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Indirect Field Oriented Control of Induction Motors is Robustly Globally Stable *

Paul de Wit†, Romeo Ortega**, Iven Mareels‡

†University of Twente
Faculty of Mechanical Engineering
PO Box 217, 7500 AE ENSCHEDE
p.a.s.dewit@wb.utwente.nl
THE NETHERLANDS

**Université de Compiègne
URA C.N.R.S. 817 , BP 649
60206 Compiègne
rortega@hds.univ-compiegne.fr
FRANCE

‡Australian National University
Department of Systems Engineering
Canberra, ACT 0200
AUSTRALIA

Abstract

It is generally accepted that field orientation, in one of its many forms, is the most promising control method for high dynamic performance AC drives. In particular, for induction motors indirect field oriented control is a simple and highly reliable scheme which has become an industry standard. In spite of its widespread popularity no rigorous stability proof for this controller was available in the literature. In a recent paper [15] we have shown that, in speed regulation tasks with constant load torque and current fed machines, indirect field oriented control is globally asymptotically stable provided the motor rotor resistance is exactly known. It is well known that this parameter is subject to significant changes during the machine operation, hence the question of the robustness of this stability result remained to be established. In this paper we provide some answers to this question. First, we give necessary and sufficient conditions for uniqueness of the equilibrium point of the (nonlinear) closed loop, which interestingly enough allow for a 200% error in the rotor resistance estimate. Then, we give conditions on the motor and controller parameters, and the speed and rotor flux norm reference values that insure either global boundedness of all solutions, or (global or local) asymptotic stability or instability of the equilibrium. The analysis is carried out using classical Lyapunov stability theory and some basic input-output theory.

1. Problem formuation

We carry out in this paper the stability analysis of an indirect field oriented controller (FOC) that regulates the velocity and the rotor flux norm of a current fed induction motor in the presence of an unknown constant load torque and rotor resistance uncertainty. For further details and motivation of induction motors and FOC the reader is refered to [2], [9], in the electrical machines literature, and to [1], [15], [17] in the control journals.

The dynamic model of the current fed induction motor in its simplest formulation expresses the rotor flux and the stator currents in a reference frame rotating at the rotor angular speed[1]

$$\dot{x} = -R_r x + R_r u \tag{1}$$
$$\dot{y} = \tau - \tau_L \tag{2}$$
$$\tau = u^T J x \tag{3}$$

where

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{R}^2$ – rotor flux vector

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{R}^2$ – stator currents

y – rotor velocity

τ – generated torque

$R_r > 0$ – rotor resistance

τ_L – load torque

and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

To simplify the expressions below, and without loss of generality for the purposes of this study, all motor parameters have been set to unity except the rotor resistance and the load torque, which are assumed constant but unknown.

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[1]See [12] for the derivation of this model from the classical textbook models, and the transformations required to obtain other representations studied in the control literature.

In indirect FOC the stator currents are chosen as

$$u = e^{J\rho_d} \begin{bmatrix} \beta \\ \frac{\tau_d}{\beta} \end{bmatrix} \qquad (4)$$

where $\beta > 0$ is the constant desired value of the rotor flux norm, ρ_d – which may be interpreted as the angle of the desired rotor flux (or an estimate of the actual rotor flux angle)– is given by

$$\dot{\rho}_d = \frac{\hat{R}_r}{\beta^2}\tau_d \qquad (5)$$

with $\hat{R}_r > 0$ the estimated rotor resistance, and τ_d the desired torque. In velocity regulation applications the latter is typically defined via a PI velocity loop as

$$\tau_d = -\left(K_P + \frac{K_I}{p}\right)(y - y_d) \qquad (6)$$

where y_d is the desired velocity, which we assume constant, $p = \frac{d}{dt}$ is the derivative operator, and $K_P, K_I \geq 0$ are the PI tuning gains.

In summary, the closed loop is described by a fourth order nonlinear autonomous system, whose block diagram description is given in Fig. 1.

The problems we solve in this paper are formulated as follows:

Stability analysis of indirect FOC
Given the motor model (1),(2),(3) in closed loop with the indirect FOC (4), (5), (6) find sufficient conditions on the motor parameters R_r, τ_L, the controller parameters $\hat{R}_r, K_P, K_I$, and the reference values y_d, β such that:

1. All solutions of the system are globally bounded;

2. The system is (globally or locally) asymptotically stable. That is, such that

$$\lim_{t\to\infty} y = y_d, \quad \lim_{t\to\infty} |x| = \bar{\beta}$$

where $|\cdot|$ is the Euclidean norm, and $\bar{\beta}$ denotes a constant value for the rotor flux norm;

3. The system has unstable equilibria.

Discussion

• It is important to underscore the fact that x is a vector quantity. This model should not be confused with the machine model in decoupling control, e.g., (7.75), (7.82), (2.78) of [2], which describes the asymptotic behaviour of the motor in closed-loop with an ideal direct FOC.

• As discussed in [15] one way of explaining the rationale underlying indirect FOC is to compare it with direct FOC, which is described by

$$u = e^{J\rho} \begin{bmatrix} \beta \\ \frac{\tau_d}{\beta} \end{bmatrix}$$

with $\rho = \arctan(\frac{x_2}{x_1})$ the rotor flux angle. In indirect FOC we simply replace ρ by ρ_d. Further, notice from (1)-(3) that, whenever $|x| \neq 0$, ρ satisfies

$$\dot{\rho} = \frac{R_r}{|x|^2}\tau$$

which motivates the choice of ρ_d given in (5). That is, (5) follows replacing now $|x|$ and τ by their desired values β, τ_d respectively, and replacing R_r by its estimate. In the electrical machines literature ρ_d is sometimes refered to as slip angle, hence (5) shows that the desired torque is proportional to the slip speed.

• It is interesting to remark that indirect FOC is obtained as a particular case of the passivity-based controller first proposed in [12] when the stator current dynamics are neglected.

• For the known parameter case, GAS of indirect FOC has been proved in [15].

The remaining of the paper is organized as follows. In section 2 we then represent the closed loop as the feedback interconnection of a linear time invariant (LTI) system and a sector bounded nonlinearity. This allows us, invoking the small gain theorem [5], to derive simple conditions of global boundedness. Necessary and sufficient conditions for uniqueness of equilibrium are given in section 3. Finally, conditions for local and global asymptotic stability are derived using Lyapunov techniques in sections 4 and 5 respectively.

2. Global boundedness: Input-output approach

In this section we use input-output techniques to derive sufficient conditions that ensure all signals of the closed loop remain uniformly bounded. The result is established showing that the closed loop system can be viewed as a feedback interconnection of an LTI system and a nonlinear sector bounded gain whose inputs are in $\mathcal{L}_\infty$, where $\mathcal{L}_\infty$ denotes the space of (essentially) bounded signals. Conditions for global boundedness are then obtained via a direct application of the $\mathcal{L}_\infty$ small gain theorem [5].

Proposition 3.1
The system (1)-(3) in closed loop with (4)-(6) may be written as (see Fig. 2)

$$\begin{aligned} \tau_d &= G(p)e \\ e &= v - b(t)\tau_d \end{aligned}$$

where

$$G(p) = \frac{pK_P + K_I}{p^2 + (pK_P + K_I)\frac{R_r}{\hat{R}_r}}$$

the external signal $v \in \mathcal{L}_\infty$, and

$$|b(t)| \leq |\frac{\hat{R}_r - R_r}{\hat{R}_r}|$$

□□□

The proof of the proposition, being a little technical and very lengthy, has not been included. Now, we state a corollary whose proof follows inmediately from the $\mathcal{L}_\infty$ small gain theorem and the fact that the $\mathcal{L}_\infty$ gain of $G(p)$ equals $\frac{\hat{R}_r}{R_r}$.

Corollary 3.1

If $0 < \hat{R}_r < 2R_r$ then all signals of the closed loop are uniformly bounded. □□□

3. Coordinate changes and uniqueness of equilibrium

To carry out the asymptotic stability analysis in the general case we find convenient to work with a state space representation of the system, and introduce some coordinate transformations. First, let us define the (nonlinear) coordinate transformation

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} x_d^T J x \\ x_d^T x \\ \tau_d \\ \tilde{y} \end{bmatrix}$$

This results in the following dynamic model.

$$\dot{v} = \qquad (7)$$

$$\begin{bmatrix} -R_r & \hat{R}_r \frac{v_3}{\beta^2} & -R_r & 0 \\ -\hat{R}_r \frac{v_3}{\beta^2} & -R_r & 0 & 0 \\ -K_P & 0 & -K_P \frac{v_2}{\beta^2} & -K_I \\ 1 & 0 & \frac{v_2}{\beta^2} & 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ R_r \beta^2 \\ K_P \tau_L \\ -\tau_L \end{bmatrix}$$

Now, we shift the equilibrium to the origin. To this end, we define the new coordinates $w = v - \bar{v}$ where $\bar{v} \in \mathcal{R}^4$ is an equilibrium of (7). Below we will show that, for all practical purposes, the equilibrium is unique. The transformed dynamic model becomes

$$\dot{w} = \qquad (8)$$

$$\begin{bmatrix} -R_r & \hat{R}_r \frac{\bar{v}_3 + w_3}{\beta^2} & -R_r + \frac{\hat{R}_r}{\beta^2} \bar{v}_2 & 0 \\ -\hat{R}_r \frac{\bar{v}_3 + w_3}{\beta^2} & -R_r & -\frac{\hat{R}_r}{\beta^2} \bar{v}_1 & 0 \\ -K_P & -\frac{K_P}{\beta^2} \bar{v}_3 & -K_P \frac{\bar{v}_2 + w_2}{\beta^2} & -K_I \\ 1 & \frac{1}{\beta^2} \bar{v}_3 & \frac{\bar{v}_2 + w_2}{\beta^2} & 0 \end{bmatrix} w$$

Now we will prove the following.

Proposition 4.1

The equilibria of (7) are independent of K_P, K_I. Further, the equilibrium is unique for all values of τ_L if and only if $0 < \hat{R}_r \leq 3R_r$.

The actual proof is not given here. It is based on the calculation of real roots of the third-order polynomial

$$1 - \tau_L \frac{\hat{R}_r}{R_r} \frac{\bar{v}_3^2 + \frac{R_r^2}{\hat{R}_r^2} \beta^4}{\bar{v}_3 (\bar{v}_3^2 + \beta^4)} = 0$$

In the polynomial, τ_L enters linearly, which permits the application of the root locus technique to visualize the possibility of multiple equilibria for certain values of τ_L. In Fig. 3 and 4 we depict the τ_L-root locus for the cases when $\hat{R}_r = 3R_r$ and $\hat{R}_r > 3R_r$.

The uniqueness of equilibrium for $\hat{R}_r = 3R_r$ is evident from Fig. 3, since there are three *coinciding* real roots for one value of τ_L, while for any other value of τ_L there is only one real root. The non-uniqueness of the equilibria for $\hat{R}_r > 3R_r$ causes the locus of Fig. 4 to have three distinct real roots for a certain range of τ_L.

4. Local asymptotic stability

In this section we will study, via the first Lyapunov method, the local asymptotic stability of (8). Towards this end, we see that the systems first order approximation is obtained simply from (8) by leaving out the quadratic terms, which results in the following characteristic polynomial:

$$\begin{vmatrix} s + R_r & -\hat{R}_r \frac{\bar{v}_3}{\beta^2} & R_r - \frac{\hat{R}_r}{\beta^2} \bar{v}_2 & 0 \\ \hat{R}_r \frac{\bar{v}_3}{\beta^2} & s + R_r & \frac{\hat{R}_r}{\beta^2} \bar{v}_1 & 0 \\ K_P & \frac{K_P}{\beta^2} \bar{v}_3 & s + K_P \frac{\bar{v}_2}{\beta^2} & K_I \\ -1 & -\frac{1}{\beta^2} \bar{v}_3 & -\frac{\bar{v}_2}{\beta^2} & s \end{vmatrix} = 0$$

Given the complexity of the expression above, (recall that $\bar{v}$ is itself a nonlinear function of the motor parameters), we are unable at this point to make a general statement concerning the stability of the roots of this polynomial. An interesting case for which we have been able to calculate the poles of the system is the zero load torque case. We will show that, even with zero load torque, the equilibrium may become unstable.

Proposition 5.1

Assume $\tau_L = 0$. Then, the system is LOCALLY asymptotically stable if $0 < \hat{R}_r \leq R_r + K_P$. On the other hand, the equilibrium will be unstable if $\hat{R}_r$ overestimates R_r by K_P and a large integral gain is used.

Proof

When $\tau_L = 0$ the characteristic equation reduces to

$$\begin{aligned} (s + R_r)^2 (s(s + K_P) + K_I) \\ + K_P (s + R_r)(\hat{R}_r - R_r) s \\ + (s + R_r)(\hat{R}_r - R_r) K_I = 0 \end{aligned}$$

The proof is completed noting that his equation has one root at $s = -R_r$, while from Routh-Hurwitz we know that the other roots are on the open the left hand plane if and only if

$$\hat{R}_r R_r K_P + \hat{R}_r K_P^2 > (\hat{R}_r - R_r - K_P) K_I \qquad (9)$$

□□□

The proposition above shows that the system can be destabilized, in the sense of having unstable equilibria, if the rotor resistance is overestimated, the proportional gain is small, and a large integral gain is used.

5. Global asymptotic stability

In this section we will investigate GAS of the equilibrium using Lyapunov's second method. Namely, we will construct Lyapunov functions of the form

$$V(w) = \frac{1}{2} w^T P w$$

where P is a positive definite symmetric constant matrix. To select P we first find positive semidefinite matrices P_i that lead to expressions *without cubic terms* in the derivative of V. Second, linear combinations of these positive semi-definite matrices are constructed that lead to a negative-definite $\dot{V}(w)$. Finally, the positive definiteness of P is checked.

To illustrate the procedure we first construct a Lyapunov function for the case where $R_r = \hat{R}_r$. Then, we treat the case when $\tau_L = 0$, $R_r \neq \hat{R}_r$, and derive a sufficient condition on $\hat{R}_r$ for GAS. Finally, the general case, is illustrated with a numerical example.

Positive semi-definite matrices to avoid cubic terms

To construct our Lyapunov-function candidate $V(w) = w^T P w$, we can only use for P linear combinations of the following positive semi-definite matrices P_i, $i = 1, \cdots, 4$.

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; P_2 = \begin{bmatrix} \frac{1}{\hat{R}_r} & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \hat{R}_r \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & K_P \\ 0 & 0 & K_P & K_P^2 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} K_P^2 & 0 & K_P h & 0 \\ 0 & 0 & 0 & 0 \\ K_P h & 0 & h^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding functions $V_i(w) = \frac{1}{2} w^T P_i w$, $i = 1, \cdots, 4$ have time-derivatives $\dot{V}_i(w) = w^T P_i \dot{w}$, $i = 1, \cdots, 4$ with only quadratic terms in w. Particularly interesting is the expression for $\dot{V}_1(w)$:

$$\dot{V}_1(w) = w^T P_1 \dot{w} =$$

$$-R_r w_1^2 - \tfrac{R_r \beta^2 - \hat{R}_r \bar{v}_2}{\beta^2} w_1 w_3 - R_r w_2^2$$
$$- \tfrac{\hat{R}_r \bar{v}_1}{\beta^2} w_2 w_3$$

in which cross-terms appear only if $\hat{R}_r \neq R_r$ (see [2]), which constrains the construction of a Lyapunov function.

[2] the equilibrium for $\hat{R}_r = R_r$ is $\bar{v}_1 = 0$, $\bar{v}_2 = \beta^2$, and $\bar{v}_3 = \tau_L$

As a result, the derivative of $V(w) = w^T P w$ will also have only quadratic terms if P is a linear combination of P_i, $i = 1, \cdots, 4$. In that case, $\dot{V}(w) = -w^T Q w$ with Q a symmetric constant matrix, and $\dot{V}(w)$ is negative definite iff all leading principal minors of Q are positive.

Lyapunov function for $\hat{R}_r = R_r$

For the nominal case $R_r = \hat{R}_r$, a Lyapunov function can be constructed that is valid for all τ_L. The construction is easy since $\dot{V}_1(w) = -R_r w_1^2 - R_r w_2^2$ in the nominal case. Consider the matrix

$$P_a = P_3 + P_4 + \frac{K_I + R_r^2 K_I}{R_r} P_2$$

which results in the candidate Lyapunov function $V_a(w)$ with derivative of the form

$$\dot{V}_a(w) =$$

$$-a_1 w_1^2 - a_3 w_3^2 - a_4 w_4^2 - 2b_{13} w_1 w_3 + 2b_{14} w_1 w_4$$

This derivative can always be rendered negative definite by adding a component $(z_3 + z_4) P_1$ to the matrix P_a:

$$P_b = P_3 + P_4 + \frac{K_I + R_r^2 K_I}{R_r} P_2 + (z_3 + z_4) P_1$$

where the coefficients z_3, z_4 are chosen to compensate for the cross-terms of $\dot{V}_a$ as follows:

$$z_3 = \frac{1}{R_r} \frac{b_{13}^2}{a_3}, \; z_4 = \frac{1}{R_r} \frac{b_{14}^2}{a_4}$$

so that the derivative of the Lyapunov function $V_b(w) = \frac{1}{2} w^T P_b w$ becomes

$$\begin{aligned} \dot{V}_b(w) &= -a_1 w_1^2 - a_3 w_3^2 \\ &\quad -a_4 w_4^2 - 2b_{13} w_1 w_3 + 2b_{14} w_1 w_4 \\ &\quad -\frac{b_{13}^2}{a_3} w_1^2 - \frac{b_{14}^2}{a_4} w_1^2 - \left(\frac{b_{13}^2}{a_3} + \frac{b_{14}^2}{a_4}\right) w_2^2 \end{aligned}$$

The function $V_b(w)$ is positive definite and its derivative is negative definite, therefore it is a strict Lyapunov function for $R_r = \hat{R}_r$.

Lyapunov functions for $\hat{R}_r \neq R_r$, $\tau_L = 0$

For the case $\tau_L = 0$ and $R_r \neq \hat{R}_r$, the cross-term $(R_r - \hat{R}_r) w_1 w_3$ appears in $\dot{V}_1(w)$. This constrains the construction of a Lyapunov-function. However, using the approach of the previous subsection to construct a Lyapunov function, the following expression for negative-definiteness of $\dot{V}(w) = -w^T Q w$ has been derived from checking the leading principal minors of Q:

$$\frac{a}{4\xi^2} - \frac{b}{\xi} + c > 0$$

where $\xi = (R_r - \hat{R}_r)/R_r$ and $a > 0, b, c$ are complicated functions of system parameters.

Lyapunov functions for $\hat{R}_r \neq R_r$ and $\tau_L \neq 0$ The approach of the previous subsection can also be applied to the case where $\tau_L \neq 0$, but the resulting sufficient conditions for GAS are not given here since they are rather complicated. Instead, a numerical example is given of a Lyapunov function that ensures GAS for the particular parameter values $K_P = 1$, $K_I = 0.1$, $\beta = 1$, $R_r = 2$, $\hat{R}_r = 1$ and all τ_L.

Consider the candidate Lyapunov function

$$V(w) = w^T\left(\frac{1}{2}P_1 + 0.1P_2 + P_3 + P_4\right)w$$

This function is positive definite, and its derivative is

$$\dot{V}(w) = -w^T Q(\xi_1, \xi_2) w$$

where

$$\xi_1 = \frac{R_r\beta^2 - \hat{R}_r\bar{v}_2}{2R_r\beta^2}$$

$$\xi_2 = \frac{\hat{R}_r\bar{v}_1}{2R_r\beta^2}$$

and $Q(\xi_1, \xi_2)$ is a constant symmetric matrix whose off-diagonal coefficients depend on ξ_1 and ξ_2. For $V(w)$ to be a Lyapunov function, $\dot{V}(w)$ must be negative definite, and $Q(\xi_1, \xi_2)$ must therefore be positive definite. For the particular parameter values of the numerical example, this positive definiteness can be proved using the property that ξ_1 and ξ_2 are bounded functions of $\bar{v}_3$.

6. Conclusions

We have shown that the widely used indirect field oriented control is globally asymptotically stable if the estimated rotor resistance estimate is estimated close enough to the real value. Unique equilibria are guaranteed if the estimated rotor resistance is within a 200% error range. Also, all signals in the system remain uniformly bounded if the estimated rotor resistance is within a 100% error range. The system becomes locally unstable if the rotor resistance is overestimated and a large integral gain is used.

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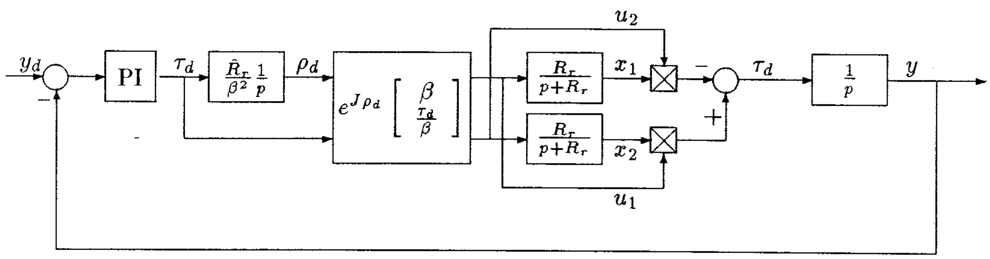

Figure 1: Closed loop velocity control of induction motor using indirect FOC

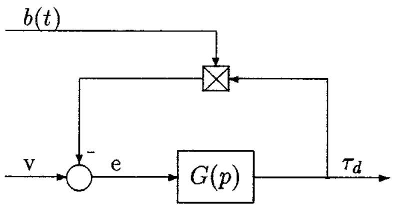

Figure 2: Input-output description of closed loop system

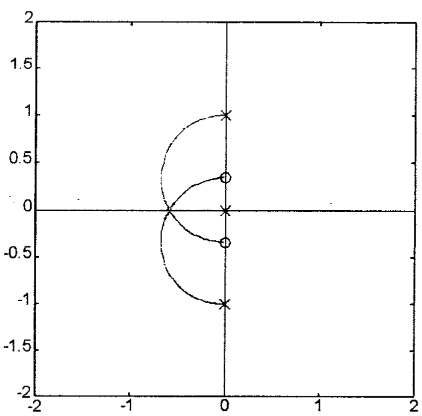

Figure 3: Root locus of the system equilibria for $\hat{R}_r = 3R_r$

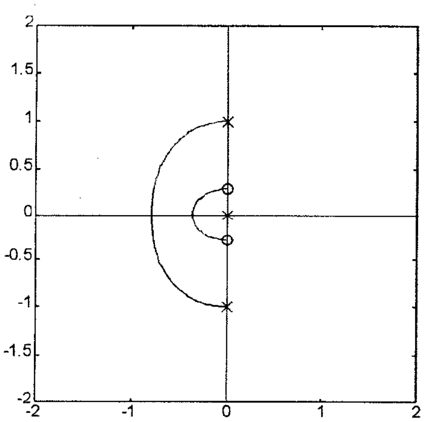

Figure 4: Root locus of the system equilibria for $\hat{R}_r > 3R_r$