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Removable Edges and Chords of Longest Cycles in 3-Connected Graphs

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Abstract We verify two special cases of Thomassen's conjecture of 1976 stating that every longest cycle in a 3-connected graph contains a chord. We prove that Thomassen's conjecture is true for two classes of 3-connected graphs that have a bounded number of removable edges on or off a longest cycle. Here an edge e of a 3-connected graph G is said to be removable if $G - e$ is still 3-connected or a subdivision of a 3-connected (multi)graph. We give examples to show that these classes are not covered by previous results.

Keywords 3-Connected graph · Removable edge · Chord

Mathematics subject classifications (2000) 05C40 · 05C38 · 05C75

1 Introduction

All graphs considered here are finite and simple. For notations and terminology not defined here, we refer the reader to [3].

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The key concept in our study is the concept of a removable edge of a 3-connected graph. Let G be a 3-connected graph and let e be an edge of G. Consider the graph $G - e$ obtained by deleting e from G. If $G - e$ has vertices of degree 2, we suppress them, i.e., we remove the vertices of degree 2 from $G - e$ and we join their two neighbors by an edge unless they are already adjacent (so we avoid multiple edges). The resultant graph is denoted by $G \ominus e$. Note that if there is no vertex of degree 2 in $G - e$, then $G \ominus e$ is simply the graph $G - e$.

An edge e of a 3-connected graph G is called *removable* if $G \ominus e$ is still 3-connected; otherwise e is called *unremovable*. The set of all removable edges of G is denoted by $E_R(G)$, whereas the set of all unremovable edges of G is denoted by $E_N(G)$.

In 1976, Thomassen conjectured that every longest cycle in a 3-connected graph has a chord (See, e.g., conjecture 6.11 in [2]). Here a longest cycle is a cycle of maximum length and a chord of a cycle C is an edge between two vertices of C that are nonadjacent on C. Thomassen [10] proved that his conjecture is true for cubic graphs. The conjecture has also been verified for graphs embeddable in several surfaces [5–7]. In particular, Zhang [11] proved that the conjecture is true for planar graphs with minimum degree at least four. More recently, Birmelé [1] verified the conjecture for the larger class of $K_{3,3}$-minor free graphs.

If C is a longest cycle in a 3-connected graph G and e is a removable edge of G, then in some cases C will still be a longest cycle of $G \ominus e$, and one could try to use induction on the order and size of G in an attempt to prove Thomassen's conjecture. In many cases, however, C will not be a longest cycle of $G \ominus e$ and it is not clear what to do in such cases. There is some hope that the approach could still be successful if one could show that many edges of G are removable, especially in case many edges of $E(G) - E(C)$ are removable. Our results make a first step in this direction.

Here we prove that Thomassen's conjecture is true for two classes of 3-connected graphs that have a bounded number of removable edges on or off a longest cycle. We give examples to show that these classes are not covered by previous results. We first introduce some additional notation and terminology.

In the following G always denotes a 3-connected graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The order and size of G are denoted by $|G|$ and $|E(G)|$, respectively. Instead of $x \in V(G)$ we simply write $x \in G$. The neighborhood of $x \in G$ is denoted by $\Gamma_G(x)$. If x and y are the two vertices incident with an edge e, we write $e = xy$. For sets $A, B \subset V(G)$ such that $A \neq \emptyset \neq B$ and $A \cap B = \emptyset$, we define $[A, B] = \{xy \in E(G) \mid x \in A, y \in B\}$. For a subset S of $V(G)$, $G - S$ denotes the graph obtained by deleting all the vertices of S from G together with all the edges incident with vertices of S. If $G - S$ is disconnected, we say that S is a vertex-cut of G, and an s-cut if $|S| = s$.

It is easy to check and folklore knowledge that for every edge e of a 3-connected graph G, the graph $G - e$ is at least 2-connected. Moreover, if in this case $G - e$ has a 2-cut S, then $(G - e) - S$ consists of precisely two components. If one of these components has only one vertex, this vertex has degree 2 in $G - e$ and will disappear in $G \ominus e$. This motivated the following definitions.

For $e \in E(G)$ and $S \subset V(G)$ with $|S| = 2$, we say that (e, S) is a *separating pair* if $(G - e) - S$ has exactly two components, say A and B, such that $|A| \geq 2$ and $|B| \geq 2$; then $(e, S; A, B)$ is called a *separating group*, in which A and B are called the *fragments*. If moreover $|A| = 2$, then A is called an *atom*. If A is an atom, let $A = \{x, z\}$, and $S = \{a, b\}$ such that x is an end vertex of edge e. If $ax \in E(G)$, $bx \notin E(G)$, then A is called a *1-atom*; if $ax, bx \in E(G)$, then A is called a *2-atom*. It is easy to see that an atom is either a 1-atom or a 2-atom.

Su [9] studied the distribution of removable edges in 3-connected graphs and obtained the following result that we will use in the sequel.

Theorem 1 *Let $(xy, S; A, B)$ be a separating group of a 3-connected graph G such that $A = \{x, z\}$, $y \in B$ and $S = \{a, b\}$. If A is a 1-atom, then $xa, za \in E_R(G)$, $zb \in E_N(G)$; if A is a 2-atom, then $xa, xb, xz \in E_R(G)$.*

Let $E_0 \subset E_N(G)$ such that $E_0 \neq \emptyset$, and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$ and $y \in B$. If $xy \in E_0$, then A and B are called E_0*-fragments*. An E_0-fragment is called an E_0*-end-fragment* of G if it does not contain any other E_0-fragment of G as a proper subgraph. It is easy to see that any E_0-fragment of G contains such an E_0-end-fragment. Similarly, if $|A| = 2$, then A is called an E_0*-atom*.

Before we present our main results, we first list two additional known results that will be used in the proofs of our results. These results are due to Holton et al. [4] and give useful relationships between (un)removable edges and separating groups.

Theorem 2 ([4], Theorem 1) *Let G be a 3-connected graph of order at least six and $e \in E(G)$. Then e is unremovable if and only if there exists a separating group $(e, S; A, B)$ in G.*

Theorem 3 ([4], Theorem 2) *Let G be a 3-connected graph of order at least six and let $(xy, S; A, B)$ be a separating group. Then every edge joining S and $\{x, y\}$ is removable.*

2 Main Results

We verified the following two cases of Thomassen's conjecture. The proofs will be presented in the next section.

Theorem 4 *Let G be a 3-connected graph and let C be a longest cycle of G. If $|E(C) \cap E_R(G)| \leq 5$, then C has a chord.*

Theorem 5 *Let G be a 3-connected graph and let C be a longest cycle of G. If $|(E(G) - E(C)) \cap E_R(G)| \leq 7$, and there is no atom of G vertex-disjoint with C, then C has a chord.*

At first sight, the condition on the atom in the Theorem 5 seems a bit artificial. However, it is quite natural, as we explain next and as pointed out to us by one of the referees.

Assuming that Thomassen's conjecture is not true, consider a counterexample G with the number of edge minimum to Thomassen's conjecture, that is, there exists a longest cycle C of G without chords. So G is neither a hamiltonian graph nor a 3-regular graph by the results of [10]. Then the following holds.

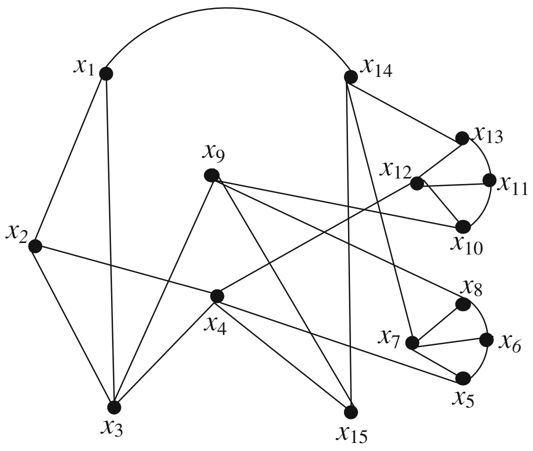
Observation 1 *There is no atom of G vertex-disjoint with C.*

Proof of Observation 1 To the contrary, assume that there exists an atom $A = \{x, z\}$ vertex-disjoint with C. We take the corresponding separating group $(xy, S; A, B)$ such that $x \in A$, $y \in B$, $S = \{a, b\}$, $B = G - xy - S - A$. We may assume $xa \in E(G)$. Then using Theorem 3 we have that $ax \in E_R(G)$. Let $G' = G \ominus ax$. Suppose that $d(a) \geq 4$. If $d(x) = 3$, then according to the definition of removable edges, we know that $G' = G - x + zy$. Since $\{x, z\} \cap V(C) = \emptyset$, we know that zy is not a chord of C in G'. Let C' be a longest cycle containing the edge zy in G'. We see that $|V(C)| > |V(C')|$ in G'; otherwise we use the path $P = yxz$ instead of the edge yz to obtain a cycle longer than C in G, a contradiction. Hence C is still a longest cycle in G'. So, by the minimality of G, C has a chord in G', but it is also a chord in G, a contradiction. If $d(x) = 4$, we have that $G' = G - ax$. It is easy to see that C is still a longest cycle in G'. So, by similar arguments we can deduce a contradiction. If $d(a) = 3$, we have to suppress a after removing ax. Since $z \notin V(C)$, by similar arguments we get that it has no effect on the length and the chordlessness of C. This completes the proof of Observation 1. □

It is not difficult to come up with examples of graphs that satisfy the conditions of Theorem 4 or 5 but not of previous results that verify special cases of Thomassen's conjecture. However, up to now we were unable to find an infinite family of graphs as examples for Theorem 5.

Example 1 Let $C = a_1x_1^1x_2^1 \ldots x_{k_1}^1a_2x_1^2x_2^2 \ldots x_{k_2}^2a_3x_1^3x_2^3 \ldots x_{k_3}^3 \ldots a_lx_1^l \ldots x_{k_l}^la_1$, with all $k_i \geq 3$. Let $E_1 = \{a_1x_1^2, a_1x_2^2, \ldots, a_1x_{k_2}^2\}$, $E_2 = \{a_2x_1^3, a_2x_2^3, \ldots, a_2x_{k_3}^3\}$, $\ldots, E_l = \{a_lx_1^1, a_lx_2^1, \ldots, a_lx_{k_1}^1\}$, with $l \geq 3$. Now we construct a graph G as follows. Let $V(G) = \{a\} \cup V(C)$ and $E(G) = E(C) \cup E_1 \cup \ldots \cup E_l \cup \{aa_1, aa_2, \ldots, aa_l\}$. Clearly, G is not hamiltonian: deleting $a_1, \ldots, a_l$ we obtain $l + 1$ components, so G is not 1-tough. So C is a longest cycle of G. Considering the orders of the components of $G - \{a_1, \ldots, a_l\}$, it is not difficult to argue that C is the cycle of length $|G| - 1$. It is also easy to check that G is a 3-connected graph. Next we show that $E(C) \subset E_N(G)$. Both $(a_1x_1^1, \{a_l, a_2\})$ and $(x_1^1x_2^1, \{a_l, a_2\})$ are separating pairs of G, so $a_1x_1^1, x_1^1x_2^1 \in E_N(G)$. For any other edge e on the segment between a_1 and a_2 on C, we have that $(e, \{a_1, a_l\})$ is a separating pair. By symmetry, this shows that $E(C) \subset E_N(G)$. Clearly, G satisfies the conditions of Theorem 4, and the longest cycle C of G has chords.

Example 2 It is easy to check that the graph G sketched in the figure is a 3-connected graph, and that $(x_2x_4, \{x_3, x_{14}\})$, $(x_3x_9, \{x_1, x_4\})$, $(x_7x_{14}, \{x_5, x_8\})$, $(x_6x_8, \{x_4, x_7\})$, $(x_{11}x_{13}, \{x_9, x_{12}\})$, and $(x_4x_{12}, \{x_{10}, x_{13}\})$ are separating pairs of G. Hence the edges $x_2x_4, x_3x_9, x_7x_{14}, x_6x_8, x_{11}x_{13}, x_4x_{12}$ are unremovable. Deleting the vertices x_4, x_9, x_{14}, we obtain four components with only one having exactly one vertex x_{15}. This shows that the cycle $C = x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13}x_{14}x_1$ is the unique longest cycle of G. There are six removable edges: $x_1x_3, x_5x_7, x_{10}x_{12}, x_4x_{15}, x_9x_{15}$, $x_{14}x_{15}$, outside cycle C. It is not difficult to check that G satisfies the conditions of Theorem 5. Obviously, C has chords.



3 Proofs

Before we present the proofs of the two main results of this paper, we introduce and prove the following technical lemma which is a key ingredient in the proofs of both main results.

Lemma 6 *Let G be a 3-connected graph of order at least 6, $E_0 \subset E_N(G)$ and $E_0 \neq \emptyset$. Let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$, $S = \{a, b\}$, $xy \in E_0$. If A is an E_0-end-fragment of G such that $|A| \geq 3$ and $(E(A) \cup [A, S]) \cap E_0 \neq \emptyset$, then the following conclusions hold:*

there exists a separating group $(x'y', S'; A', B')$ of G such that $x' \in A'$, $y' \in B'$, $x'y' \in E_0$, B' is a 1-atom, $A \cap B' = \{y'\}$, and $|A \cap B'| = |B' \cap S| = 1$, and $|A \cap S'| = |B \cap S'| = 1$.

Proof Since $(E(A) \cup [A, S]) \cap E_0 \neq \emptyset$, we have either $E(A) \cap E_0 \neq \emptyset$ or $[A, S] \cap E_0 \neq \emptyset$. We will distinguish the following cases and subcases to complete the proof.

Case 1 There exists an edge $uz \in E(A) \cap E_0$.

We consider the separating group $(uz, T; C, D)$ such that $u \in C$, $z \in D$. Then $u \in A \cap C$, $z \in A \cap D$. Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T)$$

$$X_2 = (A \cap T) \cup (S \cap T) \cup (D \cap S)$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T)$$

$$X_4 = (B \cap T) \cup (S \cap T) \cup (C \cap S)$$

Then $x \in A \cap C$, $x \in A \cap T$, or $x \in A \cap D$. Obviously, the subcases with $x \in A \cap C$ and $x \in A \cap D$ are symmetric. So, we only consider the subcases $x \in A \cap C$ and $x \in A \cap T$.

Subcase 1.1 $x \in A \cap C$.

Then either $y \in B \cap C$ or $B \cap T$. We next prove a number of claims to complete Subcase 1.1.

Claim 1 $y \notin B \cap T$.

Proof of Claim 1 To the contrary, assume that $y \in B \cap T$. Since $A \cap D \neq \emptyset$, X_2 is a vertex-cut of $G - uz$. So $|X_2| \geq 2$, and hence $|D \cap S| \geq |B \cap T| \geq 1$. Since $|S| = 2$, $|C \cap S| \leq 1$. Noticing that $|X_2| + |X_4| = |S| + |T| = 4$, we get $|X_4| \leq 2$. Since G is 3-connected, we have $B \cap C = \emptyset$. If $C \cap S = \emptyset$, then $C = A \cap C$ and it is easy to see that C is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. Hence $C \cap S \neq \emptyset$. Then it is easy to see that $|C \cap S| = |D \cap S| = 1 = |B \cap T|$ and $|S \cap T| = 0$. Then $B \cap D = \emptyset$, and so $B = \{y\}$, which contradicts that $|B| \geq 2$. This completes the proof of Claim 1.

Using Claim 1, we conclude that $y \in B \cap C$.

Claim 2 $A \cap D = \{z\}$ *and* $|D \cap S| = |B \cap T| = 1$.

Proof of Claim 2 Since $A \cap D \neq \emptyset$, X_2 is a vertex-cut of $G - uz$. Since G is 3-connected, we have $|X_2| \geq 2$. By a similar argument, we get $|X_4| \geq 2$. Noticing that $|X_2| + |X_4| = |S| + |T| = 4$, we obtain $|X_2| = |X_4| = 2$, and so $|S \cap C| = |A \cap T|$ and $|B \cap T| = |D \cap S|$. First, we claim $A \cap D = \{z\}$. Otherwise, $|A \cap D| \geq 2$. Let $A_1 = A \cap D$, $S_1 = X_2$, $B_1 = G - uz - S_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G. Since $uz \in E_0$, A_1 is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. Hence, as we claimed $A \cap D = \{z\}$.
Since $|D| \geq 2$ and D is a connected subgraph of G, we have $D \cap S \neq \emptyset \neq B \cap T$. If $|D \cap S| = |B \cap T| = 2$, then noticing that $|S| = |T| = 2$, it is easy to see that $|X_1| = 0$. Then $\{z, y\}$ would be a 2-cut of G, a contradiction. This completes the proof of Claim 2.

Using Claim 2, we conclude that $|S \cap T| \leq 1$.

Claim 3 $S \cap T = \emptyset$.

Proof of Claim 3 Supposing the contrary, we have $|S \cap T| = 1$, hence $|C \cap S| = |A \cap T| = 0$. If $x \neq u$, let $A_1 = A \cap C$, $S_1 = (S \cap T) \cup \{z\}$, and $B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G. Since $xy \in E_0$, A_1 is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. If $x = u$, then, from the assumption $|A| \geq 3$, we know $|A \cap C| \geq 2$, and we conclude that in this case $(S \cap T) \cup \{x\}$ is a 2-cut of G, a contradiction. This completes the proof of Claim 3.

Since $|S| = |T| = 2$, using Claims 2 and 3 we obtain the following.

Claim 4 $|A \cap T| = |S \cap C| = 1$.

We have $|X_3| = 2$, *and so* $B \cap D = \emptyset$. *It is easy to see that* D *is a 1-atom,* $|A \cap D| = |S \cap D| = 1$ *and* $|A \cap T| = |B \cap T| = 1$. *Let* $D = B'$, $T = S'$, $C = A'$ *and* $u = x'$, $z = y'$. *Then clearly the conclusion of the lemma holds, thus completing Subcase 1.1.*

Subcase 1.2 $x \in A \cap T$.

First, we claim that $y \notin B \cap T$. Otherwise, if $y \in B \cap T$, we have $S \cap T = \emptyset$. Since X_1 is a vertex-cut of $G - uz$, $|X_1| \geq 2$, and so $C \cap S \neq \emptyset$. Similarly, $D \cap S \neq \emptyset$. It is easy to see that now $|C \cap S| = |D \cap S| = 1$. Since G is a 3-connected graph, $|B \cap C| = |B \cap D| = 0$. So we have $B = \{y\}$, which contradicts $|B| \geq 2$. Therefore, as we claimed $y \notin B \cap T$. By symmetry, we may assume $y \in B \cap C$. Since $A \cap D \neq \emptyset$, X_2 is a vertex-cut of

$G - uz$, and so $|X_2| \geq 2$. By similar arguments, we obtain $|X_4| \geq 2$. Since $|X_2| + |X_4| = |S| + |T| = 4$, we have $|X_2| = |X_4| = 2$, and so $|S \cap C| = |A \cap T|$ and $|B \cap T| = |D \cap S|$. Since $|X_2| = 2$ implies $|A \cap D| = 1$, we have $A \cap D = \{z\}$. Since $|D| \geq 2$ and D is a connected subgraph of G, we have $D \cap S \neq \emptyset$. Since G is 3-connected, we have $B \cap D = \emptyset$. Noticing that $|A \cap T| \geq |D \cap S|$, we have $|D \cap S| = 1$, and so $|B \cap T| = 1$ and $|A \cap T| = 1$. Obviously, D is a 1-atom. Let $D = B', T = S', C = A'$ and $u = x', z = y'$. Then clearly the conclusion of the lemma holds, thus completing Subcase 1.2 and Case 1.

Case 2 There exists an edge $uz \in [A, S] \cap E_0$.

From Theorem 3, we conclude that $u \neq x$. Analogously to Case 1, we now consider the separating group $(uz, T; C, D)$ such that $u \in C, z \in D$. We use symmetry to obtain $u \in A \cap C$, and $z \in S \cap D$. We adopt the definitions of X_1, X_2, X_3, X_4 from Case 1. We prove the following claims to complete Case 2.

Claim 1 $x \notin A \cap C$.

Proof of Claim 1 To the contrary, assume that $x \in A \cap C$. Then we have $y \in B \cap C$ or $y \in B \cap T$.

First suppose $y \in B \cap C$. Since $B \cap C \neq \emptyset$, X_4 is a vertex-cut of $G - xy$, and so $|X_4| \geq 2$. Since $|X_2| + |X_4| = |S| + |T| = 4$, we have $|X_2| \leq 2$, and so $A \cap D = \emptyset$. We claim that $A \cap T \neq \emptyset$. Otherwise, we have $A = A \cap C$, and so $|A \cap C| \geq 3$. Since X_1 is a vertex-cut of $G - uz - xy$, we have $|X_1| \geq 1$. Noting that $D \cap S \neq \emptyset$, we get $|X_1| = |S \cap (C \cup T)| = 1$. We let $A_1 = A - \{u\}, S_1 = X_1 \cup \{u\}$ and $B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. So, as we claimed $A \cap T \neq \emptyset$, and hence $|T \cap (B \cup S)| \leq 1$. If $S \cap D = \{z\}$, then $|X_3| \leq 2$, and so $B \cap D = \emptyset$, hence $D = \{z\}$, which contradicts $|D| \geq 2$. Thus $|D \cap S| = 2$, and $|S \cap (C \cup T)| = 0$. Noting that $|A \cap T| \geq 1$, we get $|X_4| \leq 1$, which contradicts $|X_4| \geq 2$. Therefore, $y \notin B \cap C$.

We conclude that $y \in B \cap T$. Since X_1 is a vertex-cut of $G - xy - uz$, we have $|X_1| \geq 1$. First, we claim that $A \cap T = \emptyset$. Assume to the contrary that $A \cap T \neq \emptyset$. Since $y \in B \cap T$, we have $|A \cap T| = 1$ and $T \cap S = \emptyset$. First assume $C \cap S = \emptyset$. Since C is a connected subgraph of G, we have $B \cap C = \emptyset$. It is easy to see that C is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. So $C \cap S \neq \emptyset$. Since $z \in D \cap S$, we get $|C \cap S| = |D \cap S| = 1$. Here we have $|X_3| = |X_4| = 2$. Since G is a 3-connected graph, $B \cap C = \emptyset = D \cap B$, and so $B = \{y\}$, a contradiction. So we proved our claim that $A \cap T = \emptyset$. Since A is a connected subgraph of G, we have $A \cap D = \emptyset$. Then $|A \cap C| = |A| \geq 3$. We let $A_1 = A - \{u\}, S_1 = \{u\} \cup S - \{z\}$, and $B_1 = G - xy - A_1 - S_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. This completes the proof of Claim 1.

Claim 2 $x \notin A \cap T$.

Proof of Claim 2 To the contrary, assume that $x \in A \cap T$. We can apply similar arguments as in Subcase 1.2 to show that $y \notin B \cap T$. Hence we have $y \in B \cap C$ or $y \in B \cap D$.

First suppose $y \in B \cap C$. Since C is a connected subgraph of G, $C \cap S \neq \emptyset$. Then $D \cap S = \{z\}$. Since $B \cap C \neq \emptyset$, X_4 is a vertex-cut of $G - xy$, so $|X_4| \geq 2$. Since $|X_2| + |X_4| = |S| + |T| = 4$, we have $|X_2| \leq 2$, and so $A \cap D = \emptyset$. Since X_1 is a vertex-cut of $G - uz$, we have $|X_1| \geq 2$. So $|X_3| \leq 2$, and hence $B \cap D = \emptyset$. Now $D = D \cap S = \{z\}$, which contradicts $|D| \geq 2$. So, $y \notin B \cap C$.

We conclude that $y \in B \cap D$. Since X_1 is a vertex-cut of $G - uz$, we have $|X_1| \geq 2$. Similarly, we have $|X_3| \geq 2$. Since $|X_1| + |X_3| = |S| + |T| = 4$, we have $|X_1| = |X_3| = 2$. Then we obtain $|A \cap T| = |D \cap S|$ and $|C \cap S| = |B \cap T|$. First, we claim $A \cap C = \{u\}$. Otherwise, $|A \cap C| \geq 2$. We then let $A_1 = A \cap C$, $S_1 = X_1$ and $B_1 = G - uz - S_1 - A_1$. In this case $(uz, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. This proves our claim that $A \cap C = \{u\}$. Since C is a connected subgraph and $|C| \geq 2$, we have $|C \cap S| = |B \cap T| \geq 1$. Noticing that $|S| = 2$ and $z \in D \cap S$, we have $|C \cap S| = |B \cap T| = 1$. So $|D \cap S| = |A \cap T| = 1$. We have $|X_2| = 2$. From $|A| \geq 3$ we know that $A \cap D \neq \emptyset$, and so X_2 is a 2-cut of G, a contradiction. This completes the proof of Claim 2.

Using Claims 1 and 2 we conclude that $x \in A \cap D$.

Claim 3 $y \notin B \cap T$.

Proof of Claim 3 To the contrary, assume that $y \in B \cap T$. Since X_2 is a vertex-cut of $G - xy$, we have $|X_2| \geq 2$. From $|X_2| + |X_4| = |S| + |T| = 4$, we know $|X_4| \leq 2$. Thus $B \cap C = \emptyset$. By similar arguments, we get $B \cap D = \emptyset$. Hence we have $|B| = |B \cap T| \geq 2$. Noticing that $|T| = 2$, we have $A \cap T = \emptyset$, which contradicts that A is a connected subgraph of G. This completes the proof of Claim 3.

Using Claims 1, 2 and 3 we conclude that $x \in A \cap D$ and $y \in B \cap D$. Since X_2 is a vertex-cut of $G - xy$, we have $|X_2| \geq 2$. From $|X_2| + |X_4| = |S| + |T| = 4$, we know $|X_4| \leq 2$, and so $B \cap C = \emptyset$. We claim $C \cap S \neq \emptyset$. Assuming the contrary, it is easy to check that C is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. So, $C \cap S \neq \emptyset$. Noticing that $z \in D \cap S$, we get that $|C \cap S| = 1$ and $S \cap T = \emptyset$. Since X_1 is a vertex-cut of $G - uz$, we have $|X_1| \geq 2$. Similarly, we have $|X_3| \geq 2$. From $|X_1| + |X_3| = |S| + |T| = 4$, we deduce $|X_1| = |X_3| = 2$, and so $|C \cap S| = |B \cap T| = 1$ and $|A \cap T| = |D \cap S| = 1$. We claim $A \cap C = \{u\}$. Otherwise, if $|A \cap C| \geq 2$, we let $A_1 = A \cap C$, $S_1 = X_1$ and $B_1 = G - uz - X_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment, which contradicts that A is an E_0-end-fragment. So, $A \cap C = \{u\}$. Let $z = x'$, $u = y'$ and $C = B'$, $T = S'$, $D = A'$. Then clearly the conclusion of the lemma holds. This completes the proof of the lemma. □

We will also frequently use the following easy lemma.

Lemma 7 *Let C be a longest cycle of a 3-connected graph G, and assume that C passes through an edge of a triangle. Then C has a chord.*

Proof The statement is obvious if C passes through two edges of a triangle. So, let $C' = x_1x_2x_3x_1$ be a triangle and suppose C passes through exactly one edge of C'. We may assume that $x_1x_2 \in E(C)$. Since C is a longest cycle of G, we have that $x_3 \in V(C)$; otherwise, there is a longer cycle C'' with $V(C'') = V(C) \cup \{x_3\}$, and $E(C'') = (E(C) - x_1x_2) \cup \{x_1x_3, x_3x_2\}$, a contradiction. Clearly, C has chords x_1x_3 and x_3x_2. □

Proof of Theorem 4

Let C be a longest cycle of G and let $E_0 = E(C) \cap E_N(G)$, $E_1 = E(C) \cap E_R(G)$. By Menger's Theorem, the length of C is at least 6; otherwise we find three internally-disjoint paths from a vertex outside C to three distinct vertices on C, and we can easily construct a longer cycle, a contradiction. So the length of C is at least 6, implying that $E_0 \neq \emptyset$. Consider a separating group $(xy, S; A, B)$ of G such that $x \in A$, $y \in B$, $S = \{a, b\}$, $xy \in E_0$. From $|E_1| \leq 5$ we deduce that either $|(E(A) \cup [A, S]) \cap E_1| \leq 2$ or $|(E(B) \cup [B, S]) \cap E_1| \leq 2$. Without loss of generality, we may assume that $|(E(A) \cup [A, S]) \cap E_1| \leq 2$. Since A is an E_0-fragment of G and every E_0-fragment contains an E_0-end-fragment as its subgraph, we may assume that A is an E_0-end-fragment with $|(E(A) \cup [A, S]) \cap E_1| \leq 2$.

We distinguish two cases.

Case 1 $|A| = 2$.

Let $A = \{x, z\}$.

If A is a 1-atom, we may assume that $xa \in E(G)$, $xb \notin E(G)$. Since $xy \in E(C)$, either $xz \in E(C)$ or $xa \in E(C)$. Since xz, xa belong to a triangle $xzax$, using Lemma 7 we conclude that C has a chord.

If A is a 2-atom, then $\{xa, xb, xz\} \cap E(C) \neq \emptyset$. Since all of $\{xa, xb, xz\}$ are on triangles $zxaz$, $xzbx$, using Lemma 7 we conclude that C has a chord. This completes the proof of Case 1.

Case 2 $|A| \geq 3$.

First we claim that $(E(A) \cup [A, S]) \cap E_0 \neq \emptyset$. Assuming the contrary, by the assumption that $|(E(A) \cup [A, S]) \cap E_1| \leq 2$ we know that there is at most one edge on C the end-vertices of which lie in A. We distinguish the following subcases according to the value of $|(E(A) \cup [A, S]) \cap E_1|$ and derive contradictions in all subcases.

Subcase 2.1 $|(E(A) \cup [A, S]) \cap E_1| = 2$.

Then $E(C) \cap E(A) = \{xx_1\}$ for some vertex x_1. We may assume that $E(C) \cap (E(A) \cup [A, S]) = \{xx_1, x_1a\}$. If $A - xx_1$ is a connected subgraph, then xx_1 is on a cycle in A and it is easy to find a longer cycle C' of G than C, a contradiction. So we may assume that $A - xx_1$ is a disconnected graph with two components. Let $A - xx_1 = A_1 \cup A_2$ such that $x \in A_1$, and $x_1 \in A_2$, where A_1 and A_2 are two disjoint connected subgraphs of G. If $|A_2| \geq 2$, then $(xx_1, \{a, b\})$ is a separating pair of G, which contradicts $xx_1 \in E_R(G)$. So $|A_2| = 1$. Then $A_2 = \{x_1\}$, and so $x_1a, x_1b \in E(G)$.

If $b \in V(C)$, then since $x_1a \in E(C)$, $x_1b \notin E(C)$. Then x_1b is a chord of C.

Next we assume $b \notin V(C)$. We claim that $\Gamma_G(b) \cap A_1 - \{x\} \neq \emptyset$; otherwise, $\{x, a\}$ is a 2-cut of G, a contradiction. So $\Gamma_G(b) \cap A_1 - \{x\} \neq \emptyset$. Let $x_3 \in \Gamma_G(b) \cap A_1$. Then in A_1 there is a path P_1 from x to x_3. We use the path $P = P_1 \cup \{x_3b, bx_1\}$ instead of the edge xx_1 to obtain a longer cycle than C, a contradiction.

Subcase 2.2 $|(E(A) \cup [A, S]) \cap E_1| = 1$.

Let $xz \in E(C) \cap (E(A) \cup [A, S])$. Then $z \neq y$. If $z \in A$, then since C is a cycle, we have $(E(A) \cup [A, S]) \cap E_0 \neq \emptyset$, which contradicts the assumption. So we may assume that $z \in S = \{a, b\}$. Without loss of generality, let $z = a$. Let x' be a neighbor of a on C and $x' \neq x$. Since $(E(A) \cup [A, S]) \cap E_0 = \emptyset$, we have $x' \notin A$. Noticing that C is a cycle, we know that $V(C) \cap V(A) = \{x\}$. We claim that $\Gamma_G(a) \cap V(A) - \{x\} \neq \emptyset$; otherwise, $\{x, b\}$ is a 2-cut of G, a contradiction. Since A is a connected subgraph and

$|A| \geq 3$, we can find a path P_1 from x to a in $A \cup \{a\}$. We use P_1 instead of the edge xa to find a longer cycle than C, a contradiction.

Subcase 2.3 $|(E(A) \cup [A, S]) \cap E_1| = 0$.

This clearly contradicts our assumption that $(E(A) \cup [A, S]) \cap E_0 = \emptyset$.

These subcases confirm our claim that $(E(A) \cup [A, S]) \cap E_0 \neq \emptyset$. Using Lemma 6 we conclude that there exists another separating group $(x'y', S'; A', B')$ of G such that $x' \in A'$, $y' \in B'$, $x'y' \in E_0$, B' is a 1-atom, $A \cap B' = \{y'\}$, and $|A \cap B'| = |B' \cap S| = 1$, and $|A \cap S'| = |B \cap S'| = 1$. Let $A \cap S' = \{a'\}$, $B' \cap S = \{z'\}$. Since $x'y' \in E(C)$, $\{y'a', y'z'\} \cap E(C) \neq \emptyset$. However, $y'a'$ and $y'z'$ belong to the triangle $y'a'z'y'$. Using Lemma 7 we conclude that C has a chord. This completes the proof of Case 2 and Theorem 4. □

Proof of Theorem 5

Let $E_0 = (E(G) - E(C)) \cap E_N(G)$, $E_1 = (E(G) - E(C)) \cap E_R(G)$. Then $|E_1| \leq 7$. We complete the proof by contradiction. We know that G is neither a hamiltonian graph nor a 3-regular graph by the results of [10].

It is easy to verify that the result holds if $|G| \leq 9$. So we assume that $|G| \geq 10$. Since C is a longest cycle of G without a chord and C is not a Hamilton cycle, we have that $|E(G) - E(C)| \geq 8$. And so $E_0 \neq \emptyset$. Consider a separating group $(xy, S; A, B)$ of G such that $x \in A$, $y \in B$, $S = \{a, b\}$, and $xy \in E_0$. Since $|E_1| \leq 7$ we know that either $|(E(A) \cup [A, S]) \cap E_1| \leq 3$ or $|(E(B) \cup [B, S]) \cap E_1| \leq 3$. Without loss of generality, we may assume that $|(E(A) \cup [A, S]) \cap E_1| \leq 3$. By similar arguments as in the proof of Theorem 4, we may assume that A is an E_0-end-fragment. We distinguish the following two cases according to the value of $|A|$ to complete the proof.

Case 1 $|A| = 2$.

Let $A = \{x, z\}$. Since there is no atom vertex-disjoint with C, we have $\{x, z\} \cap V(C) \neq \emptyset$.

First suppose that $x \in V(C)$. Since $xy \notin E(C)$, obviously C passes through an edge of a triangle. Using Lemma 7, we conclude that C has a chord, a contradiction.

So, $x \notin V(C)$. Then $z \in V(C)$, and so $za, zb \in E(C)$. Since za is an edge of a triangle, using Lemma 7 we conclude that C has a chord, a contradiction. This completes Case 1.

Case 2 $|A| \geq 3$.

Since $|(E(A) \cup [A, S]) \cap E_1| \leq 3$, it is easy to see that $|(E(A) \cup [A, S]) \cap E_0| \neq \emptyset$. Then by Lemma 6 there exists a separating group $(x'y', S'; A', B')$ of G such that $x' \in A'$, $y' \in B'$, $x'y' \in E_0$, B' is a 1-atom, $A \cap B' = \{y'\}$, $|A \cap B'| = |B' \cap S| = 1$, and $|A \cap S'| = |B \cap S'| = 1$. We can apply similar arguments as in Case 1 to complete this case. □

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