



Mathematical formulations and improvements for the multi-depot open vehicle routing problem

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Abstract

The multi-depot open vehicle routing problem (MDOVRP) is a hard combinatorial optimization problem belonging to the vehicle routing problem family. It considers vehicles departing from different depots to deliver goods to customers without requiring to return to the depots once all customers have been served. The aim of this work is to provide a new two-index-based mathematical formulation for the MDOVRP as well as enhancements on alternative sub-tour elimination constraints. Through numerical experiments, using traditional benchmarks for the MDOVRP, we show that the proposed formulation outperforms existing ones. In addition, we provide insights with regards to problem instance defining parameters on the models performance.

Keywords Multi-depot open vehicle routing problem · Mathematical formulation · Optimization model · Sub-tour elimination constraints

1 Introduction

The multi-depot open vehicle routing problem (MDOVRP) considers a set of vehicles departing from different depots that are not required to return to the depot once they have delivered the goods to customers. This problem was firstly proposed by Tarantilis and Kiranoudis [14] within the context of a real-world application involving the distribution of fresh meat in a real application. They solved the problem by using a list-based threshold-accepting algorithm. Later, Liu et al. [8] proposed the first mixed integer programming (MIP) formulation for the MDOVRP. Due to the extensive time and memory requirements when applying this formulation through a general-purpose

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solver such as CPLEX, the authors additionally proposed a hybrid genetic algorithm (HGA). Lalla-Ruiz et al. [6] proposed an improved formulation based upon adapting and lifting the Miller–Tucker–Zemlin (MTZ) sub-tour elimination constraints as well as additional constraints.

Besides the two above-mentioned formulations, no additional formulation nor exact approach has been developed for this problem. In this sense, having efficient formulations permits advancing the knowledge of the problem as well as assessing more accurately the performance of approximate approaches. With regards to approximate approaches, Soto et al. [13] proposed multiple neighborhood search, tabu search and ejection chains for the MDOVRP. Yao et al. [15] proposed an ant colony optimization approach in the context of product seafood delivery. Lahyani et al. [5] developed a hybrid adaptive large neighborhood search for the MDOVRP. Recently, Brandao [1] proposed a memory-based iterated local search algorithm (MBILSA) that uses the information on the history of the search to solve the MDOVRP.

The goal of this paper is to provide a new two-index-based mathematical formulation for the MDOVRP as well as assess the contribution of alternative constraints for handling flow conservation and sub-tours. In this sense, the comparison with the state-of-the-art formulations reported in the related literature so far indicates that our approach outperforms previous ones in terms of both solution quality and computational time. Furthermore, for some problem instances, our proposed model is capable of solving them to optimality for the first time. Finally, we analyze the relationship between the main defining parameters for this problem (i.e., number of customers, number of depots, demand) and the models' performance.

The remainder of this paper is organized as follows. The proposed mathematical formulation for the MDOVRP as well as additional enhancements are introduced in Sect. 2. The numerical experiments carried out in this work is reported and discussed in Sect. 3. We end with conclusions in Sect. 4.

2 Two-index mathematical formulation

The MDOVRP can be defined as follows. Let $G = (V, A)$ be a directed complete graph, where $V = N \cup D$ is the vertex or node set and $A = V \times V$ is the arc set. Vertex set $D = \{1, 2, \dots, m\}$ represents the set of uncapacitated depots, whereas vertex set $N = \{m + 1, m + 2, \dots, m + n\}$ represents the customers to be served. A symmetric travelling cost, $c_{ij} > 0$, is defined for each arc between each pair of vertices (i, j) , $i, j \in V$, $i \neq j$. Without loss of generality, the traveling cost can represent, according to the application environment, the distance, time, fuel consumption, etc. between each pair of vertices. In this regard, as in [6,8], $c_{ij} = 0$, $\forall i \in N$, $j \in D$, that is, the travelling cost from each customer to each depot is set to zero. This way the flow constraints are considered but they do not have an impact on the objective function. Also, in this work we do not consider distance limitations (i.e., maximum route length per vehicle) as already done in the work proposing models for the MDOVRP [6] and [8] in order to provide comparable objective and time values for the same scenarios. Moreover, each depot $d \in D$ stores and supplies enough goods to serve all the customers. With this goal in mind, each depot has an unlimited fleet of identical

vehicles with the same positive capacity, denoted as Q . Each customer $i \in N$ has a certain demand of goods, denoted as q_i , where $0 < q_i \leq Q$. The MDOVRP pursues to determine the routes of minimum traveling cost satisfying the following conditions:

1. Each customer must be visited in exactly one route.
2. Each customer has to be fully served when visited.
3. Each vehicle departs from one of the available depots and finishes at the last customer it serves.
4. The fleet of vehicles is homogeneous.
5. The total demand of the customers on any route does not exceed the capacity of the vehicles, that is, Q .

The above-mentioned assumptions are those related to the well-known open vehicle routing problem (OVRP, [12]) with the exception of assumption 3 that only applies to cases with more than one depot.

In the following, we introduce a new mathematical formulation for solving the MDOVRP where instead of using three indices, two are used. The following are the decision variables:

x_{ij} 1 if a vehicle travels from node $i \in V$ to node $j \in V$, 0 otherwise

y_i 1 if the minimum travelling cost to visit a customer $i \in N$ is from a depot, 0 otherwise

$u_i \geq 0$ Upper bound on the load already assigned to a vehicle upon leaving node $i \in V$

As can be observed, x_{ij} has two indices, while in the formulations of [6,8] three indices are used, as they include the departing depot as additional index: $x_{ij}^k, k \in D$.

The following parameters are used:

\bar{q}_i For each customer $i \in N$, maximum value of $q_j, \forall j \in N, i \neq j$, i.e., $\max(q_j)_{j \in N, j \neq i}$

d_i Minimum traveling cost from any depot $k \in D$ to customer $i \in N$, i.e., $\min(c_{ji})_{j \in D}$. Note that $d_k = 0, \forall k \in D$

r_i Minimum traveling cost from any customer $j \in N$ to customer $i \in N, i \neq j$, i.e., $\min(c_{ji})_{j \in N, j \neq i}$

The mathematical model proposed in this work is defined as follows:

$$(MDOVRP_{2i}) \text{ minimize } \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} \tag{1}$$

subject to:

$$\sum_{i \in V, i \neq j} x_{ij} = 1, \forall j \in N \tag{2}$$

$$\sum_{i \in V, i \neq j} x_{ij} = \sum_{i \in V, i \neq j} x_{ji}, \forall j \in V \tag{3}$$

$$u_i - u_j + Qx_{ij} + (Q - q_i - q_j)x_{ji} \leq Q - q_j, \forall i, j \in N, i \neq j \tag{4}$$

$$u_i \geq q_i + \sum_{j \in N, i \neq j} q_j x_{ji}, \forall i \in N \tag{5}$$

$$u_i \leq Q - (Q - q_i - \bar{q}_i) \sum_{k \in D} x_{ki} - \sum_{j \in N, i \neq j} q_j x_{ij}, \forall i \in N \quad (6)$$

$$u_k = 0, \forall k \in D \quad (7)$$

$$d_i + M y_i \geq r_i, \forall i \in N \quad (8)$$

$$x_{k'i} \geq y_i, \forall i \in N, k' = \operatorname{argmin}_{k \in D} (c_{ki}) \quad (9)$$

$$\sum_{i \in N} \sum_{k \in D} x_{ki} \geq \left\lceil \frac{\sum_{i \in N} q_i}{Q} \right\rceil \quad (10)$$

$$x_{ij} \in \{0, 1\}, \forall i, j \in V \quad (11)$$

$$y_i \in \{0, 1\}, \forall i \in N \quad (12)$$

$$u_i \geq 0, \forall i \in V \quad (13)$$

The objective function (1) aims at minimizing the total traveling cost of the routes used to deliver the goods to the customers. Constraints (2) guarantee that each customer is visited exactly once. Constraints (3) impose the degree balance of each node, including both customers and depots. Constraints (4)–(7) eliminate the sub-tours in the solutions found by considering the capacity of the vehicles and ensuring that their capacity is not exceeded. These constraints were those studied in [6] and proposed considering the improvements and extensions of the Miller–Tucker–Zemlin subtour elimination constraints (see Miller et al. [9]) to the VRP proposed by Desrochers and Laporte [3] and Kara et al. [4]. Furthermore, as investigated in [6], constraints (8) and (9) establish that if the minimum travelling cost to visit a customer is achieved directly from a depot, then a vehicle will visit the customer directly by departing from that depot. The minimum travelling cost to visit a customer $i \in N$ is determined by comparing all distances to that customer from any other node $j \in V$, $j \neq i$. Constraint (10) sets a lower bound over the number of vehicles required to serve the customers. Finally, constraints (11)–(13) are the integrality and non-negativity constraints for the different kinds of variables. It is worth mentioning that constraints (2)–(7) are necessary to model the MDOVRP, while constraints (8)–(10) are additional ones devoted to strengthen the formulation.

Note that one might but need not to add a redundant constraint (14) which would explicitly rule out arcs connecting pairs of depots from the solutions.

$$\sum_{i \in D} \sum_{j \in D} x_{ij} = 0 \quad (14)$$

Proposition 1 Any solution from the MDOVRP_{2i} formulation can be transformed into a solution for the model of Lalla-Ruiz et al. [6] (i.e., MDOVRP_{3i}) and vice-versa.

Proof For transforming a solution from MDOVRP_{3i} to MDOVRP_{2i}, the following expression has to be applied $x_{ij} = \sum_k x_{ij}^k$, $\forall i, j \in V$. On the other hand, given a solution from MDOVRP_{2i}, variables x_{ij} can be transformed into x_{ij}^k variables by using the following expressions:

$$x_{kj} = x_{kj}^k, \forall k \in D, \forall j \in V \quad (15)$$

$$x_{ij}^k \leq \sum_{l \in V, l \neq i} x_{li}^k, \forall i \in N, \forall j \in V, \forall k \in D, i \neq j \tag{16}$$

$$x_{ij} - \sum_{k \in D} x_{ij}^k = 0, \forall i \in N, \forall j \in V, i \neq j \tag{17}$$

Since all solutions from $MDOVRP_{2i}$ can be transformed into solutions for $MDOVRP_{3i}$ and vice-versa, thus the considered integer formulations are equivalent. \square

2.1 Alternative flow-constraints

In the formulations presented in [6] and [8], the flow constraints consider the return to depots, this consideration is valid for the MDOVRP bearing that both formulations consider $c_{ki} = 0, \forall k \in D$ and $\forall i \in V$, is set. Nevertheless, in the context of open routes, this set of constraints can be refined to avoid the return to depots. Thus, constraints (3) can be replaced by the following new set of constraints:

$$\sum_{i \in V, i \neq j} x_{ij} - \sum_{i \in N, i \neq j} x_{ji} \geq 0, \forall j \in N \tag{18}$$

$$x_{ij} + x_{ji} \leq 1, \forall i, j \in V, i \neq j \tag{19}$$

$$\sum_{k \in D} \sum_{j \in V} x_{jk} = 0 \tag{20}$$

Constraints (18) establish that when a customer is visited, either the route can finish there or continue to another customer but no return to a depot is now contemplated. Constraints (19) forbid symmetries. Finally, constraints (20) set variables x to 0 when returning to depots. This formulation considering the alternative flow constraints is termed as $MDOVRP_{2i-f}$.

Proposition 2 Any solution of $MDOVRP_{3i}$ exists in $MDOVRP_{2i-f}$.

Proof To proof this, consider a feasible solution for a given problem instance composed of (x, y, u) for $MDOVRP_{3i}$. Since $x_{ik}, \forall i \in V, k \in D$ are not considered in $MDOVRP_{2i-f}$, and bearing in mind that $c_{ik} = 0, \forall i \in V, k \in D$ in $MDOVRP_{3i}$, then the partial solution (i.e., without considering x_{ik}) of $MDOVRP_{3i}$ exists in $MDOVRP_{2i-f}$ with the same objective function value. \square

Proposition 3 Any solution of $MDOVRP_{2i-f}$ exists in $MDOVRP_{3i}$ once that solution is extended to incorporate the corresponding x_{ik} values satisfying constraints (3).

Proof Since $x_{ik}, \forall i \in V, k \in D$ are not used in $MDOVRP_{2i-f}$, their corresponding values have to be set. Nevertheless, as $c_{ik} = 0, \forall i \in V, k \in D$ in $MDOVRP_{3i}$, any value set to x_{ik} from the $MDOVRP_{2i-f}$ leads to having a solution in $MDOVRP_{3i}$ with the same objective function value as in $MDOVRP_{2i-f}$. \square

Proposition 4 Any solution of $MDOVRP_{2i}$ exists in $MDOVRP_{2i-f}$ and vice-versa.

Proof Considering Propositions 3 and 4 with the procedure discussed in Proposition 1. \square

Proposition 5 $MDOVRP_{2i}$, $MDOVRP_{2i-f}$, and $MDOVRP_{3i}$ are equivalent.

Proof Consider Propositions 3, 4, and 5. \square

2.2 Incorporating arc-based loading constraints to $MDOVRP_{2i-f}$

As can be checked in the $MDOVRP_{2i}$, the MTZ elimination constraints [i.e., constraints (4)–(7)] are adapted to hold capacities as opposed to [8]. This allows to ensure the continuity of vehicles in terms of demand. Nevertheless, similarly as done in [10,11], remaining load on the vehicles among arcs can be explicitly be considered and, hence, used to avoid sub-tours.

$$\sum_{i \in V, i \neq j} u_{ij} - \sum_{i \in V, i \neq j} u_{ji} \geq q_j, \forall j \in N \quad (21)$$

$$(Q - q_i) \cdot x_{ij} \geq u_{ij}, \forall i, j \in N \quad (22)$$

$$Q \cdot x_{kj} \geq u_{kj}, \forall k \in D, j \in N \quad (23)$$

Constraints (21) set a lower-bound on the remaining quantity of the arcs. That is, if a customer j is visited, the current load of the vehicle has to be larger than the requested quantity as well as the next customer requested quantity in the route. Constraints (22–23) set an upper bound on variables u . The formulation $MDOVRP_{2i-f}$ replacing (4)–(7) by (21)–(23) is termed as $MDOVRP_{2i-flv}$.

Proposition 6 The LP relaxation of $MDOVRP_{2i-flv}$ is tighter than that of $MDOVRP_{2i}$.

Proof Since constraints (22)–(23) and (5)–(7) establish the bounds on u variables in the corresponding formulations, it is sufficient to show that constraints (21) are stronger than those relating u for two different customers, namely constraints (4). In doing so, consider constraints (4) in the LP relaxation of $MDOVRP_{2i}$, as it includes variables x , it leads to the following cases:

- when $x_{ij} > 0$ and $x_{ji} = 0$; $u_j \geq u_i + q_j - Q + Qx_{ij}$. This can be rewritten as $u_j \geq u_i + q_j - \Delta Q$.
- Similarly, when $x_{ji} > 0$ and $x_{ij} = 0$; $u_j \geq u_i - Q + q_j + (Q - q_i - q_j)\Lambda x_{ij}$, that is, $u_j \geq u_i - \Lambda Q + \Lambda q_j - \Lambda q_i$.
- Finally, $x_{ji}, x_{ij} > 0$; $u_j \geq u_i - Q + q_j - \Delta Q + \Lambda Q - \Lambda q_i - \Lambda q_j$.

Where Δ and Λ represent the fractional value due to the multiplication with x_{ij} and x_{ji} , respectively. Notice that in the first two cases, related constants are considered.

With regards to constraints (21), it is necessary to convert u_{ij} into u_i as considered in $MDOVRP_{2i}$. In doing so, the following transformation has to be conducted for each

customer $j \in N$: $u_j = Q - \sum_{l \in V} u_{lj} + q_j$. That can be rewritten as $\sum_{l \in V} u_{lj} = Q - u_j + q_j$. The previous permits to transform constraints (21) in: $Q - u_j \geq \sum_{l \in V} u_{jl} \implies Q - u_j \geq Q - u_l + q_l \implies u_l \geq u_j + q_l$. Where for a customer j, l represents the customer visited after j in that route.

As can be checked, the previous expression is tighter than those obtained for MDOVRP_{2i} as x variables are not incorporated. This completes the proof. \square

Proposition 7 *The LP relaxation of MDOVRP_{2i-flv} is tighter than MDOVRP_{3i}.*

Proof Similar to that for Proposition 6. \square

3 Computational results

This section presents the computational experiments carried out to assess the performance of the proposed mathematical model as well as improvements proposed in this work. With this goal in mind, we compare the performance of our formulations with that of the mathematical models proposed by Liu et al. [8] and Lalla-Ruiz et al. [6]. All computational experiments were conducted on a computer equipped with an Intel Core i7 3.7 GHz and 32 GB of RAM and all models, including the ones from the literature, were implemented in a general-purpose solver, CPLEX version 12.9, with a maximum computational time of 2 hours (7,200 seconds) and limited to one-thread.

The problem instances used in this work were initially proposed by Cordeau et al. [2] for the MDVRP. Liu et al. [8] used them for assessing their MDOVRP formulation and algorithm. These and subsequent works do not consider the distance limitation. To provide a fair comparison, i.e., using comparable values and times, as well as an equal comparison with approximate approaches proposed for this problem, we do the same. Moreover, the main parameters ruling the instances are the number of customers $|N|$, number of depots $|D|$, and capacity of the vehicles Q . This as well as additional information is provided in detail in Sect. 3.2, in Table 3. This section also provides insights on how the instance parameters influence performance.

3.1 Optimization models' performance

The computational results obtained by the linear relaxation of the models proposed by Liu et al. [8], Lalla-Ruiz et al. [6], and the ones proposed in this paper are depicted in Table 1. The first column shows the identifier of each problem instance to solve, *Inst.* For each formulation, the column *Value* provides the optimal value and $t(s.)$ the computation time in seconds.

As can be seen, for all instances, the linear relaxation is improved with the exception of $p12$ that provides the same value also in the 3 index formulation. As later discussed in Figs. 1 and 2, it can be inferred that a larger number of constraints is beneficial at the expense of higher computational times. Nevertheless, comparing the case without additional improvements MDOVRP_{2i} with MDOVRP_{3i}, it can be observed that the quality of the relaxation remains the same but the average computational time is reduced by 78%.

Table 1 Computational results obtained from the relaxation of the models

Inst.	MDOVRP _{Liu} [8]		MDOVRP _{3i} [6]		MDOVRP _{2i}		MDOVRP _{2i-flv}	
	Value	t (s.)	Value	t (s.)	Value	t (s.)	Value	t (s.)
p01	350.88	0.10	375.06	0.13	375.06	0.05	378.41	0.11
p02	350.88	0.10	374.92	0.13	374.921	0.04	374.93	0.10
p03	440.17	0.32	468.35	0.44	468.35	0.09	469.18	0.49
p04	561.49	0.20	607.05	0.38	607.05	0.18	628.35	1.28
p05	557.61	0.20	592.90	0.43	592.90	0.19	596.91	1.85
p06	546.96	0.33	582.00	0.57	582.00	0.18	595.41	1.33
p07	547.36	0.66	575.77	0.67	575.77	0.18	588.87	1.42
p08	1867.48	1.45	2350.55	2.70	2350.55	1.40	2530.02	52.68
p09	1840.72	2.51	2216.97	3.67	2216.97	1.23	2369.93	54.20
p10	1827.73	3.78	2190.97	5.05	2190.97	1.28	2294.66	53.16
p11	1807.86	6.64	2145.14	6.39	2145.14	1.29	2284.66	55.13
p12	915.98	0.16	953.26	0.17	953.26	0.10	953.26	0.12
p15	1815.39	1.61	1881.67	1.60	1881.67	0.43	1881.95	0.55
p18	2714.80	8.84	2810.07	6.52	2810.07	1.15	2810.64	1.94
pr01	617.20	0.09	633.31	0.11	633.31	0.04	635.86	0.07
pr02	867.37	0.40	962.77	0.60	962.77	0.15	966.18	0.95
pr03	1225.69	0.95	1392.53	1.54	1392.53	0.35	1397.96	5.96
pr04	1287.08	1.95	1449.24	3.46	1449.24	0.70	1464.45	17.07
pr05	1393.43	6.14	1591.55	6.14	1591.55	1.34	1615.36	52.26
pr06	1682.19	10.83	1892.23	10.02	1892.23	2.01	1917.74	106.54
pr07	765.71	0.36	817.31	0.40	817.31	0.09	818.05	0.26
pr08	1101.74	1.72	1231.75	1.94	1231.75	0.34	1237.00	5.53
pr09	1390.40	7.59	1531.43	5.73	1531.43	0.86	1544.33	19.55
pr10	1638.47	12.65	1858.56	11.27	1858.56	1.79	1892.94	101.16
Avg.	1171.44	2.90	1311.89	2.92	1311.89	0.64	1343.63	22.24

Best values are reported in bold

With regards to the results obtained by the non-relaxed optimization models they are reported in Table 2. The layout of this table is similar to the previous one but in this case, we also provide columns *UB* and *LB* for reporting the upper and lower bounds reported by CPLEX within the time limit set by default, respectively. Column *Gap* (%) reports the relative error of the bounds in percentage.

From the computational results reported in Table 1, having a two-index formulation greatly accelerates the solving time when compared to the three-index formulation. Moreover, MDOVRP_{2i-flv} provides better optimal values, indicating thus a better relaxation.

Table 2 Computational results obtained by the mathematical models proposed by Liu et al. [8] and that proposed in this paper for the instances introduced by Cordeau et al. [2]. Best values are reported in bold

Inst.	MDOVRP _{Liu} [8]				MDOVRP _{3i} [6]				MDOVRP _{2i}				MDOVRP _{2i-flv}			
	UB	LB	gap (%)	t (s.)	UB	LB	gap (%)	t (s.)	UB	LB	gap (%)	t (s.)	UB	LB	gap (%)	t (s.)
p01	386.18	386.18	0.00	696.78	386.18	386.18	0.00	47.26	386.18	386.18	0.00	1.05	386.18	386.18	0.00	0.91
p02	375.93	375.93	0.00	14.30	375.93	375.93	0.00	0.92	375.93	375.93	0.00	0.20	375.93	375.93	0.00	0.20
p03	474.57	474.57	0.00	4699.56	474.57	474.57	0.00	42.04	474.57	474.57	0.00	0.95	474.57	474.57	0.00	1.38
p04	704.81	599.68	14.92	7200	662.85	619.40	6.56	7200	664.21	624.26	6.01	7200	662.22	652.07	1.73	7200
p05	615.24	599.04	2.63	7200	609.07	599.41	1.59	7200	607.53*	607.53	0.00	6891.12	607.53*	607.53	0.00	72.95
p06	631.11	581.22	7.91	7200	611.99	590.76	3.47	7200	613.96	597.79	2.63	7200	611.99	611.99	0.00	411.96
p07	638.97	573.49	10.25	7200	610.55	583.48	4.43	7200	609.74	592.19	2.88	7200	608.28*	608.28	0.00	1031.42
p08	4022.37	2162.40	46.24	7200	3088.58	2402.09	22.23	7200	2877.46	2404.54	16.44	7200	2870.21*	2617.06	10.93	7200
p09	5328.53	2119.61	60.22	7200	2994.12	2269.96	24.19	7200	2680.05	2277.41	15.02	7200	2660.45*	2461.48	7.89	7200
p10	5793.60	2068.24	64.30	7200	2824.81	2246.16	20.48	7200	2551.24	2252.20	11.72	7200	2528.98*	2379.37	7.72	7200
p11	5872.87	2068.24	64.78	7200	2845.15	2192.32	22.95	7200	2537.19	2191.49	13.63	7200	2499.25*	2366.46	6.80	7200
p12	953.26	953.26	0.00	2.45	953.26	953.26	0.00	0.58	953.26	953.26	0.00	0.23	953.26	953.26	0.00	0.46
p15	1885.81	1885.81	0.00	655.88	1885.81	1885.81	0.00	17.77	1885.81	1885.81	0.00	3.15	1885.81	1885.81	0.00	3.63
p18	2826.64	2787.29	1.39	7200	2818.36	2818.36	0.00	321.05	2818.36	2818.36	0.00	9.04	2818.36	2818.36	0.00	14.68
pr01	647.03	647.03	0.00	6.81	647.03	647.03	0.00	0.47	647.03	647.03	0.00	0.17	647.03	647.03	0.00	0.14
pr02	979.82	979.82	0.00	2258.29	979.82	979.82	0.00	18.21	979.82	979.82	0.00	2.42	979.82	979.82	0.00	1.97
pr03	1435.40	1395.44	2.78	7200	1423.48	1423.48	0.00	5998.46	1423.48	1423.48	0.00	26.91	1423.48	1423.48	0.00	16.84
pr04	1714.20	1441.03	15.94	7200	1521.33	1478.25	2.83	7200	1514.07	1498.71	1.01	7200	1514.07	1514.07	0.00	1738.18
pr05	2084.07	1560.73	25.11	7200	1736.38	1608.33	7.37	7200	1706.80	1620.03	5.08	7200	1716.02	1645.67	3.96	7200
pr06	3230.27	1867.42	42.19	7200	2049.15	1916.42	6.48	7200	1984.32	1931.20	2.68	7200	1978.46	1956.22	1.74	7200
pr07	821.25	821.25	0.00	82.11	821.25	821.25	0.00	2.33	821.25	821.25	0.00	0.30	821.25	821.25	0.00	0.30

Table 2 continued

Inst.	MDOVRP _{Liu} [8]			MDOVRP _{3i} [6]			MDOVRP _{2i}			MDOVRP _{2i-flu}						
	UB	LB	gap (%)	t (s.)	UB	LB	gap (%)	t (s.)	UB	LB	gap (%)	t (s.)	UB	LB	gap (%)	t (s.)
pr08	1261.31	1230.94	2.41	7200	1254.45	1252.84	0.13	7200	1254.45*	1254.45	0.00	18.46	1254.45*	1254.45	0.00	8.45
pr09	1940.03	1495.91	22.89	7200	1594.17	1558.50	2.24	7200	1591.78	1579.10	0.80	7200	1591.78*	1591.78	0.00	326.22
pr10	2731.53	1643.15	39.85	7200	2394.83	1887.49	21.18	7200	1977.33	1905.72	3.62	7200	1997.96	1934.29	3.31	7200
Avg.	1973.12	1279.90	17.66	5151.43	1481.80	1332.13	6.09	4468.71	1413.99	1337.59	3.40	3889.75	1411.14	1373.60	1.84	2551.24

* Solved to proven optimality for the first time

With regards to the non-relaxed models reported in Table 2, it can be noted that our proposed mathematical model, either with or without enhancements, provides better performance in terms of narrower linear bounds and computational times than MDOVRP_{Liu} and MDOVRP_{3i} . The proposed models improve or at least equal the quality of the linear bounds obtained by the model MDOVRP_{3i} in the majority of the cases. In terms of gap, the model that considers the enhancements reports a gap of only 1.84% on average in comparison with the gap of 3.40% on average reported by the model without enhancements or 6.09% obtained using MDOVRP_{3i} . This reduction of gap finds its rationale behind the improved lower bounds provided by our model. In this regard, it is worth highlighting that the lower bounds provided by MDOVRP_{2i-flv} are equal or better than the best ones of the rest of formulations.

Furthermore, our mathematical model solves to optimality for the first time the problem instances $p05$, $p07$, $p08$, $p09$, $p10$, $p11$, $pr08$, and $pr09$. It is noticeable that in the cases where state-of-the-art mathematical models provide the optimal solution, our proposed models require remarkably less computational time. For example, for the problem instance $p01$, the models MDOVRP_{Liu} and MDOVRP_{3i} require 696.78 and 47.26, respectively while the new models require about 1 second. Another example from the last reported model is $pr03$, the state-of-the-art model requires around 5998 seconds to solve to optimality, whereas our proposed models require 26.91 and 19.84 seconds. Finally, in those cases where the optimal solution has not yet been achieved, both proposed models enhance the quality of the bounds for all cases.

Figures 1 and 2 show the structure of the mathematical models in terms of number of rows and binaries generated by the solver when dealing with the problem instances under analysis. As can be observed, MDOVRP_{2i-flv} requires a larger amount of number of rows than MDOVRP_{2i} and MDOVRP_{3i} which is mainly caused by the additional flow and arc-load based constraints. Nevertheless, it can be seen that both, MDOVRP_{2i} and MDOVRP_{2i-flv} , greatly reduce the number of binaries with regards to state-of-the-art formulations. Considering this information along with Table 1, we see that the more constraints provide a tighter LP relaxation. It is worth noting that this is just a comparison by means of the number of constraints, however, some constraints are different and restrict the feasible region in a different way.

The comparison of the proposed model with and without enhancements leads to the following conclusions:

- Including additional constraints as in MDOVRP_{2i-flv} , instead of the Miller–Tucker–Zemlin subtour elimination constraints (4)–(6), that keep track of the remaining space along arcs, improves the quality of the solutions in comparison to MDOVRP_{2i} and MDOVRP_{3i} . Furthermore, as can be observed, the average computational times are reduced while the quality of bounds is improved. Also, it overall leads to a large number of best solutions found.
- The model with and without enhancements reports better performance in terms of computational time and linear bounds. In this regard, the gap difference with respect to those already reported in the literature is substantial. For instance,

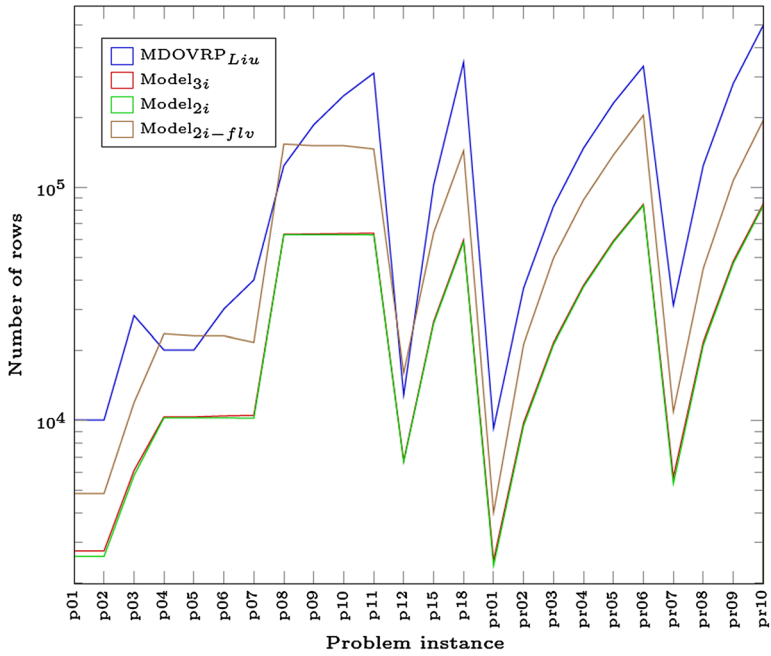


Fig. 1 Number of rows processed by CPLEX for solving each instance

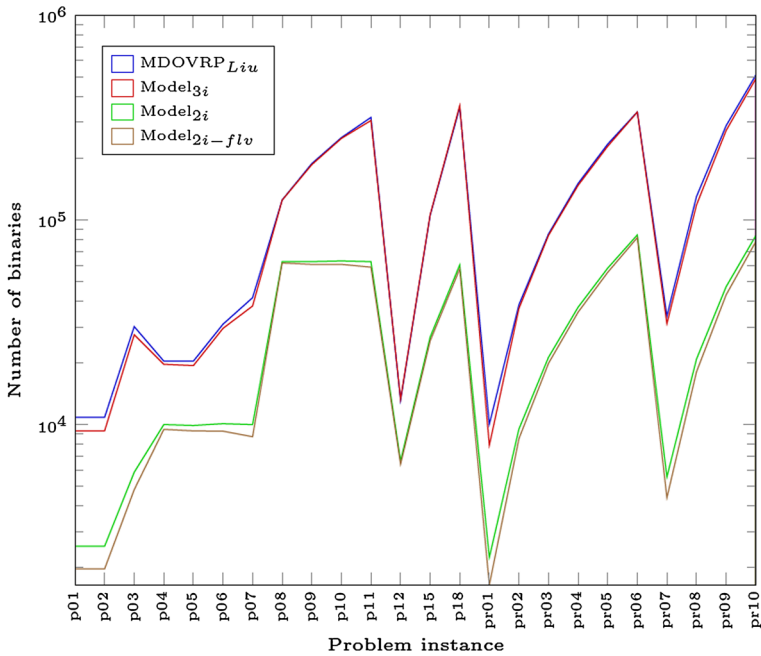


Fig. 2 Number of binaries processed by CPLEX for solving each instance

the gap of the hardest instances so far, i.e., $p08 - p11$, has been reduced to less than half in $MDOVRP_{2i-flv}$. On other cases, the enhancement in terms of time performance is noticeable, e.g., $pr03$, where the time to provide the optimal solution is reduced from 5998.46 to less than 27 seconds with $MDOVRP_{2i}$, and 17 with $MDOVRP_{2i-flv}$.

- For some instances, the inclusion of the enhancements accelerates the convergence to optimal solutions. For example, in $p05$, $MDOVRP_{2i-flv}$ requires 72.96 while $MDOVRP_{2i}$ requires 6891.12. Considering the information provided in Figs. 1 and 2, we see the positive influence of adding the proposed additional constraints without increasing the number of variables.

3.2 Analysis considering problem instances' structure

In this section we provide insights into the effects of the instance parameter settings on the performance of the models. In Table 3, the following scenario parameters are reported: number of customers $|N|$, number of depots $|D|$, and capacity of the vehicles Q . Also, we report the ratio between the number of customers and depots (customer-depots= $|N|/|D|$), total amount of customers' demand ($\sum q_i$), and the minimum number of vehicles needed to satisfy that demand (minimum vehicles= $\sum q_i/Q$). Finally, we also report the gap performance of all MDOVRP optimization models.

As can be concluded from the table, in those cases where the ratio customer-depots $|N|/|D|$ is relatively low, i.e., less than or equal to 15, all formulations provide the optimal solution. When this ratio increases from 15 to 40, we see that not in all cases the optimal solutions are provided. In some cases, when they have the same ratio, the minimum number of vehicles influences, like for example, $p12$ and $p15$, where the three formulations are solved to optimality. Those scenarios have a ratio of 40. However, when the minimum number of vehicles $\sum q_i/Q$ increases like in $p18$, the problem becomes harder to solve for $MDOVRP_{Liu}$.

When considering only the best state-of-the-art performing model (i.e., $MDOVRP_{3i}$) and our current proposals, we observe that the hardest instances to solve in terms of gap are $p08$, $p09$, $p10$, and $p11$. Those instance have in common that they have the highest $\sum q_i/Q$. Also, they have the highest ratio customer-depots $|N|/|D|$ (except for $pr06$ that has a higher value than $p10$ and $p11$). Thus, considering the previous and current observation, we see an influence of those parameters on the performance of the models.

Finally, it should be noted that these observations only take into account structural instance parameters but the model constraints, variables, and specific instance information play a relevant role on the models performance. Therefore, a more in-depth study in this research direction might be valuable.

Table 3 Problem instances' structure and optimization models gap performance

Instance	Instance structure			Exact approaches' gap (%)						
	$ N $	$ D $	Q	$\sum q_i$	$ N / D $	$\sum q_i/Q$	MDOVRP _{Liu}	MDOVRP _{3i}	MDOVRP _{2i}	MDOVRP _{2i-fiv}
p01	50	4	80	777	12.50	9.71	0.00	0.00	0.00	0.00
p02	50	4	160	777	12.50	4.856	0.00	0.00	0.00	0.00
p03	75	5	140	1364	15.00	9.74	0.00	0.00	0.00	0.00
p04	100	2	100	1458	50.00	14.58	14.92	6.56	6.01	1.73
p05	100	2	200	1458	50.00	7.29	2.63	1.59	0.00	0.00
p06	100	3	100	1458	33.33	14.58	7.91	3.47	2.63	0.00
p07	100	4	100	1458	25.00	14.58	10.25	4.43	2.88	0.00
p08	249	2	500	12106	124.50	24.21	46.24	22.23	16.44	10.93
p09	249	3	500	12106	83.00	24.21	60.22	24.19	15.02	7.89
p10	249	4	500	12106	62.25	24.21	64.30	20.48	11.72	7.72
p11	249	5	500	12106	49.80	24.21	64.78	22.95	13.63	6.80
p12	80	2	60	432	40.00	7.20	0.00	0.00	0.00	0.00
p15	160	4	60	864	40.00	14.40	0.00	0.00	0.00	0.00
p18	240	6	60	1296	40.00	21.60	1.39	0.00	0.00	0.00
pr01	48	4	200	657	12.00	3.29	0.00	0.00	0.00	0.00
pr02	96	4	195	1220	24.00	6.26	0.00	0.00	0.00	0.00
pr03	144	4	190	1788	36.00	9.41	2.78	0.00	0.00	0.00
pr04	192	4	185	2477	48.00	13.39	15.94	2.83	1.01	0.00
pr05	240	4	180	3351	60.00	18.62	25.11	7.37	5.08	3.96
pr06	288	4	175	3671	72.00	20.98	42.19	6.48	2.68	1.74
pr07	72	6	200	948	12.00	4.74	0.00	0.00	0.00	0.00
pr08	144	6	190	2006	24.00	10.56	2.41	0.13	0.00	0.00
pr09	216	6	180	2736	36.00	15.20	22.89	2.24	0.80	0.00
pr10	288	6	170	3850	48.00	22.65	39.85	21.18	3.62	3.31

4 Conclusions and further research

In this work, we have presented a novel formulation and enhancements for the well-known Multi-Depot Open Vehicle Routing Problem. Both contributions provide better results in terms of computational time and number of best solutions provided, and provide tighter linear bounds than those mathematical formulations previously reported in the literature. The incorporation of constraints to track vehicles remaining capacity as well as symmetry constraints to consider empty routes remarkably improves the convergence speed over those problem instances where the optimal solutions are reached. Furthermore, the numerical results indicate that our mathematical model reduces the computational burden while achieving better solutions for those instances where the optimal solutions are not yet known. In this regard, the new formulation with and without enhancements provides new best values for all those problem instances and provides optimal solutions for some instances that had not been solved to optimality so far. Also, the analysis of the gaps provided by the models and the instance parameters showed that the relation between customers, depots, and the minimum number of required vehicles have a relevant influence for the studied instances.

Even though our proposed formulation and improvements have been tested only for the MDOVRP, we believe they can contribute and improve other variants of the VRP and multi-depot VRP (e.g., [2,7]) in future research. Finally, a more in-depth study on the influence of instance parameters on the solution approaches performance will be a topic for further research.

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