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High gain adaptive control revisited: first and second order case

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Abstract

Adaptive controllers based on high gain feedback suffer from lack of robustness with respect to bounded disturbances. Existing modifications prevent the feedback gain from drifting away, but at the same time introduce solutions that, even in the absence of disturbances, do not converge to zero. In this paper we investigate a further modification that maintains the robustness and rules out undesirable solutions when disturbances are not present. For clarity of presentation and because of space limitations we restrict ourselves to first and second order systems.

1 Introduction and problem statement

We consider SISO systems described by

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u. \tag{1}$$

Here $p(\xi), q(\xi)$ are polynomials with real coefficients. The assumptions that we make are:

- The polynomial $p(\xi)$ is monic of degree n and $q(\xi)$ has degree $n-1$.
- The system defined by (1) is controllable, i.e., $p(\xi)$ and $q(\xi)$ have no nontrivial common factors (see [1, Chapter 5]).
- The system is minimum phase, i.e., $q(\xi)$ has all its roots in the open left half plane.
- The high-frequency gain, q_{n-1}, is positive.

Otherwise the coefficients of the polynomials $p(\xi), q(\xi)$ are unknown.

It is a well established fact that the system (1) can be stabilized by output feedback $u = -ky$, provided that k is sufficiently large. To ensure that k grows beyond the bound after which the system is asymptotically stable, k is driven by the truncated $\mathcal{L}_2$ norm of the output via $\frac{d}{dt}k = y^2$. See [2, Chapter 6] for a self-contained analysis of the resulting adaptive control system.

An annoying feature of the adaptation law $\frac{d}{dt}k = y^2$ is that the slightest error in the output, e.g. caused by measurement noise, yields k unbounded. To avoid this it has been proposed in the literature, see e.g. [3, chapter 4], [4] and the references therein, to incorporate a damping term in the update law:

$$\frac{d}{dt}k = -\sigma k + y^2 \tag{2}$$

Here σ is a (small) positive damping term. This is usually referred to as the sigma modification. One effect of this modification is clear: bounded y yields bounded k. Whereas the unmodified algorithm has an infinite number of equilibria, namely $(y, k) = (0, \bar{k})$ with $\bar{k}$ arbitrary, the sigma modification reduces the set of equilibria to at most three. For instance in the first order case, if the system is unstable, there are three equilibria. Unfortunately, two of these equilibria correspond to nonzero output so that even in the absence of measurement and control errors the output may not converge to zero. In particular for the case that the controlled system is not asymptotically stable for $k = 0$, undesired behavior may result. Indeed, for as long as y^2 is large, k will grow until it reaches a value for which the system stabilizes. Consequently y^2 will start to decrease. As soon as $-\sigma k$ dominates y^2, k will decrease thus destabilizing the system so that y will grow again. For a detailed study of the nonlinear behavior of the system (1,2) the reader is referred to [5]. We conclude that there are two problems caused by the sigma modification. Firstly, the damping factor not only prevents k from growing unbounded, it is also responsible for a destabilizing effect as soon as y^2 becomes small. Secondly, the sigma modification introduces equilibria for which the output is nonzero.

The aim of the present paper is to propose a further modification that eliminates these two drawbacks while maintaining the robustness properties with respect to measurement errors.

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The first step is to remove the destabilizing effect. This is easily achieved by preventing k to decrease. So instead of (2) we propose

$$\frac{d}{dt}k = \max(0, -\sigma k + y^2) \tag{3}$$

The effect of the maximization is clear. Since $\frac{d}{dt}k$ is always non-negative, k can never return to a lower, possibly destabilizing, value. Admitted, k can, due to transients become larger than required or desired. The above modification disables the possibility of recovering from that effect. At this stage we accept that as an inevitable price that has to be paid.

Unfortunately, (3) introduces new difficulties. An example illustrates this.

1.1 EXAMPLE
Let the system be given by

$$\frac{d}{dt}y = 2y + u \tag{4}$$

$$\frac{d}{dt}k = \max(0, -k + y^2) \tag{5}$$

$$u = -ky \tag{6}$$

Obviously $(y, k) = (\bar{y}, 2)$ is an equilibrium for all $\bar{y}$ with $|\bar{y}| \leq 1$.

Notice that $k = 2$ is exactly the value for which the system in Example 1.1 is marginally stable. Let us use the example to find a way out. A first idea could be to add a time varying part to the control law to disable equilibria for which the output is nonzero but which does not influence the average behavior of the system.

1.2 EXAMPLE (EXAMPLE 1.1 CONTINUED)
Instead of $u = -ky$ we could take $u = -(k + \sin t)y$. The effect of the periodic part of the gain is that the only equilibrium now is $(y, k) = (0, \bar{k})$. Also, one may hope that the periodic gain at $k = 2$ yields alternately stable and unstable behavior so that the resulting output will increase k beyond $k = 2$. However, this is not true. For $k = 2$ we can find periodic solution within the dead zone for k:

$$(y(t), k(t)) = (ce^{-\cos(t)}, 2) \tag{7}$$

is a solution for all c sufficiently small.

Let us take a closer look at the effect of a time varying gain in the feedback.

1.3 EXAMPLE (EXAMPLE 1.2 CONTINUED)
Let us leave the time-varying part of the feedback gain unspecified and consider the marginally stable situation, $k = 2$,

$$u(t) = (-2 + f(t))y(t) \tag{8}$$

The resulting output trajectory is given by

$$y(t) = y(0)e^{\int_0^t f(\tau)d\tau}, \tag{9}$$

whereas for other constant values of k the output is given by

$$y(t) = y(0)e^{(2-k)t + \int_0^t f(\tau)d\tau}, \tag{10}$$

It is natural to require that y converges to zero for $k > 2$ and that y diverges for $k < 2$. This is achieved if f is such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt = 0 \tag{11}$$

Moreover, f should be such that for $k = 2$ periodic solutions are not possible. This is guaranteed if we choose f such that in addition to (11) there holds

$$\limsup_{T>0} \int_0^T f(t)dt = \infty \quad \text{and} \quad \liminf_{T>0} \int_0^T f(t)dt = -\infty \tag{12}$$

In fact the second requirement in (12) is an inevitable consequence of (11) and the first property in (12). Intuitively speaking the effect of a time-varying feedback gain satisfying (11,12) is as follows. If k would converge to a value for which the closed-loop system is only marginally stable, then the variations of the feedback gain are such that it alternately stabilizes and destabilizes the system. The destabilization is strong enough to eventually yield unbounded output so that the dead-zone defined by (5) becomes ineffective contradicting the convergence of k. On the other hand, a limit of k that would yield an unstable closed-loop system cannot be stabilized by the time-varying gain due to the fact that its average value is zero. Finally, if k converges to a value that yields an asymptotically stable closed-loop system, then for the very same reason the time varying gain cannot destabilize the system. All of this is formalized in the following theorem.

1.4 THEOREM
Consider the adaptive system

$$\frac{d}{dt}y = ay + u \quad \frac{d}{dt}k = \max(0, -\sigma k + y^2)$$
$$u(t) = (-k(t) + f(t))y(t), \tag{13}$$

where $\sigma > 0$ and f is a function for which (11,12) hold. Then, for every (y_0, k_0) the solution of (1.4) with $(y(0), k(0)) = (y_0, k_0)$ exists for all $t \geq 0$ and is unique. Moreover,

$$\lim_{t\to\infty} y(t) = 0 \quad \lim_{t\to\infty} k(t) = k_\infty \tag{14}$$

PROOF Let $[0, t')$ be the maximal interval of existence of $(y(t), k(t))$. Since k is monotonically non-decreasing, it either converges or it diverges to infinity on $[0, t')$. Let

$t_1 \in [0, t')$ be such that $a + 1 + \frac{1}{t}\int_0^t f(\tau) - k(\tau) d\tau \leq -1$ for all $t > t'$. Since k tends to infinity and f satisfies (11) such a t_1 exists. On the interval of existence, the output is given by

$$|y(t)| = |y_0 e^{t(a + \frac{1}{t}\int_0^t -k(\tau) + f(\tau) d\tau)}| \leq |y_0| e^{-t} \quad t > t_1 \quad (15)$$

From (15) and the adaptation law for k it follows immediately that k is bounded on $[0, t')$. This contradicts the assumption that k grows without bound. The conclusion is that $k(t)$ converges to a finite limit, k_∞, say.

Assume now that $t' < \infty$. Since we have already established that k is bounded on $[0, t')$, we conclude that y is unbounded on $[0, t')$. But then, since $|\frac{d}{dt} y| \leq c|y|$ for some positive constant c, the update law for k implies that k cannot converge to a finite limit. Therefore $t' = \infty$.

There are two possibilities for k_∞, each of which we investigate below.

1. $a - k_\infty \geq 0$.

In this case, since $a - k(t) \geq 0$ for all t, we would have

$$\limsup_{t \geq 0} |y(t)| \geq |y_0| e^{\limsup_{t \geq 0} \int_0^t f(\tau) d\tau} = \infty \quad \text{(by (12))} \quad (16)$$

As in part 1. this implies that that $k(t)$ cannot converge to a finite limit.

2. $a - k_\infty < 0$.

Choose $t_1 \in [0, \infty)$ such that for $t \geq t_1$ $a - \frac{1}{t}\int_{t_1}^t k(\tau) + \frac{1}{t}\int_{t_1}^t f(\tau) d\tau \leq -\epsilon$, for some positive ϵ. Then

$$|y(t)| \leq |y(t_1)| e^{t(-\epsilon + \frac{1}{t}\int_{t_1}^t f(\tau) d\tau)} \quad (17)$$

This shows that $|y(t)|$ converges to zero, as claimed. □

1.5 EXAMPLE

We mention two examples of functions that satisfy (11,12).

$$f_1(t) = \sin\sqrt{t} \quad f_2(t) = \begin{cases} 1 & (2k)^2 \leq t < (2k+1)^2 \\ -1 & (2k+1)^2 \leq t < (2k+2)^2 \end{cases} \quad k \in \mathbb{N} \quad (18)$$

One would expect that the idea for the first order case carries over to the general case. However, this is not quite obvious and in fact nor true either. One reason why the time-varying gain works for the first order case is that it acts symmetrically about the nominal gain induced by k. In the higher case this is not true and therefore it could happen that a stable period is just compensated by an unstable period thus yielding a bounded solution for y within the dead zone for k. In fact it is possible to construct examples for which both $-k + 1$ and $-k - 1$ yield closed-loop systems that have all their poles in the closed-left half plane and not all in the open left half plane.

1.6 EXAMPLE

Let $p(\xi) = \xi^4 + 3/2\xi^3 + 8\xi^2 + 27/2\xi + 11$ and $q(\xi) = 3/2\xi^3 + 3\xi^2 + 27/2/xi + 7$. Then $p(\xi) + q(\xi) = (\xi + 2)(\xi + 3)(\xi + 3i)(\xi - 3i)$ and $p(\xi) - q(\xi) = (\xi - 2i)(\xi + 2i)(\xi - i)(\xi + i)$. Notice that $q(\xi)$ is a Hurwitz polynomial.

Example 1.6 shows that we may end up in a situation where for $u = -(\bar{k} + 1)y$ and $u = -(\bar{k} - 1)y$ allow bounded solutions within the dead zone for k, thus preventing y from going to zero. A direct generalization of the first order case is therefore not possible. A first step towards a better understanding of this phenomenon is therefore to study the second order case. In what follows $p(\xi)$ is a monic polynomial of degree two and $q(\xi)$ is a Hurwitz polynomial of degree one. To simplify the analysis we restrict our attention to feedback laws of the form (1.4) for which f is piecewise constant and takes on the values plus and minus one only.

Since the switching times at which f changes sign are of crucial importance, we discuss the conditions on f first. The function f should be piecewise constant and for all t $|f(t)| = 1$. Denote the time instants at which f switches sign by a_k.

$$f(t) = \begin{cases} 1 & a_{2k} \leq t < a_{2k+1} \\ -1 & a_{2k+1} \leq t < a_{2k+2} \end{cases} \quad (19)$$

The sequence $\{a_k\}$ should grow sufficiently fast so as to provide time for the system to either stabilize or destabilize and prevent bounded solutions that are bounded from below by a positive constant. Let Ω be a compact subset of $\mathbb{R}$ that does not contain zero, and let $\alpha_\ell \in \Omega \cup \{0\}$ for all integers ℓ and such that $\alpha_\ell \neq 0$ for an infinite number of ℓs. Then we require that the sequence defined by

$$b_k = \sum_{\ell=0}^{k} \alpha_\ell (a_{k+1} - a_k) \quad (20)$$

contains no bounded subsequence. It is not difficult to check that an example of a sequence $\{a_k\}$ that has this property is $a_k = k!$.

1.7 THEOREM

Consider the adaptive system

$$\frac{d^2}{dt^2} y + p_1 \frac{d}{dt} y + p_0 y = q_1 \frac{d}{dt} u + q_0 u$$

$$\frac{d}{dt} k = \max(0, -\sigma k + y^2) \quad u(t) = (-k(t) + f(t)) y(t),$$

where $\sigma > 0$ and f is a function for which (19,20) hold. Moreover $q_1 > 0$ and $q_0 > 0$ and $p(\xi)$ and $q(\xi)$ are coprime. Then, for every (y_0, k_0) the solution of (1.4) with $(y(0), k(0)) = (y_0, k_0)$ exists for all $t \geq 0$ and is unique. Moreover,

$$\lim_{t \to \infty} y(t) = 0 \quad \lim_{t \to \infty} k(t) = k_\infty \quad (21)$$

PROOF The proof of uniqueness and existence of solution is completely analogous to the case of Theorem 1.4 and is therefore skipped. First we notice that k remains bounded. For, assume the contrary, then k grows beyond the value after which the frozen system is asymptotically stable. In exactly the same way as in [6] and [2, Section 6.4] a contradiction is obtained. Denote the limit of $k(t)$ as t tends to ∞ by $\bar{k}$. Now three situation may occur:

1. $p(\xi) + (\bar{k} - 1)q(\xi)$ and therefore also $p(\xi) + (\bar{k} + 1)q(\xi)$ are Hurwitz.
2. $p(\xi) + (\bar{k} + 1)q(\xi)$ and $p(\xi) - (\bar{k} + 1)q(\xi)$ is not Hurwitz.
3. $p(\xi) + (\bar{k} - 1)q(\xi)$ is not Hurwitz and $p(\xi) + (\bar{k} + 1)q(\xi)$ is Hurwitz.

The first two cases are easy. In the first case, due to the property of f that it is constant during arbitrarily long intervals of time, we conclude that the overall system is asymptotically stable. Similarly, in the second case, we conclude that the system is unstable. In it is not difficult to prove that the unstable modes corresponding to both dynamic regimes are excited persistently, and therefore the output is unbounded. Since it is the output of a linear system with a bounded feedback gain, the output is well outside the dead zone for k for time intervals of positive lengths, $k(t)$ could not have converged. This yields a contradiction, so the second case cannot occur.

The third case is less easy to analyze. The limiting system dynamics is now alternatingly stable and unstable. We show that either the output converges to zero or it is unbounded. In the first case we have proved the theorem and in the latter case we have obtained a contradiction.

Assume that $y(t)$ does not converge to zero. Because $p(\xi)$ and $q(\xi)$ are co-prime, so are $p(\xi)$ and $p(\xi) + \ell q(\xi)$ for every value of ℓ (in state space terms: observability is preserved under static output feedback). Since by assumption y does not converge to zero, its behavior is eventually captured by the limiting dynamics of the controlled system. This limiting dynamics is as follows. On each interval on which $f(t)$ is constant, the output behaves exponentially. Since the intervals on which f is constant become arbitrarily large, we may neglect the transients towards this exponential behavior. As a consequence the 'tail' of the output is a product of exponentials corresponding to the roots of $p(\xi) + (\bar{k} + 1)q(\xi)$ and $p(\xi) + (\bar{k} - 1)q(\xi)$ respectively. At switching times of f:

$$c \exp\Big(\sum_{\ell=k}^{K} \alpha_\ell (a_{\ell+1} - a_\ell)\Big) \leq$$
$$|y(a_K)| \leq C \exp\Big(\sum_{\ell=k}^{K} \beta_\ell (a_{\ell+1} - a_\ell)\Big) \qquad (22)$$

for suitable positive constants c and C and k sufficiently large. Furthermore α_ℓ and β_ℓ are roots of $p(\xi) + (\bar{k} + 1)q(\xi)$ or $p(\xi) + (\bar{k} - 1)q(\xi)$. Since at least one of these two polynomials has no roots on the imaginary axis, and by the growth condition (20) on $a_{\ell+1} - a_\ell$ we conclude that either the upper bound in (1) converges to zero or the lower bound is unbounded. As remarked earlier the latter leads to a contradiction. This establishes the proof. □

2 Simulations

To gain some insight in the various modifications, we have simulated six different situations for the system $\frac{d}{dt} y = y + u + d$. Here d is a constant disturbance. It is clear that by applying $u = -ky$, the marginally stabilizing value for k equals unity. In the six figures below we have plotted the behavior of k only. The initial condition are $y(0) = 1$ and $k(0) = 0$. We number the six simulations from left to right and from top to bottom. The various algorithms and conditions are:

1. $\dfrac{d}{dt}k = y^2$, no disturbance. It is clear that k converges to a stabilizing value.
2. Same situation, but now with $d = 1$. The plot strongly suggests that k indeed drifts away.
3. $\frac{d}{dt}k = -0.1k + y^2$ and $d = 1$. The plot shows that k now remains bounded.
4. Same situations, but now without disturbance, $d = 0$. After the output has become small enough, k tries return to its equilibrium value which is destabilizing, so it starts growing again. And so on. To illustrate this we have simulated this for a longer period.
5. $\frac{d}{dt}k = \max(0, -0.1k + y^2)$, $u = -(k + \sin\sqrt{t})y$, without disturbance. It is clear that k grows well beyond unity and eventually enters a dead zone.
6. Same situations, but now with $d = 1$. This and the previous plot shows that our modification, at least for the first order case, combines the best of two worlds. It is robust with respect to (constant) disturbances and it still works well if there are no disturbances.

3 Conclusions

For first and second order systems we have derived a further modification of the sigma modification for high gain adaptive control. Although this limited study shows quite clearly on what kind principles our modification is based, the difficulties that arise when trying to generalize the second order strategy to systems of arbitrary order are severe. The main problem that has to be overcome is that the algorithm may get stuck at values for $k(t)$ for which $p(\xi)+(k+f(t))q(\xi)$ has all its eigenvalues in the closed left-half plane and some on the imaginary axis. In principle, at the cost of increasing complexity, this problem may be circumvented. Partial results towards a more systematic approach have been obtained and will be reported in the near future.

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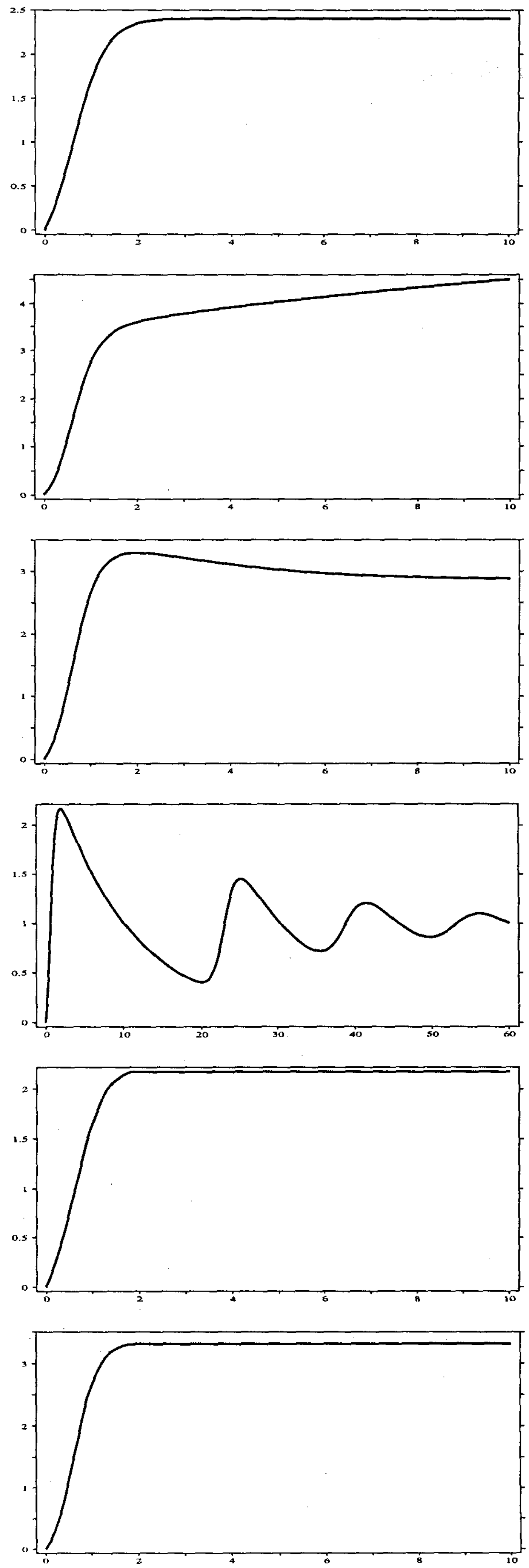

Figure 1: Top to bottom: Simulations 1 — 6