Analysis of birth-death fluid queues

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Abstract We present a survey of techniques for analysing the performance of a reservoir which receives and releases fluid at rates which are determined by the state of a background birth-death process. The reservoir is assumed to be infinitely large, but the state space of the modulating birth-death process may be finite or infinite.

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1 Introduction

In this paper we shall study the stationary behaviour of the content of a fluid reservoir which receives and releases fluid flows at rates which are determined by the actual state of an ergodic birth-death process evolving in the background. The reservoir is assumed to be infinitely large, which implies that for the stationary distribution of the content of the reservoir to exist it is necessary that some stability condition be satisfied.

The state space of the background birth-death process will be denoted by $\mathcal{N}$ and may be finite or infinite; in the former case $\mathcal{N} = \{1, 2, \ldots, N\}$ for some natural number $N \geq 2$, in the latter case $\mathcal{N}$ is the set of positive integers. We shall denote the state of the background process at time t by $X(t)$ and the content of the reservoir at time t by $C(t)$. The obvious approach to obtaining the stationary distribution of $\{C(t),\ t \geq 0\}$ is by analysing the two-dimensional process $\{(X(t), C(t)),\ t \geq 0\}$, which is Markovian.

Our assumption that the flow rates of fluid into and out of the reservoir are determined by the current state of the background process, entails that for each $i \in \mathcal{N}$ there is a real number r_i, the *drift* in state i, such that r_i is the slope of $\{C(t)\}$ when the birth-death process is in state i, as long as this is physically possible. That is, the rate of change of the content of the reservoir (or the *net input rate*) at time t is $r_{X(t)}$, provided $r_{X(t)} \geq 0$, or $r_{X(t)} < 0$ and $C(t) > 0$; if the reservoir has emptied at time t it stays empty as long as the drift remains negative. We shall assume throughout this paper that $r_i \neq 0$ for all states i. We shall also assume that $r_i > 0$ for at least one $i \in \mathcal{N}$, since otherwise the reservoir is always empty.

When $\mathcal{N} = \{1, 2, \ldots, N\}$ for some natural number N, the model at hand is a generalization of the fluid flow models studied in the famous papers by Anick, Mitra and Sondhi [3] and Gaver and Lehoczky [13] in the early eighties, where specific birth-death processes and *drift vectors* $(r_1, r_2, \ldots, r_N)$ are considered. In van Doorn, Jagers and de Wit [9] and Coffman, Igelnik and Kogan [7] the authors allow the rate-modulating process to be an arbitrary birth-death process, but require the drift vector to have a particular sign structure. Background processes (on a finite state space) of a more general type than birth-death processes have been considered in many papers, notably [16, 17, 19, 23, 11, 12, 4, 14]. However, it appears that in the birth-death context some structural properties prevail which are lost in more general contexts. Therefore, it is of interest to analyse the model in the setting where the modulating process is a birth-death process with a finite state space, thereby generalizing some of the results in [7] and [9]. This will be done in Section 3.

Relatively few results are available in the literature dealing with (variants of) our model when $\mathcal{N}$ is infinite. We know of one reference in which, for a specific model, the approach is taken of letting N tend to infinity in the expressions obtained for the truncated model in which $\mathcal{N} = \{1, 2, \ldots, N\}$, see [1]. It appears that this is a viable procedure whenever the number of positive components of the drift vector $(r_1, r_2, \ldots)$ is finite. This case will therefore be treated in Section 4.

As far as we know only one model has been treated in the literature which fits into our setting and for which both $\mathcal{N}$ and the number of positive components of the drift vector is infinite, see [24] and [2]. In this model, however, the number of negative components of the drift vector is finite. In Section 5 we shall outline a general procedure for solving models with this property. A complete analysis of this case will be elaborated elsewhere [10]. The solution procedure we propose is entirely different from the approaches chosen in [24] and [2].

The analyses in Sections 3, 4 and 5 amount to solving a finite (in Section 3) or infinite (in Sections 4 and 5) system of differential equations under certain boundary conditions. The derivation of this system of differential equations will be outlined in Section 2.

We finally note that there are several papers dealing with approximations for our model when $\mathcal{N}$ is infinitely large, see, e.g., [15, 21, 22]. In this paper we restrict ourselves to exact solutions.

2 Preliminaries

We shall let λ_i denote the birth rate and μ_i the death rate in state i, $i \in \mathcal{N}$, of the birth-death process $\{X(t),\ t \geq 0\}$ with state space $\mathcal{N}$ which regulates the content of the reservoir. We shall assume that the birth and death rates are positive with the exception of the death rate μ_1 in the lowest state and, if $\mathcal{N} = \{1, 2, \ldots, N\}$, the birth rate λ_N in the highest state. It will be convenient to interpret λ_i and μ_i as zero if $i \notin \mathcal{N}$. We let

$$\pi_i \equiv \prod_{j=1}^{i-1} \frac{\lambda_j}{\mu_{j+1}}\ , \quad i \in \mathcal{N}, \tag{2.1}$$

where the empty product is interpreted as unity. The stationary state probabilities p_i, $i \in \mathcal{N}$, of the birth-death process can then be represented as

$$p_i = \frac{\pi_i}{\sum_{j \in \mathcal{N}} \pi_j}\ , \quad i \in \mathcal{N}. \tag{2.2}$$

When $\mathcal{N}$ is infinite we shall always assume that the stationary distribution of the birth-death process exists, that is, $\sum_{i \in \mathcal{N}} \pi_i$ is finite. In order that a stationary distribution for $C(t)$, the content of the reservoir at time t, exists, the mean drift should evidently be negative, that is, $\sum_{i \in \mathcal{N}} p_i r_i < 0$, or, equivalently,

$$\sum_{i \in \mathcal{N}} \pi_i r_i < 0\ . \tag{2.3}$$

We shall assume throughout that this stability condition is satisfied.

In what follows we let

$$\mathcal{N}^{+} \equiv \{i \in \mathcal{N} |\ r_i > 0\} \ , \quad \mathcal{N}^{-} \equiv \{i \in \mathcal{N} |\ r_i < 0\} \ , \tag{2.4}$$

and

$$d_{+} \equiv |\mathcal{N}^{+}| \ , \quad d_{-} \equiv |\mathcal{N}^{-}| \ . \tag{2.5}$$

Obviously, $\mathcal{N}^{+} \bigcup \mathcal{N}^{-} = \mathcal{N}$, since we have assumed that the drift in each state is nonzero. Also, when $\mathcal{N}$ is infinite at least one of d_{+} or d_{-} is infinity.

Putting

$$F_i(t,u) \equiv \Pr[X(t) = i,\ C(t) \le u] \ , \quad t \ge 0,\ u \ge 0,\ i \in \mathcal{N},$$

and $F_i(t,u) \equiv 0$ if $i \notin \mathcal{N}$, it is not difficult to show that the Kolmogorov forward equations for the Markov process $\{(X(t), C(t)),\ t \ge 0\}$ are given by

$$\begin{aligned} \frac{\partial F_i(t,u)}{\partial t} &= -r_i \frac{\partial F_i(t,u)}{\partial u} \ - \ (\lambda_i + \mu_i) F_i(t,u) \\ &\quad + \ \lambda_{i-1} F_{i-1}(t,u) \ + \ \mu_{i+1} F_{i+1}(t,u) \ , \quad i \in \mathcal{N}. \end{aligned} \tag{2.6}$$

But assuming that the process is in equilibrium, we may set $F_i(t,u) \equiv F_i(u)$ and $\partial F_i(t,u)/\partial t \equiv 0$ and, hence, obtain the system

$$r_i F_i^{'}(u) = \lambda_{i-1} F_{i-1}(u) - (\lambda_i + \mu_i) F_i(u) + \mu_{i+1} F_{i+1}(u) \ , \quad i \in \mathcal{N}, \tag{2.7}$$

where $F_i(u)$ denotes the equilibrium probability that the birth-death process is in state i and the content of the reservoir does not exceed u, again with the convention $F_i(u) \equiv 0$ if $i \notin \mathcal{N}$.

Since the content of the reservoir is increasing whenever the drift is positive, the solution to (2.7) must satisfy the boundary conditions

$$F_i(0) = 0 \ , \quad i \in \mathcal{N}^{+}. \tag{2.8}$$

Also, we must obviously have

$$F_i(\infty) \equiv \lim_{u \to \infty} F_i(u) = p_i \ , \quad i \in \mathcal{N}. \tag{2.9}$$

3 Finite state space

In this section we will describe the procedure for solving the differential equations (2.7), subject to the boundary conditions (2.8) and (2.9), assuming $\mathcal{N} = \{1, 2, \ldots, N\}$ with $N \ge 2$ and condition (2.3) is satisfied.

It will be convenient to write the homogeneous system (2.7) in matrix form as

$$\mathbf{F}'(u) = R^{-1} Q^{T} \mathbf{F}(u) \ , \tag{3.1}$$

where superscript T denotes transpose,

$$\mathbf{F}(u) \equiv (F_1(u), F_2(u), \dots, F_N(u))^T,$$

$$R \equiv \mathrm{diag}(r_1, r_2, \dots, r_N) \ ,$$

and Q is the generator of the modulating birth-death process, that is,

$$Q \equiv \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & \\ \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & \mu_{N-1} & -(\lambda_{N-1}+\mu_{N-1}) & \lambda_{N-1} \\ & \cdots & 0 & \mu_N & -\mu_N \end{pmatrix} . \tag{3.2}$$

Note that R^{-1} exists because of our assumption that $r_i \neq 0$ for each state $i \in \mathcal{N}$.

We start off by deriving a representation formula for the characteristic polynomial of the matrix $R^{-1}Q^T$. To this end we define the sequence of polynomials $\{\Delta_n(x)\}_{n=0}^{N-1}$ by the recurrence relations

$$\begin{aligned} &\Delta_0(x) = 1 \ , \quad \Delta_1(x) = x + \frac{\lambda_1}{r_1} + \frac{\mu_2}{r_2} \ , \\ &\Delta_n(x) = \left(x + \frac{\lambda_n}{r_n} + \frac{\mu_{n+1}}{r_{n+1}}\right)\Delta_{n-1}(x) - \frac{\lambda_n\mu_n}{r_n^2}\Delta_{n-2}(x) \ , \quad 1 < n < N, \end{aligned} \tag{3.3}$$

and observe the following, where I denotes the $N \times N$ identity matrix.

Lemma 3.1 *The characteristic polynomial* $\det[xI - R^{-1}Q^T]$ *of the matrix* $R^{-1}Q^T$ *can be represented as* $x\Delta_{N-1}(x)$.

Proof. We define another sequence of polynomials $\{T_n(x)\}_{x=0}^{N-1}$ by the recurrence relations

$$\begin{aligned} &T_0(x) = 1 \ , \quad T_1(x) = x + \frac{\lambda_1}{r_1} \ , \\ &T_n(x) = \left(x + \frac{\lambda_n + \mu_n}{r_n}\right)T_{n-1}(x) - \frac{\lambda_{n-1}\mu_n}{r_{n-1}r_n}T_{n-2}(x) \ , \quad 1 < n < N. \end{aligned} \tag{3.4}$$

It is easy to see that the polynomial $T_n(x)$, $0 < n < N$, is the characteristic polynomial of the $n \times n$ north-west corner truncation of $R^{-1}Q^T$, and hence

$$\det[xI - R^{-1}Q^T] = \left(x + \frac{\mu_N}{r_N}\right)T_{N-1} - \frac{\lambda_{N-1}\mu_N}{r_{N-1}r_N}T_{N-2}(x) \ .$$

It can readily be established by induction, however, that

$$x\Delta_n(x) = \left(x + \frac{\mu_{n+1}}{r_{n+1}}\right)T_n(x) - \frac{\lambda_n\mu_{n+1}}{r_n r_{n+1}}T_{n-1}(x) \ , \quad 1 \le n < N,$$

which proves the lemma. □

By Favard's Theorem, see, e.g., Chihara's book [6], the polynomials $\Delta_n(x)$, $n = 0, 1, \ldots, N-1$, constitute the first N elements of a sequence of orthogonal polynomials. It follows, see [6] again, that the zeros of these polynomials, and the zeros of $\Delta_{N-1}(x)$ in particular, are real and simple. We can therefore conclude from the above lemma that the eigenvalues of $R^{-1}Q^T$ are real and simple, with the possible exception of the eigenvalue 0. Since it has been shown, in a more general setting, in [19] and [23], that the matrix $R^{-1}Q^T$ must have d_+ negative eigenvalues, $d_- - 1$ positive eigenvalues and one eigenvalue 0, we can conclude the following.

Lemma 3.2 *The eigenvalues ξ_j, $j \in \mathcal{N}$, of $R^{-1}Q^T$ are all real and simple; ordering them in increasing magnitude one has $\xi_j < 0$, $j = 1, \ldots, d_+$, $\xi_{d_++1} = 0$, $\xi_j > 0$, $j = d_+ + 2, \ldots, N$.*

Knowing that all eigenvalues are simple it is straightforward to verify that the solution of (3.1) must be of the form

$$\mathbf{F}(u) = \sum_{j \in \mathcal{N}} c_j \exp\{\xi_j u\} \mathbf{y}^{(j)}, \quad u \geq 0, \tag{3.5}$$

where, for each $j \in \mathcal{N}$, the vector $\mathbf{y}^{(j)} \equiv \left(y_1^{(j)}, y_2^{(j)}, \ldots, y_N^{(j)}\right)$ is the suitably normalized eigenvector corresponding to the eigenvalue ξ_j, and c_j is a constant. However, since $\mathbf{F}(u)$ is bounded the coefficients c_j corresponding to positive eigenvalues must vanish, that is, $c_j = 0$ for $j = d_+ + 1, \ldots, N$, by Lemma 3.2. Also, boundary condition (2.9) tells us that $c_{d_++1}\mathbf{y}^{(d_++1)} = \mathbf{p}$, where $\mathbf{p} \equiv (p_1, \ldots, p_N)^T$ and p_i is given in (2.2). Consequently, (3.5) reduces to

$$\mathbf{F}(u) = \mathbf{p} + \sum_{j=1}^{d_+} c_j \exp\{\xi_j u\} \mathbf{y}^{(j)}, \quad u \geq 0. \tag{3.6}$$

The d_+ negative eigenvalues ξ_j in (3.6) can be found by determining the negative zeros of the polynomial $\Delta_{N-1}(x)$ of Lemma 3.1. Since $\Delta_{N-1}(x)$ is an element of a sequence of orthogonal polynomials, very efficient methods exist for finding these zeros, which can be interpreted as eigenvalues of a *symmetric* tridiagonal matrix, see, e.g., [5] and [18]. Since $R^{-1}Q^T$ is a tridiagonal matrix, the eigenvectors $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(d_+)}$ have nonzero first components. Hence, for $j = 1, \ldots, d_+$, we can normalize $\mathbf{y}^{(j)}$ to have $y_1^{(j)} = 1$ and subsequently find the remaining components by solving the recurrence relations

$$\begin{aligned} y_1^{(j)} &= 1\,, \qquad \mu_2 y_2^{(j)} = r_1\xi_j + \lambda_1 \\ \mu_i y_i^{(j)} &= (r_{i-1}\xi_j + \lambda_{i-1} + \mu_{i-1}) y_{i-1}^{(j)} - \lambda_{i-2} y_{i-2}^{(j)}\,, \quad i = 3, 4, \ldots, N. \end{aligned} \tag{3.7}$$

Finally, the constants $c_1, \ldots, c_{d_+}$ must be determined by the boundary condition (2.8), which translates into

$$p_i + \sum_{j=1}^{d_+} c_j y_i^{(j)} = 0 \ , \quad i \in \mathcal{N}^+. \tag{3.8}$$

As an aside we note that the system (3.8) can be solved explicitly when the drift vector has a particular sign structure, see [7] and [9].

The above is summarized in the following theorem.

Theorem 3.3 *The stationary distribution* $F_i(u) \equiv \Pr[X(t) = i,\ C(t) \le u],\ i \in \mathcal{N} = \{1, 2, \ldots, N\},\ u \ge 0,$ *of the process* $\{(X(t), C(t)),\ t \ge 0\}$ *is given by*

$$F_i(u) = p_i + \sum_{j=1}^{d_+} c_j y_i^{(j)} \exp\{\xi_j u\} \ , \tag{3.9}$$

where $\xi_j,\ j = 1, \ldots, d_+$, *are the negative eigenvalues of* $R^{-1}Q^T$, *or, equivalently, the negative zeros of the polynomial* $\Delta_{N-1}(x)$ *defined in (3.3), and the constants* p_i, $y_i^{(j)}$ *and* c_j, *are determined by (2.2), (3.7) and (3.8), respectively.*

Once the probabilities $F_i(u)$ are known it is usually a matter of routine to obtain various performance measures of interest, such as the probability of the content of the reservoir exceeding a particular level, cf. [3, 13, 9, 7].

4 Infinite state space with $d_+ < \infty$

In this section our goal is to obtain the solution of the differential equations (2.7), subject to the boundary conditions (2.8) and (2.9), assuming $\mathcal{N} = \{1, 2, \ldots\}$ and $d_+ \equiv |\mathcal{N}^+| < \infty$. Our approach involves truncation of the state space of the birth-death process to the set $\{1, 2, \ldots, N\}$ for some sufficiently large N and letting N tend to infinity in the expressions found for the ensuing finite model by the procedure of the previous section. As we shall see, the viability of this approach hinges on the fact that d_+ is finite.

Concretely, we choose N such that $N > \max \mathcal{N}^+$ and

$$\sum_{i=1}^{n} \pi_i r_i < 0 \ , \quad \text{for all } n \ge N, \tag{4.1}$$

which is always possible since stability condition (2.3) is assumed to be satisfied. Next we truncate the state space of the birth-death process to $\{1, 2, \ldots, N\}$ and make state N reflecting by setting $\lambda_N = 0$. The results of the previous section then tell us that for the truncated system, which is stable because of (4.1), the stationary probability that the birth-death process is in state i and the content of the reservoir does not exceed u is given by

$$F_i^{(N)}(u) = p_i^{(N)} + \sum_{j=1}^{d_+} c_j^{(N)} y_i^{(N,j)} \exp\{\xi_j^{(N)} u\} \ , \quad u \geq 0,\ i = 1, 2, \ldots, N, \quad (4.2)$$

where we have indicated dependence on N, see Theorem 3.3. Of crucial importance is the fact that the number of terms in the summation appearing in (4.2) equals $d_+ < \infty$ independent of N, which allows us to interchange limit and summation when we let N tend to infinity in (4.2). Before doing so, however, we must determine the limiting behaviour as $N \to \infty$ of the quantities $p_i^{(N)},\ \xi_j^{(N)},\ y_i^{(N,j)}$ and $c_j^{(N)}$.

First, it is obvious from (2.2) that

$$p_i^{(\infty)} \equiv \lim_{N\to\infty} p_i^{(N)} = p_i \ , \quad i \in \mathcal{N}. \quad (4.3)$$

Subsequently turning to the eigenvalues $\xi_j^{(N)},\ j = 1, 2, \ldots, d_+$, we can show the following.

Lemma 4.1 *The limits*

$$\xi_j^{(\infty)} \equiv \lim_{N\to\infty} \xi_j^{(N)}, \quad j = 1, 2, \ldots, d_+,$$

exist and satisfy $-\infty < \xi_1^{(\infty)} < \xi_2^{(\infty)} < \cdots < \xi_{d_+}^{(\infty)} < 0.$

Proof. We recall that the polynomial $\Delta_{N-1}(x)$ defined in (3.3) has negative zeros $\xi_j^{(N)},\ j = 1, 2, \ldots, d_+$, while its other zeros are positive. By lifting the restriction $n < N$ in (3.3) we can make $\Delta_{N-1}(x)$ element of an infinite sequence $\{\Delta_n(x)\}_{n=0}^{\infty}$ which, by Favard's Theorem, constitutes a sequence of orthogonal polynomials, see [6]. The lemma can now be established with the help of two results about zeros of orthogonal polynomials, see [6] again. Letting x_{ni} denote the ith zero in ascending order of the nth polynomial in an orthogonal polynomial sequence, the first result says that for any fixed i the sequence $\{x_{ni}\}_{n=i}^{\infty}$ is decreasing, so that its limit exist (possibly $-\infty$). Letting $X_i \equiv \lim_{n\to\infty} x_{ni}$, the second result says that if $X_i = X_{i+1}$ for some i then $X_i = X_{i+k}$ for all $k = 1, 2, \ldots$. Considering that the $(d_+ + 1)$st zero of $\Delta_{N-1}(x)$ is positive for all N, the validity of the lemma is now evident. □

Remark. Interestingly, the sequence $\{\Delta_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to a positive measure which has point masses precisely at the points $\xi_1^{(\infty)}, \xi_2^{(\infty)}, \ldots, \xi_{d_+}^{(\infty)}$, but no other mass on the negative axis.

Having established the existence of the limits $\xi_j^{(\infty)}$ we can obviously let N tend to infinity in the recurrence relations (3.7) for $y_i^{(N,j)}$, $i = 1, 2, \ldots, N$, by which we get the infinite system

$$\begin{aligned} y_1^{(j)} &= 1 \ , \qquad \mu_2 y_2^{(j)} = r_1 \xi_j^{(\infty)} + \lambda_1 \\ \mu_i y_i^{(j)} &= (r_{i-1}\xi_j^{(\infty)} + \lambda_{i-1} + \mu_{i-1}) y_{i-1}^{(j)} - \lambda_{i-2} y_{i-2}^{(j)} \ , \quad i \in \mathcal{N}\backslash\{1,2\}, \end{aligned} \tag{4.4}$$

where, for convenience, we have written $y_i^{(j)} \equiv \lim_{N\to\infty} y_i^{(N,j)}$.

Next, we turn to the d_+ equations (3.8) for the constants $c_j^{(N)}$, $j = 1, 2, \ldots, d_+$. Assuming that the matrix of coefficients remains nonsingular as $N \to \infty$, it is clear that $c_j^{(\infty)} \equiv \lim_{N\to\infty} c_j^{(N)}$, $j = 1, 2, \ldots, d_+$, exists and is the unique solution of the d_+ equations (3.8), where $y_i^{(j)}$ must now satisfy (4.4).

Finally, we can let N tend to infinity in the right-hand side of (4.2) and check that the resulting expressions indeed represent the solution to (2.7) - (2.9). Summarizing we have the following.

Theorem 4.2 *The stationary distribution* $F_i(u) \equiv \Pr[X(t) = i,\ C(t) \le u]$, $i \in \mathcal{N} = \{1, 2, \ldots\}$, $u \ge 0$, *of the process* $\{(X(t), C(t)),\ t \ge 0\}$ *when* d_+ *is finite, is given by*

$$F_i(u) = p_i + \sum_{j=1}^{d_+} c_j y_i^{(j)} \exp\{\xi_j^{(\infty)} u\} \ , \tag{4.5}$$

where $\xi_1^{(\infty)}, \xi_2^{(\infty)}, \ldots, \xi_{d_+}^{(\infty)}$ *are the limits in Lemma 4.1 and the constants* p_i, $y_i^{(j)}$ *and* c_j *are determined by (2.2), (4.4), and (3.8).*

As in the finite case, it is evident that we cannot find explicit expressions for the quantities ξ_j, p_i, $y_i^{(j)}$ and c_j in general. However, in special cases explicit results can be obtained. The following is an example.

Example. We consider a simplification of the model in [1] in which

$$r_1 \equiv 1 \ , \quad r_i \equiv -r < 0 \ , \ i = 2, 3, \ldots,$$

so that

$$\mathcal{N}^+ \equiv \{1\} \ , \quad \mathcal{N}^- \equiv \{2, 3, \ldots\} \ .$$

Furthermore, the birth and death rates are constant, viz.,

$$\lambda_i \equiv \lambda \ \text{ and } \ \mu_{i+1} \equiv \mu \ , \quad i \in \mathcal{N}.$$

Since $\pi_i = (\lambda/\mu)^{i-1}$, stability of the system is ensured if

$$\frac{1}{1+r} < \frac{\lambda}{\mu} < 1 \ , \tag{4.6}$$

which we shall assume in the remainder of this example.

Our main problem is to find $\xi_1^{(\infty)} \equiv \lim_{N\to\infty} \xi_1^{(N)}$, where $\xi_1^{(N)}$ is the smallest zero of $\Delta_{N-1}(x)$ defined in (3.3). For the given parameters, however, the sequence $\{\Delta_n(x)\}_{n=0}^{\infty}$ can, after appropriate renormalization, be recognized as a sequence of *perturbed Chebysev polynomials*, see [6] and [20]. Since the zeros of these polynomials converge for $N \to \infty$ to limits for which explicit expressions exist, we can easily solve our problem. Indeed, from [6, p. 204] we obtain after some calculations, see [1],

$$\xi_1^{(\infty)} \equiv \lim_{N\to\infty} \xi_1^{(N)} = -\left(\lambda - \frac{\mu}{1+r}\right) . \tag{4.7}$$

Writing $y_i \equiv y_i^{(j)}$ the recurrence relations (4.4) reduce to

$$\begin{aligned} y_1 &= 1 , \qquad y_2 = (1+r)^{-1} \\ \mu y_i &= \left((1+r)\lambda + \frac{\mu}{1+r}\right) y_{i-1} - \lambda y_{i-2} , \quad i \in \mathcal{N}\backslash\{1,2\}, \end{aligned}$$

which immediately yields

$$y_i = \left(\frac{1}{1+r}\right)^{i-1} , \quad i \in \mathcal{N}. \tag{4.8}$$

Since (3.8) becomes

$$p_1 + c_1 y_1 = 0 ,$$

so that $c_1 = -p_1$, and evidently

$$p_i = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{i-1} , \quad i \in \mathcal{N}, \tag{4.9}$$

we finally obtain, for $u \geq 0$ and $i \in \mathcal{N}$,

$$F_i(u) = \left(1 - \frac{\lambda}{\mu}\right)\left[\left(\frac{\lambda}{\mu}\right)^{i-1} - \left(\frac{1}{1+r}\right)^{i-1} \exp\left\{-\left(\lambda - \frac{\mu}{1+r}\right)u\right\}\right] . \tag{4.10}$$

5 Infinite state space with $d_- < \infty$

We finally consider the case in which $\mathcal{N} = \{1, 2, \ldots\}$ and $d_+ \equiv |\mathcal{N}^+| = \infty$, but $d_- \equiv |\mathcal{N}^-| < \infty$. As announced in the Introduction we shall outline an approach to obtain the equilibrium distribution of the content of the reservoir under these circumstances. The approach will be elaborated elsewhere [10].

As a starting point we take the (infinite) system of differential equations (2.7) again, but, for the time being, we forget about the boundary conditions (2.8) and

(2.9). Instead, we shall try to obtain, for each $j \in \mathcal{N}$, the solution of (2.7) under the initial conditions

$$F_i(0) = \delta_{ij} \ , \quad i \in \mathcal{N}, \tag{5.1}$$

where δ_{ij} is Kronecker's delta. We shall assume that, for each $j \in \mathcal{N}$, this solution is unique and denote it by $\{F_i^{(j)}(u),\ i \in \mathcal{N}\}$. Later on we shall try to find a linear combination of solutions of this type which fits our original boundary conditions.

We now form the infinite matrices $\mathcal{F}(u),\ u \geq 0$, with elements

$$(\mathcal{F}(u))_{ij} \equiv F_i^{(j)}(u) \ , \quad i,j \in \mathcal{N}, \tag{5.2}$$

and note that the system of differential equations and initial conditions satisfied by the functions $F_i^{(j)}(u),\ i,j \in \mathcal{N}$, may be represented by

$$\mathcal{F}'(u) = R^{-1}Q^T\mathcal{F}(u) \tag{5.3}$$

and

$$\mathcal{F}(0) = I \ , \tag{5.4}$$

respectively, where I denotes the infinite identity matrix, $R \equiv \text{diag}(r_1, r_2, \ldots)$ and Q is the generator of the birth-death process, that is,

$$Q \equiv \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & & \\ \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots & \\ 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} . \tag{5.5}$$

Writing for convenience

$$A \equiv R^{-1}Q^T \ , \tag{5.6}$$

it now follows, formally at least, that

$$\mathcal{F}(u) = \exp(uA) = \sum_{n=0}^{\infty} A^n \frac{u^n}{n!} \ , \tag{5.7}$$

and hence that

$$F_i^{(j)}(u) = \sum_{n=0}^{\infty} (A^n)_{ij} \frac{u^n}{n!} \ , \quad i,j \in \mathcal{N}. \tag{5.8}$$

To obtain an alternative expression for $F_i^{(j)}(u),\ i,j \in \mathcal{N}$, we next consider the polynomials $P_j(x),\ j \in \mathcal{N}$, recurrently defined as $P_1(x) = 1$ and

$$xP_j(x) = \sum_{k \in \mathcal{N}} (A)_{kj} P_k(x) \ , \quad j \in \mathcal{N}. \tag{5.9}$$

As an aside we note that, apart from normalization and an index shift, these polynomials are identical to the polynomials appearing in the proof of Lemma 3.1.

It is not difficult to see by induction that we actually have

$$x^n P_j(x) = \sum_{k\in\mathcal{N}} (A^n)_{kj}\, P_k(x) \ , \quad j \in \mathcal{N}, \tag{5.10}$$

for all $n = 1, 2, \ldots$, and as a consequence we can write

$$e^{xu} P_j(x) = \sum_{n=0}^{\infty} x^n \frac{u^n}{n!} P_j(x) = \sum_{n=0}^{\infty} \sum_{k\in\mathcal{N}} (A^n)_{kj} \frac{u^n}{n!} P_k(x) \ , \quad j \in \mathcal{N}, \tag{5.11}$$

which, after interchanging summation signs and substituting (5.8), reduces to

$$e^{xu} P_j(x) = \sum_{k\in\mathcal{N}} F_k^{(j)}(u) P_k(x) \ , \quad j \in \mathcal{N}. \tag{5.12}$$

Now, if the sequence of polynomials $\{P_j(x), j \in \mathcal{N}\}$, would be orthogonal with respect to some inner product $(\ . \ , \ . \ )$, then the previous result would imply

$$(e^{xu} P_j(x), P_i(x)) = F_i^{(j)}(u)(P_i(x), P_i(x)) \ , \quad i, j \in \mathcal{N}, \tag{5.13}$$

that is,

$$F_i^{(j)}(u) = \frac{(e^{xu} P_j(x), P_i(x))}{(P_i(x), P_i(x))} \ , \quad i, j \in \mathcal{N}. \tag{5.14}$$

Fortunately, it so happens that the sequence $\{P_j(x), \ j \in \mathcal{N}\}$ constitutes a sequence of *chain-sequence polynomials*, see [8], and therefore is orthogonal with respect to the inner product defined by

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) d\psi(x) \ , \tag{5.15}$$

where ψ is a signed measure of total mass one which has positive mass on the positive axis and negative mass on the negative axis. Actually, we have

$$\int_{-\infty}^{\infty} P_i(x) P_j(x) d\psi(x) = \frac{r_i}{r_1 \pi_i} \delta_{ij} \ , \quad i, j \in \mathcal{N}. \tag{5.16}$$

It thus follows that $F_i^{(j)}(u)$ can be represented as

$$F_i^{(j)}(u) = \frac{r_1 \pi_i}{r_i} \int_{-\infty}^{\infty} e^{xu} P_i(x) P_j(x) d\psi(x) \ , \quad u > 0, \ i, j \in \mathcal{N}. \tag{5.17}$$

As announced our next step is to assume that the solution of the system (2.7) with boundary conditions (2.8) and (2.9) is a linear combination of the solutions

$\{F_i^{(j)}(u),\ i \in \mathcal{N}\}$, that is, we assume that there are constants $a_j,\ j \in \mathcal{N}$, such that

$$F_i(u) = \frac{r_1 \pi_i}{r_i} \sum_{j \in \mathcal{N}} a_j \int_{-\infty}^{\infty} e^{xu} P_i(x) P_j(x) d\psi(x) \ , \quad u \geq 0,\ i \in \mathcal{N}, \tag{5.18}$$

and our next task is to use the boundary conditions to determine these constants, which, by (5.16) and (5.18), have the interpretation

$$a_j = F_j(0) \ , \quad j \in \mathcal{N}. \tag{5.19}$$

At this point our assumption $d_- \equiv |\mathcal{N}^-| < \infty$ starts playing its crucial role. Indeed, it follows from boundary condition (2.8) that

$$F_i(0) = a_i = 0 \ , \quad i \in \mathcal{N}^+, \tag{5.20}$$

so that, in fact,

$$F_i(u) = \frac{r_1 \pi_i}{r_i} \int_{-\infty}^{\infty} e^{xu} P_i(x) \left(\sum_{j \in \mathcal{N}^-} a_j P_j(x) \right) d\psi(x) \ , \quad u \geq 0,\ i \in \mathcal{N}. \tag{5.21}$$

Also, it can be shown that the mass of the measure ψ on the positive axis actually consists of $d_- - 1$ isolated point masses at, say, the points $\zeta_1, \zeta_2, \ldots, \zeta_{d_- - 1}$. Considering that $F_i(u)$ is a probability, and hence uniformly bounded, it follows that the constants $a_j,\ j \in \mathcal{N}^-$, must be such that

$$\sum_{j \in \mathcal{N}^-} a_j P_j(\zeta_k) = 0 \ , \quad k = 1, 2, \ldots, d_- - 1. \tag{5.22}$$

The missing equation for the constants $a_j,\ j \in \mathcal{N}^-$, comes from the observation that the average amount of fluid flowing into the reservoir should balance the average amount of fluid flowing out, that is

$$\sum_{j \in \mathcal{N}^+} p_j r_j = - \sum_{j \in \mathcal{N}^-} (p_j - a_j) r_j \ , \tag{5.23}$$

or, with (2.2),

$$\sum_{j \in \mathcal{N}^-} a_j r_j = \frac{\sum_{j \in \mathcal{N}} \pi_j r_j}{\sum_{j \in \mathcal{N}} \pi_j} \ . \tag{5.24}$$

Alternatively, we may use boundary condition (2.9) to conclude that

$$p_i = \frac{r_1 \pi_i}{r_i} P_i(0) \left(\sum_{j \in \mathcal{N}^-} a_j P_j(0) \right) \psi(\{0\}) \ .$$

But since it can be shown that

$$\psi(\{0\}) = \frac{r_1}{\sum_{j \in \mathcal{N}} \pi_j r_j} \ , \tag{5.25}$$

while it is easy to see that

$$P_j(0) = r_j/r_1 \ , \quad j \in \mathcal{N}, \tag{5.26}$$

we can use (2.2) to obtain (5.24) again.

The above results, which are proven rigorously under mild regularity conditions in [10], can be summarized as follows.

Theorem 5.1 *The stationary distribution* $F_i(u) \equiv \Pr[X(t) = i,\ C(t) \leq u],\ i \in \mathcal{N} = \{1, 2, \ldots\},\ u \geq 0$, *of the process* $\{(X(t), C(t)),\ t \geq 0\}$ *when* d_- *is finite, can be represented as*

$$F_i(u) = p_i + \frac{r_1 \pi_i}{r_i} \int_{-\infty}^{0-} e^{xu} R(x) P_i(x) d\psi(x) \ . \tag{5.27}$$

Here $P_i(x),\ i \in \mathcal{N}$, *are the polynomials defined in (5.9) and* ψ *is the unique signed measure with positive mass on the positive axis and negative mass on the negative axis with respect to which they are orthogonal; furthermore,*

$$R(x) = \sum_{j \in \mathcal{N}^-} a_j P_j(x) \ , \tag{5.28}$$

with constants $a_j,\ j \in \mathcal{N}^-$, *such that*

$$R(0) = r_1 \frac{\sum_{j \in \mathcal{N}} \pi_j r_j}{\sum_{j \in \mathcal{N}} \pi_j} \tag{5.29}$$

and

$$R(\zeta_j) = 0 \ , \quad j = 1, 2, \ldots, d_- - 1, \tag{5.30}$$

where $\zeta_1, \zeta_2, \ldots, \zeta_{d_- - 1}$ *are the spectral points of the measure* ψ *on the positive axis.*

Evidently, the main problem in concrete examples is to find the signed measure ψ with respect to which the polynomials $P_j(x)$ are orthogonal. At least in some cases, such as the model studied in [2] and [24] it is possible to find this measure explicitly, see [10].

References

[1] I.J.B.F. Adan, E.A. van Doorn, J.A.C. Resing and W.R.W. Scheinhardt, A queueing system with two different service speeds. Memorandum No. 1316, University of Twente, Faculty of Applied Mathematics, Enschede, The Netherlands, 1996.

[2] I.J.B.F. Adan and J.A.C. Resing, A note on a fluid queue driven by an M/M/1 queue. *Queueing Systems*, to appear.

[3] D. Anick, D. Mitra and M.M. Sondhi, Stochastic theory of a data-handling system with multiple sources. *Bell System Tech. J.* **61** (1982) 1871-1894.

[4] S. Asmussen, Stationary distributions for fluid flow models with or without Brownian noise. *Stochastic Models* **11** (1995) 21-49.

[5] Baoxin Li, A.G. Law, H. Raafat, P.H. Nguyen and Y.F. Yan, Eigenvalues of tridiagonal matrices: an alternative to Givens' method. *Computers Math. Appl.* **19** (1990) 89-94.

[6] T.S. Chihara, *An Introduction to Orthogonal Polynomials*. Gordon and Breach, New York, 1978.

[7] E.G. Coffman, Jr., B.M. Igelnik and Y.A. Kogan, Controlled stochastic model of a communication system with multiple sources. *IEEE Trans. Inf. Theory* **37** (1991) 1379-1387.

[8] E.A. van Doorn, Some new results for chain-sequence polynomials. *J. Comput. Appl. Math.* **57** (1995) 309-317.

[9] E.A. van Doorn, A.A. Jagers and J.S.J. de Wit, A fluid reservoir regulated by a birth-death process. *Stochastic Models* **4** (1988) 457-472.

[10] E.A. van Doorn and W.R.W. Scheinhardt, in preparation.

[11] A.I. Elwalid and D. Mitra, Analysis and design of rate-based congestion control of high speed networks, Part I: Stochastic fluid models, access regulation. *Queueing Systems* **9** (1991) 29-64.

[12] A.I. Elwalid and D. Mitra, Statistical multiplexing with loss priorities in rate-based congestion control of high-speed networks. *IEEE Trans. Commun.* **42** (1994) 2989-3002.

[13] D.P. Gaver and J.P. Lehoczky, Channels that cooperatively service a data stream and voice messages. *IEEE Trans. Commun.* **30** (1982) 1153-1162.

[14] B. Igelnik, Y. Kogan, V. Kriman and D. Mitra, A new computational approach for stochastic fluid models of multiplexers with heterogeneous sources. *Queueing Systems* **20** (1995) 85-116.

[15] C. Knessl and J.A. Morrison, Heavy-traffic analysis of a data-handling system with many sources. *SIAM J. Appl. Math.* **51** (1991) 187-213.

[16] L. Kosten, Stochastic theory of data-handling systems with groups of multiple sources. In *Performance of Computer-Communication Systems*, H. Ruding and W. Bux, eds., Elsevier, Amsterdam, 1984, 321-331.

[17] L. Kosten, Liquid models for a type of information buffer problems. *Delft Prog. Rep.* **11** (1986) 71-86.

[18] A.S. Krishnakumar and M. Morf, Eigenvalues of a symmetric tridiagonal matrix: a divide-and-conquer approach. *Numer. Math.* **48** (1986) 349-368.

[19] D. Mitra, Stochastic theory of a fluid model of producers and consumers coupled by a buffer. *Adv. Appl. Probab.* **20** (1988) 646-676.

[20] G. Sansigre and G. Valent, A large family of semi-classical polynomials: the perturbed Tchebichev. *J. Comput. Appl. Math.* **57** (1995) 271-281.

[21] A. Simonian, Stationary analysis of a fluid queue with input rate varying as an Ornstein-Uhlenbeck process. *SIAM J. Appl. Math.* **51** (1991) 828-842.

[22] A. Simonian and J. Virtamo, Transient and stationary distributions for fluid queueus and input processes with a density. *SIAM J. Appl. Math.* **51** (1991) 1732-1739.

[23] T.E. Stern and A.I. Elwalid, Analysis of separable Markov- modulated rate models for information-handling systems. *Adv. Appl. Probab.* **23** (1991) 105-139.

[24] J. Virtamo and I. Norros, Fluid queue driven by an M/M/1 queue. *Queueing Systems* **16** (1994) 373-386.