1,1} = p_{-1,0} = p_{-1,-1} = p_{0,-1} = p_{1,-1} = 0$, which leads to a non-ergodic random walk. For $(\tilde{x}, \tilde{y}) = (0, 0)$, it follows from

$$\frac{\partial^2 Q(x, y)}{\partial x \partial y} = 4xyp_{-1,-1} + 2xp_{-1,0} + 2yp_{0,-1} + p_{0,0} - 1 = 0,$$

that $p_{00} = 1$, which leads to a random walk that is not irreducible. This concludes the proof that a singularity has order at most 2.

Next, we demonstrate that if (x, y) is a singularity, then x and y are branch points of $Y(x)$ and $X(y)$, respectively. By combining $Q(x, y) = 0$ with

$$\frac{\partial Q(x, y)}{\partial x} = 2x \left(\sum_{t=-1}^{1} p_{-1,t} y^{-t+1} \right) + \left(\sum_{t=-1}^{1} p_{0,t} y^{-t+1} - y \right) = 0,$$

we obtain

$$\sum_{t=-1}^{1} p_{-1,t} y^{-t+1} x^2 = \sum_{t=-1}^{1} p_{1,t} y^{-t+1},$$

which means x is the root of $\Delta_y(x) = 0$, defined in (14), and therefore a branch point of $Y(x)$. Similarly, it follows from $Q(x, y) = 0$ and $\partial Q(x, y)/\partial y = 0$ that y is a branch point of $X(y)$.

Now, we are ready to prove that a singularity (x, y) is a crunode. For more information on the classification of singularities of algebraic curves, see, for example [15]. An illustration of all possible singularities of order 2 is given in Fig. 5. Note that the figure does not include a ramphoid cusp, since it has order larger than 2. A singularity cannot be an ordinary cusp, because x and y are branch points of $Y(x)$ and $X(y)$, respectively. Moreover, (x, y) is not an acnode because Q_U^+ is non-empty due to Lemma 8. Therefore, a singularity must be a crunode.

The final result in this lemma follows from the observation that if x and y are branch points of $Y(x)$ and $X(y)$, respectively, and (x, y) is a crunode, then x and y must have multiplicity two. □

Theorem 13 *The algebraic curve Q has a singularity in $\bar{U}$ if and only if $p_{0,1} = p_{1,1} = p_{1,0} = 0$, in which case this singularity is located at the origin.*

Proof Lemma 12 states that (x, y) is a singularity of Q^+ if and only if it is a crunode of order 2 and x and y are branch points of multiplicity 2 of $Y(x)$ and $X(y)$, respectively. Therefore, we only need to consider (x, y), where x and y are the multiple roots of $\Delta_y(x) = 0$ and $\Delta_x(y) = 0$, respectively. A multiple root of $\Delta_y(x) = 0$ and $\Delta_x(y) = 0$

can only occur at $x = 0, 1$ or ∞ and $y = 0, 1$ or ∞, respectively, due to Lemma 6. Therefore, $x = 0$ and $y = 0$ must be multiple roots of $\Delta_y(x) = 0$ and $\Delta_x(y) = 0$, respectively, if there is a singularity in $\bar{U}$. We know from [13, Lemma 2.3.10] that $\Delta_y(x) = 0$ has a multiple root at 0 if and only if one of the following holds:

$$p_{-1,0} = p_{-1,1} = p_{0,1} = 0, \tag{19}$$

$$p_{1,0} = p_{1,1} = p_{0,1} = 0, \tag{20}$$

$$p_{-1,-1} = p_{0,-1} = p_{1,-1} = 0, \tag{21}$$

and $\Delta_x(y) = 0$ has a multiple root at 0 if and only if one of the following holds:

$$p_{0,-1} = p_{1,-1} = p_{1,0} = 0, \tag{22}$$

$$p_{0,1} = p_{1,1} = p_{1,0} = 0, \tag{23}$$

$$p_{-1,-1} = p_{-1,0} = p_{-1,1} = 0. \tag{24}$$

Conditions (21) and (24) lead to a singular random walk. The combinations of conditions (19) and (22), (19) and (23), (20) and (22) will lead to singular random walks as well. Since, by Assumption 1, the random walk is non-singular, the algebraic curve Q has a singularity in $\bar{U}$ if and only if $p_{0,1} = p_{1,1} = p_{1,0} = 0$, in which case it is located at the origin. □

4 Constraints on invariant measures and random walks

This section first characterizes the structure of $\Gamma \subset Q_U^+$ that may lead to an invariant measure. Then we will provide necessary conditions on the transition probabilities of a random walk to allow for a countably infinite sum of geometric terms to constitute the invariant measure. Finally, we demonstrate that it is necessary to have at least one negative coefficient in an invariant measure that is an infinite sum of geometric terms.

4.1 Structure of Γ

This section considers the structure of Γ. The proofs in this and subsequent sections are based on the notion of uncoupled partitions.

Definition 14 (*Uncoupled partition*) A partition $\{\Gamma_1, \Gamma_2, \ldots\}$ of Γ is

horizontally uncoupled	if $(\rho, \sigma) \in \Gamma_p$ and $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_q$, for $p \neq q$, implies that $\tilde{\rho} \neq \rho$,
vertically uncoupled	if $(\rho, \sigma) \in \Gamma_p$ and $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_q$, for $p \neq q$, implies that $\tilde{\sigma} \neq \sigma$, and
uncoupled	if it is both horizontally and vertically uncoupled.

Horizontally uncoupled sets are obtained by putting pairs (ρ, σ) with the same ρ into the same element of the partition. Vertically coupled sets are obtained by putting pairs (ρ, σ) with the same σ into the same element. A maximal partition is a partition

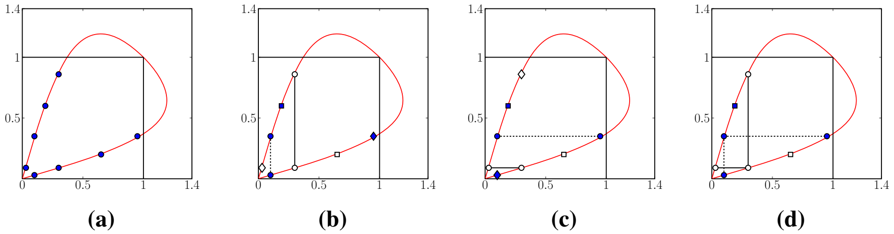

Fig. 6 Partitions of set Γ. **a** Curve Q^+ of Fig. 2d and $\Gamma \subset Q^+$ as dots. **b** Horizontally uncoupled partition with six sets. **c** Vertically uncoupled partition with six sets. **d** Uncoupled partition with four sets. Different sets are marked by different symbols

with the largest number of sets. The following result from [9] readily carries over to the case of countable Γ.

Lemma 15 ([9, Lemma 5]) *The maximal vertically uncoupled partition, the maximal horizontally uncoupled partition and the maximal uncoupled partition are unique.*

Figure 6 gives examples of a maximal horizontally uncoupled partition, of a maximal vertically uncoupled partition and of a maximal uncoupled partition. Let $H[\Gamma]$ denote the number of elements in the maximal horizontally uncoupled partition for the set Γ, and let Γ_p^h, $p = 1, \ldots, H[\Gamma]$ denote the sets themselves, where elements of Γ_p^h have common horizontal coordinate $\rho(\Gamma_p^h)$. The maximal vertically uncoupled partition of the set Γ has $V[\Gamma]$ sets, Γ_q^v, $q = 1, \ldots, V[\Gamma]$, where elements of Γ_q^v have common vertical coordinate $\sigma(\Gamma_q^v)$. The maximal uncoupled partition of the set Γ has $G[\Gamma]$ sets, Γ_k^g, $k = 1 \ldots, G[\Gamma]$. The $H[\Gamma]$, $V[\Gamma]$ and $G[\Gamma]$ are allowed to be infinite.

We now make two observations on the structure of an element Γ_k^g from the maximal uncoupled partition. Apart from the case of a singleton geometric term, for any $(\rho, \sigma) \in \Gamma_k^g$ there always exist either $(\rho, \tilde{\sigma}) \in \Gamma_k^g$ with $\tilde{\sigma} \neq \sigma$, or $(\tilde{\rho}, \sigma) \in \Gamma_k^g$ with $\tilde{\rho} \neq \rho$. Second, the degree of $Q(\rho, \sigma)$ is at most two in each variable. This means, for instance, that if $(\rho, \sigma) \in \Gamma_k^g$ and $(\rho, \tilde{\sigma}) \in \Gamma_k^g$, $\tilde{\sigma} \neq \sigma$, then there does not exist $(\rho, \hat{\sigma}) \in \Gamma_k^g$, where $\hat{\sigma} \neq \sigma$ and $\hat{\sigma} \neq \tilde{\sigma}$. By repeating the above two arguments for other elements in Γ_k^g, it follows that Γ_k^g must have a pairwise-coupled structure. An example of such a set is $\Gamma_k^g = \{(\rho_k, \sigma_k), k = 1, 2, 3 \ldots\}$, where

$$\rho_1 = \rho_2, \sigma_1 > \sigma_2, \rho_2 > \rho_3, \sigma_2 = \sigma_3, \rho_3 = \rho_4, \sigma_3 > \sigma_4, \cdots . \tag{25}$$

Finally, note that a singleton geometric term also satisfies this structure. The above discussion leads to the definition of a pairwise-coupled set in terms of the number of sets in a maximal uncoupled partition.

Definition 16 (*Pairwise-coupled set*) A set $\Gamma \subset Q_U^+$ is pairwise-coupled if and only if the maximal uncoupled partition of Γ contains only one set.

Note that the above definition implies that each of the sets in $\{\Gamma_k^g\}_{k=1}^{G[\Gamma]}$ is pairwise-coupled.

We are now ready to show that if the measure induced by Γ is the invariant measure, then Γ must be the union of finitely many pairwise-coupled sets each with countably infinite cardinality. We first introduce some additional notation. For each set Γ_p^h from the maximal horizontally uncoupled partition of Γ, let

$$B^h(\Gamma_p^h) = \sum_{(\rho,\sigma)\in\Gamma_p^h} \alpha(\rho,\sigma)\left[\sum_{s=-1}^{1}\left(\rho^{1-s}h_s + \rho^{1-s}\sigma p_{s,-1}\right) - \rho\right].$$

For each set Γ_q^v from the maximal vertically uncoupled partition of Γ, let

$$B^v(\Gamma_q^v) = \sum_{(\rho,\sigma)\in\Gamma_q^v} \alpha(\rho,\sigma)\left[\sum_{t=-1}^{1}\left(\sigma^{1-t}v_t + \rho\sigma^{1-t}p_{-1,t}\right) - \sigma\right].$$

Also, we will make use of the following result that is a special case of a result from [7].

Theorem 17 ([7, Theorem 1, Lemma 1]) *Let d be a positive integer. Consider a real finite measure $\mu : \mathbb{R} \to \mathbb{R}$ with compact support K. If*

$$\int_{\mathbb{R}} P(x)\,\mathrm{d}\mu(x) = 0 \tag{26}$$

for all polynomials P, then $\mu = 0$.

Our first result states that $B^h(\Gamma_p^h) = 0$ and $B^v(\Gamma_q^v) = 0$.

Lemma 18 *Consider the random walk R. If the invariant measure m of R is $m(i, j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j$, then we have $B^h(\Gamma_p^h) = 0$ and $B^v(\Gamma_q^v) = 0$ for all $p = 1, \ldots, H[\Gamma]$ and $q = 1, \ldots, V[\Gamma]$, respectively.*

Proof Since m is the invariant measure of R, m satisfies the balance equations (4) at state $(i, 0)$ for $i = 1, 2, 3, \ldots$. Therefore,

$$\begin{aligned}
0 &= \sum_{s=-1}^{1}\left[m(i-s,0)h_s + m(i-s,1)p_{s,-1}\right] - m(i,0)\\
&= \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\left[\sum_{s=-1}^{1}\left(\rho^{i-s}h_s + \rho^{i-s}\sigma p_{s,-1}\right) - \rho^i\right]\\
&= \sum_{p=1}^{\infty} \rho(\Gamma_p^h)^i \sum_{(\rho,\sigma)\in\Gamma_p^h} \alpha(\rho,\sigma)\left[\sum_{s=-1}^{1}\left(\rho^{-s}h_s + \rho^{-s}\sigma p_{s,-1}\right) - 1\right]\\
&= \sum_{p=1}^{\infty} \rho(\Gamma_p^h)^{i-1} B^h(\Gamma_p^h).
\end{aligned} \tag{27}$$

We now show the absolute convergence of sequence $\{B^h(\Gamma_p^h)\}$, $p = 1, \ldots, H[\Gamma]$. Because of the assumption that $m(i, j)$ is a finite measure and $(\rho, \sigma) \in (0, 1)^2$, we have

$$\begin{aligned}\sum_{p=1}^{\infty} |B^h(\Gamma_p^h)| &\le \sum_{k=1}^{\infty} |\alpha(\rho_k, \sigma_k)| \left| \sum_{s=-1}^{1} \left(\rho_k^{1-s} h_s + \rho_k^{1-s} \sigma p_{s,-1}\right) - \rho_k \right| \\ &< B \sum_{(\rho,\sigma)\in\Gamma} |\alpha(\rho, \sigma)| \frac{1}{1-\rho} \frac{1}{1-\sigma} \\ &< \infty, \end{aligned} \tag{28}$$

where the last inequality is due to assumption (10), and B is a finite positive constant.

Define a real measure μ as

$$\mu(\rho) = \begin{cases} B^h(\Gamma_p^h) & \text{if } \rho = \rho(\Gamma_p^h), \\ 0 & \text{otherwise.} \end{cases}$$

We can now write (27) as

$$\sum_{p=1}^{\infty} \rho(\Gamma_p^h)^{i-1} \mu(\rho(\Gamma_p^h)) = \int \rho(\Gamma_p^h)^{i-1} \, \mathrm{d}\mu(\rho(\Gamma_p^h)) = 0,$$

for $i = 1, 2, 3, \ldots$. This indicates that

$$\int P(\rho) \, \mathrm{d}\mu(\rho) = 0$$

for all $P(\rho) = \rho^j$, where $j = 0, 1, 2, \ldots$. Hence $\int P(\rho)\,\mathrm{d}\mu(\rho) = 0$ for all polynomials. Moreover, the condition (28) guarantees that the measure μ is finite. By definition, the support of μ is the closure of the set $\{\rho | \rho \in \rho(\Gamma_p^h)\} \subseteq \Gamma$, and since $\Gamma \subset U$, this is bounded. Hence μ has compact support. Hence, by using Theorem 17 we have $\mu = 0$, which means $B^h(\Gamma_p^h) = 0$ for $p = 1, 2, \ldots, H[\Gamma]$. Similarly, we can obtain that $B^v(\Gamma_q^v) = 0$ for $q = 1, 2, \ldots, V[\Gamma]$. □

The technique that is used in the above proof can be used to relax the condition that was made at the end of Sect. 2 that $\Gamma \subset Q$, i.e., that elements of Γ must individually satisfy the interior balance equations. In [8] we provided a weaker condition than $\Gamma \subset Q$, but an interpretation for this condition is lacking and therefore it is not included in this paper.

Next, we provide the main result of this subsection, Theorem 19, that states that Γ must be the union of at most six pairwise-coupled sets each with countably infinite cardinality.

Theorem 19 *If the measure induced by Γ, where Γ is of countably infinite cardinality, is the invariant measure of the random walk R, then $|\Gamma_k^g| = \infty$ for $k = 1, 2, \ldots, G[\Gamma]$ and $G[\Gamma] \le 6$.*

Proof Suppose that there exists a pairwise-coupled set $\Gamma_1^g \subset \Gamma$ with $|\Gamma_1^g| < \infty$ and $\Gamma_1^g \neq \Gamma$. From Lemma 18 we have $B^h([\Gamma_1^g]_p^h) = 0$ for $p = 1, \ldots, H[\Gamma_1^g]$ and $B^v([\Gamma_1^g]_q^v) = 0$ for $q = 1, \ldots, V[\Gamma_1^g]$. Now, it follows from [9, Theorem 3] that the measure induced by Γ_1^g itself is an invariant measure of R. Since the measure induced by the set Γ is also an invariant measure, this contradicts the fact that the invariant measure is unique. Therefore, $|\Gamma_k^g| = \infty$ for $k = 1, 2, \cdots, G[\Gamma]$.

Since $|\Gamma_k^g| = \infty$ and by Theorem 13 there is at most one singularity on Q in the unit square, it can be readily verified using Lemma 18 that there exists a $(\rho, \sigma) \in \Gamma$ such that $B^h(\{(\rho, \sigma)\}) = 0$ or $B^v(\{(\rho, \sigma)\}) = 0$, which corresponds to the balance equations for states at the horizontal or at the vertical axis of the state space, respectively. In other words $(\rho, \sigma) \in H$ or $(\rho, \sigma) \in V$.

Finally, [10, Lemma 1] implies that there are at most three intersections between Q_U^+ and H and between Q_U^+ and V, respectively. Therefore, we conclude that set Γ must be the union of at most six pairwise-coupled sets each with countably infinite cardinality. □

4.2 Constraints on random walks

In this section, we characterize the random walks for which the invariant measure may be an infinite sum of geometric terms. We will show that the existence of transitions to the north, northeast or east plays an essential role in distinguishing such random walks.

Theorem 20 *Let the invariant measure of the random walk R be $m(i, j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho, \sigma)\rho^i\sigma^j$ with $|\Gamma| = \infty$. Then $p_{1,0} = p_{1,1} = p_{0,1} = 0$ and Γ has a unique accumulation point in the origin.*

Proof We will demonstrate that in the absence of singularities it is not possible to have an invariant measure $m(i, j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho, \sigma)\rho^i\sigma^j$ with $|\Gamma| = \infty$. More precisely, we show that Γ must have a singularity of Q as an accumulation point. The result of the theorem then follows from Theorem 13, which states the algebraic curve Q has a singularity in $\bar{U}$ only if $p_{1,0} = p_{1,1} = p_{0,1} = 0$, in which case it is in the origin.

Suppose that Q^+ does not contain any singularities and that $m(i, j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho, \sigma)\rho^i\sigma^j$ with $|\Gamma| = \infty$ is the invariant measure of P. In the remainder of this proof we will obtain a contradiction by showing that at least one of the terms $(\rho, \sigma) \in \Gamma$ will be outside the unit square and that the measure $m(i, j)$ cannot, therefore, be finite.

To simplify the presentation we introduce the partition $\{Q_l, Q_c, Q_r\}$ of Q^+, where

$$Q_l = \left\{(x, y) \in Q^+ \;\middle|\; x \le \min\{x_t, x_b\}\right\},$$
$$Q_c = \left\{(x, y) \in Q^+ \;\middle|\; \min\{x_t, x_b\} < x \le \max\{x_t, x_b\}\right\},$$
$$Q_r = \left\{(x, y) \in Q^+ \;\middle|\; x > \max\{x_t, x_b\}\right\}.$$

This partition is illustrated in Fig. 7a. Moreover, denote the two pieces of Q_c by Q_c^t and Q_c^b satisfying $\tilde{y} > y$ if $(x, \tilde{y}) \in Q_c^t$ and $(x, y) \in Q_c^b$. Since there are no

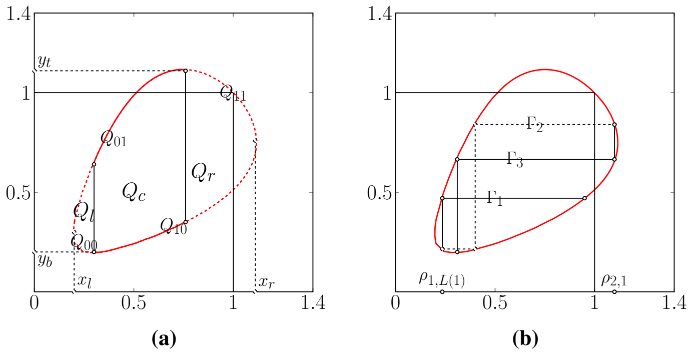

Fig. 7 **a** Different partition of Q^+ for the random walk in Fig. 2c. **b** Example of Γ_1, Γ_2 and Γ_3

singularities, Q_l, Q_c and Q_r are all non-empty. In addition, we let $\{\Gamma_1, \ldots, \Gamma_K\}$ denote a partition of Γ, with possibly $K = \infty$, where the elements of Γ_i are denoted as $\Gamma_i = \{(\rho_{i,1}, \sigma_{i,1}), \ldots, (\rho_{i,L(i)}, \sigma_{i,L(i)})\}$, with possibly $L(i) = \infty$, and each Γ_i satisfies

$$\begin{aligned} \rho_{i,1} &> \rho_{i,2}, \quad \sigma_{i,1} = \sigma_{i,2}, \\ \rho_{i,2} &= \rho_{i,3}, \quad \sigma_{i,2} > \sigma_{i,3}, \\ \rho_{i,3} &> \rho_{i,4}, \quad \sigma_{i,3} = \sigma_{i,4}, \\ &\vdots \qquad\qquad \vdots \end{aligned} \tag{29}$$

In addition the partition $\{\Gamma_1, \ldots, \Gamma_K\}$ is maximal in the sense that no $\Gamma_i \cup \Gamma_j$, $i \neq j$, satisfies (29). The partition is illustrated in Fig. 7b. In the remainder we will show that $K < \infty$ and that $L(i) < \infty$ for all $i = 1, \ldots, K$, leading to a contradiction to the assumption that $|\Gamma| = \infty$.

First, we prove $L(i) < \infty$ by demonstrating that

$$|\Gamma_i \cap Q_l| \leq 1, \quad |\Gamma_i \cap Q_c| < \infty, \quad |\Gamma_i \cap Q_r| \leq 1.$$

Suppose $|\Gamma_i \cap Q_r| \geq 2$. Then there exists (ρ, σ) and $(\tilde{\rho}, \tilde{\sigma})$ on Q_{11} or Q_{10} satisfying $\tilde{\sigma} = \sigma$, which contradicts Lemma 10. Therefore, $|\Gamma_i \cap Q_r| \leq 1$. Similarly, we have $|\Gamma_i \cap Q_l| \leq 1$. Next, observe that since Q^+ does not have any singularities, there exists a constant $D > 0$ such that, for all $(x, y) \in Q_c^t$ and $(\tilde{x}, \tilde{y}) \in Q_c^b$ with $x = \tilde{x}$, we have $y - \tilde{y} \geq D$. This implies that for three consecutive elements in $|\Gamma_i \cap Q_c|$,

$$\begin{aligned} \rho_{i,j} &> \rho_{i,j+1}, \quad \sigma_{i,j} = \sigma_{i,j+1}, \\ \rho_{i,j+1} &= \rho_{i,j+2}, \sigma_{i,j+1} > \sigma_{i,j+2}, \end{aligned}$$

we have $\sigma_{i,j+2} \leq \sigma_{i,j} - D$. Together with $y_t - y_b < \infty$ this implies that $|\Gamma_i \cap Q_c| < \infty$.

Next, we prove $K < \infty$. More precisely, we will show that $K \leq 2$. Without loss of generality, we assume $K = 3$ and $|\Gamma_i| \geq 2$. Observe that $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ forms a pairwise-coupled set. Therefore, we must have $\rho_{1,L(1)} = \rho_{2,L(2)}$ and $\rho_{2,1} = \rho_{3,1}$. This is illustrated in Fig. 7b. From the monotonicity of Q_{10} and the structure of the partition

$\{\Gamma_1, \ldots, \Gamma_K\}$ it follows that $\rho_{1,L(1)} \in Q_l$. Similarly, it follows that $\rho_{2,1} \in Q_r$ and $\rho_{3,1} \in Q_r$. Moreover, one of $\rho_{2,1}$ and $\rho_{3,1}$ must be on Q_{11}. Note that from Lemma 5 we have $y_t \geq 1$ and $x_r \geq 1$. Together with the monotonicity of Q_{11} from Lemma 10, this leads to the conclusion that Q_{11} is outside U. This implies that one of the geometric measures in Γ is outside U and leads to a contradiction to the fact the induced measure is finite.

We have shown that under the assumption that Q^+ does not contain singularities, Γ can be partitioned into a finite number of sets each with a finite number of elements, which contradicts $|\Gamma| = \infty$. □

4.3 Constraints on the coefficients

The last section was devoted to finding constraints on the random walk for which the pairwise-coupled set with infinite cardinality could be obtained. In this section, we show that it is necessary to have a geometric term with a negative coefficient in the infinite sum of geometric terms.

Theorem 21 *If the invariant measure of the random walk R is $m(i,j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j$, where $\Gamma \subset Q_U^+$, $|\Gamma| = \infty$ and $\alpha(\rho,\sigma) \in \mathbb{R}\backslash\{0\}$, then at least one $\alpha(\rho,\sigma)$ is negative.*

Proof Theorem 20 implies that $p_{1,0} = p_{1,1} = p_{0,1} = 0$ in the random walk R. Notice that there is at least one pairwise-coupled set from Γ that contains countably infinitely many geometric terms due to Theorem 19. Without loss of generality, we consider the set Γ which only contains a single pairwise-coupled set which is of the form given in (25) and assume that $\alpha(\rho_1,\sigma_1) = 1$. Since the measure induced by Γ is the invariant measure, it follows from Theorem 19 that $B^h(\Gamma_p^h) = 0$ and $B^v(\Gamma_q^v) = 0$ for all $p, q \in \{1, 2, 3, \ldots\}$. Note that $B^h\{(\rho_1,\sigma_1), (\rho_2,\sigma_2)\} = 0$ indicates that $\alpha(\rho_2,\sigma_2)$ is uniquely determined by

$$\alpha(\rho_2,\sigma_2) = -\frac{T_1}{T_2}\alpha(\rho_1,\sigma_1),$$

where

$$T_i = \left(1 - \frac{1}{\rho_i}\right)h_1 + (1-\rho_i)h_{-1} + \sum_{s=-1}^{1} p_{s,1} - \sigma_i\left(\sum_{s=-1}^{1} \rho_i^{-s} p_{s,-1}\right).$$

Next, $\alpha(\rho_3,\sigma_3)$ follows from $B^v\{(\rho_2,\sigma_2), (\rho_3,\sigma_3)\} = 0$. In similar fashion, for $k \in \{1, 2, 3, \ldots\}$,

$$\alpha(\rho_{2k},\sigma_{2k}) = -\frac{T_{2k-1}}{T_{2k}}\alpha(\rho_{2k-1},\sigma_{2k-1}), \quad \text{where} \quad \rho_{2k} = \rho_{2k-1}, \sigma_{2k} < \sigma_{2k-1}.$$

The following two facts allow us to show that there exists a positive integer N such that $\frac{T_{2k-1}}{T_{2k}} > 0$ when $k > N$. Firstly, we know $p_{1,0} = p_{1,1} = p_{0,1} = 0$ and $\lim_{k\to\infty} \rho_k = 0$

from Theorem 20. Secondly, the ergodic random walk with no drift to the northeast requires $h_1 + p_{1,-1} \neq 0$. Note that

$$\frac{T_{2k-1}}{T_{2k}} = \frac{\left(1 - \frac{1}{\rho_{2k-1}}\right) h_1 + (1 - \rho_{2k-1})h_{-1} + p_{-1,1} - \sigma_{2k-1}(\sum_{s=-1}^{1} \rho_{2k-1}^{-s} p_{s,-1})}{\left(1 - \frac{1}{\rho_{2k}}\right) h_1 + (1 - \rho_{2k})h_{-1} + p_{-1,1} - \sigma_{2k}(\sum_{s=-1}^{1} \rho_{2k}^{-s} p_{s,-1})}.$$

By using L'Hospital's rule, we can conclude

$$\lim_{k\to\infty} \frac{T_{2k-1}}{T_{2k}} = \begin{cases} \frac{h_1+\sigma_{2k-1} p_{1,-1}}{h_1+\sigma_{2k} p_{1,-1}}, & \text{if } h_1 \neq 0,\ p_{1,-1} \neq 0, \\ 1, & \text{if } h_1 \neq 0,\ p_{1,-1} = 0, \\ \frac{\sigma_{2k-1}}{\sigma_{2k}}, & \text{if } h_1 = 0,\ p_{1,-1} \neq 0. \end{cases}$$

The non-negativity of $\frac{T_{2k-1}}{T_{2k}}$ when k is large enough completes the proof. □

5 Conclusion

In this paper, we have obtained necessary conditions for the invariant measure of a random walk to be a countably infinite sum of geometric terms which individually satisfy the interior balance equations. We demonstrate that the compensation approach is the only method that may lead to such a type of invariant measure. In particular, we show that if a countably infinite sum of geometric terms is an invariant measure, then the geometric terms in an invariant measure must be the union of at most six pairwise-coupled sets of infinite cardinality each. We further show that for the invariant measure to be an infinite sum of geometric terms, the random walk cannot have transitions to the north, northeast or east. Finally, we show that for a countably infinite weighted sum of geometric terms to be an invariant measure at least one of the weights must be negative.

The multiprogramming queue and the 2×2 switch, which have been analyzed using the compensation approach in [1,6], respectively, are instances of our model. The model of [3] is more general, however, in the sense that it does not require the second condition in (2) which states that the random walk is homogeneous with respect to transition probabilities entering the same part of the state space. In future work it will be of interest to generalize our results to other models.

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