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Analysis of feedback fluid queues

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Abstract

In this article we consider the stationary behaviour of a class of feedback fluid queues. A feedback fluid queue is a natural generalisation of the well-known Markov modulated fluid queue. The essential difference is that the current state of the fluid buffer influences the behaviour of the regulating process. Therefore these models are relevant for the performance evaluation of certain regulation mechanisms in telecommunications access networks. Typically, they enable the analysis of mechanisms in which the content of a buffer is driven by some source process, while the source process itself is influenced by feedback signals coming from the buffer.

We write down the differential equations governing the stationary distribution for a large class of these models, and subsequently solve them using the spectral method. The unknown parameters in the solution are determined by an equal number of boundary conditions.

Keywords: Fluid queue, buffer content, feedback, spectral method.

AMS Subject Classifications (1991) — 60K25

1 Introduction

In the area of modern telecommunication systems, fluid queues are often used as burst scale models for multiplexers, see e.g. [8], in particular chapter 17. In such models, the content $C(t)$ of a fluid queue (or fluid buffer) changes in time at a rate which is determined by some stochastic process $(X(t), t \geq 0)$. When $(X(t))$ is a Markov process we are dealing with a so-called Markov modulated fluid model, going back to [4] and [2]. The process $(X(t))$ usually represents the sources that generate traffic that is to be carried by a telecommunications network. Thus, the behaviour of these sources is assumed to be independent of the current status of the fluid buffer, since the process $(X(t))$ behaves as an autonomous Markov process.

In [9] the concept of *feedback fluid models* was introduced as a means to deal with this shortcoming. This class of models forms a natural extension of the traditional Markov modulated fluid models. In particular, the behaviour of the process $(C(t),\, t \geq 0)$ is determined by that of the process $(X(t))$ as before. However, the evolution of $(X(t))$ is no longer autonomous, but depends on the current state of the fluid reservoir. In other words, the processes $(X(t))$ and $(C(t))$ now interact, since the dependence works both ways.

Such feedback models are particularly relevant in modeling advanced regulation mechanisms in the access network. We mention two examples in which this is demonstrated, namely [1] and [6]. In the first paper a dual leacky bucket traffic shaper is analysed, enforcing the peak rate, sustainable rate and maximum burst size of some connection. The fluid buffer models the token buffer, while the data buffer is modeled by a traditional single server queue, fed by a Poisson process of arriving jobs. In this way it is possible to capture the two-way depence between data buffer and token buffer. In a later extension of this model, the discrete queue was replaced by a fluid queue driven by an on-off source, thus leading to a model in which two fluid buffers interact, see [5, 9].

Another area of interest is that of congestion avoidance mechanisms in the acces network. One possibility for this is that the sources receive an explicit notification from the access router, telling them when and how to adapt their traffic generation rate. Such a protocol is described and analysed in [6] where an infinitely large buffer is fed by a finite number of on-off sources. The behaviour of the sources depends on whether the buffer content is below, at or above a certain threshold.

The contribution of the current article is that we show for a large class of feedback fluid models (including those in [1] and [6]) how it is possible to analyse the stationary behaviour. The type of feedback considered is one in which the regulating process behaves like some Markov process as long as the content of the fluid buffer is between two thresholds. To be more precise, we define the system to be in *regime* k whenever the buffer content is between two thresholds $B^{(k-1)}$ and $B^{(k)}, k = 1, \ldots, K$, and we assume that the regulating process behaves like a Markov chain with generator $Q^{(k)}$ in this regime. Moreover, we allow the net rate of change of the level process $(C(t))$ to depend not only on the actual state of the regulating process $(X(t))$, but also on the current regime. A possible consequence of this may be that the buffer content process remains at some threshold $B^{(k)}$ for a while. In this case, different generators $\tilde{Q}^{(k)}$ may be specified that describe the behaviour of the regulating process at the thresholds.

The threshold-like behaviour of the fluid rates has been investigated earlier in [3]. However, the regulating process in that paper was a true

Markov-process, evolving independently of the buffer content process, so that it did not constitute a feedback fluid model in the sense described above. The more general setting described here may be preferred to that in [3], due to the fact that the status of the buffer will often not only affect the sending rate of the sources, but also their dynamics: when the sending rate of a source drops, its activity period will last longer.

The remainder of this text is structured as follows. In Section 2 we formally introduce the model, including some assumptions. The differential equations that govern the stationary distribution are derived in Section 3. In Sections 4 and 5 we subsequently solve these for the finite and infinite buffer case, using the spectral method. In particular we make sure that the unknown parameters in the solution are determined by an equal number of boundary conditions. Finally, in Section 6 we illustrate the procedure for a simple example.

2 Model

We consider a feedback fluid queue as introduced in the previous section. Hence let $C(t)$ denote the fluid level in the buffer at time t and let $X(t)$ be the state of the regulating process at time t. The size of the fluid reservoir is denoted by B, which may be finite or infinite. Furthermore we identify *thresholds* $B^{(k)}, k = 0, \dots, K$, such that

$$0 = B^{(0)} < B^{(1)} < \cdots < B^{(K-1)} < B^{(K)} = B \leq \infty.$$

At times t when $B^{(k-1)} < C(t) < B^{(k)}, k = 1, \dots, K$, we say the system is *in regime* k, while we say it is *at threshold* k at time t when $C(t) = B^{(k)}, k = 0, \dots, K$. In the remainder any superscript $^{(k)}$ will refer to regime and/or threshold k.

We assume that the state space of the process $(X(t))$ is finite, and we denote it by $S = \{1, \dots, N\}$. The dynamics of the system are given as follows: When the system is in regime k (at threshold k), the process $(X(t))$ behaves like an irreducible Markov process with generator $Q^{(k)}(\tilde{Q}^{(k)})$. Furthermore, the net rate of change of the process $(C(t))$ is given by $r_i^{(k)}$ at times when the system is in regime k and $X(t) = i$. For each regime k, these rates are collected in a diagonal matrix $R^{(k)}$ with elements $(R^{(k)})_{ii} = r_i^{(k)}$.

In order to state some assumptions on the rates $r_i^{(k)}$, we define for all regimes the subsets of S consisting of up-states respectively down-states, as

$$S_+^{(k)} = \{i \in S | r_i^{(k)} > 0\} \quad (1)$$

$$S_-^{(k)} = \{i \in S | r_i^{(k)} < 0\}. \quad (2)$$

From now on we will assume the following to hold:

1. $S_+^{(k)} \cup S_-^{(k)} = S$ for all k, i.e. each fluid rate is nonzero. This assumption is made for the sake of brevity.

2. $S_+^{(k+1)} \cap S_-^{(k)} = \varnothing$ for all k, i.e. there is no state $i \in S$ for which $r_i^{(k)} < 0$ and $r_i^{(k+1)} > 0$ for some k. Not only does this assumption seem natural to make from an applications point of view, it also excludes certain ambiguity problems.

3. $S_-^{(k+1)} \cap S_-^{(k)} \neq \varnothing$ for all k, i.e. each threshold can be crossed downwards in at least one state $i \in S$.

4. $S_+^{(k+1)} \cap S_+^{(k)} \neq \varnothing$ for all k, i.e. each threshold can be crossed upwards in at least one state $i \in S$.

5. In the case when $B = \infty$, we assume the following stability condition,

$$\sum_{i=1}^{N} \pi_i^{(K)} r_i^{(K)} < 0, \quad (3)$$

   where $\pi_i^{(K)}$ is the stationary distribution of a Markov process with generator $Q^{(K)}$. Hence this assumption entails that the expected rate of change of the buffer content process, conditional on the process being above level $B^{(K-1)}$, is negative.

Notice that, when $X(t) \in S_-^{(k+1)} \cap S_+^{(k)}$ (when this set is nonempty), there is a confluence of drifts in $B^{(k)}$, so that the content process may stay at $B^{(k)}$ for a while, until $(X(t))$ jumps to a state $j \notin S_-^{(k+1)} \cap S_+^{(k)}$. This clarifies the ambiguity problem that is solved by the second assumption, since this assumption ensures that it is clearly determined what happens to the buffer content immediately after $(X(t))$ jumps from $j \in S_-^{(k+1)} \cap S_+^{(k)}$ to $i \notin S_-^{(k+1)} \cap S_+^{(k)}$.

Under the assumptions above it can be shown that the joint process $(X(t), C(t))$ is regenerative with finite expected cycle length. Hence it converges in distribution to a pair of random variables (X, C). Its distribution will be denoted by

$$F_i(y) = Pr[X = i, C \le y], \qquad i = 1, \dots, N, \;\; 0 \le y \le B.$$

Our goal in the next section is to find explicit expressions for this distribution.

3 The differential equations

In this section we give the equations that determine the stationary distribution we are looking for, both for the finite- and the infinite-buffer case. We start off by defining for $k = 1, \dots, K$, the following functions,

$$F_i^{(k)}(t, y) = Pr[X(t) = i, \; C(t) \le y], \quad t \ge 0, \; B^{(k-1)} < y < B^{(k)}, \; i \in S. \tag{4}$$

If we express $F_i^{(k)}(t+h, y)$ in the $F_j^{(\ell)}(t, \cdot)$, $j = 1, \dots, N$, $\ell = 1, \dots, k$, (and an $o(h)$ term) in the usual way, see e.g. [2], and subsequently let $h \to 0$, we find the Kolmogorov forward equations for regime k,

$$\begin{aligned}
\frac{\partial F_i^{(k)}(t,y)}{\partial t} + r_i^{(k)} \frac{\partial F_i^{(k)}(t,y)}{\partial y} = & \\
& \sum_{j=1}^{N} q_{ji}^{(k)} \left(F_j^{(k)}(t,y) - F_j^{(k)}(t, B^{(k-1)}) \right) \\
+ & \sum_{j=1}^{N} \tilde{q}_{ji}^{(k-1)} \left(F_j^{(k)}(t, B^{(k-1)}) - F_j^{(k-1)}(t, B^{(k-1)}) \right) \\
+ & \sum_{j=1}^{N} q_{ji}^{(k-1)} \left(F_j^{(k-1)}(t, B^{(k-1)}) - F_j^{(k-1)}(t, B^{(k-2)}) \right) \\
& \vdots \\
+ & \sum_{j=1}^{N} q_{ji}^{(1)} \left(F_j^{(1)}(t, B^{(1)}) - F_j^{(1)}(t, 0) \right) \\
+ & \sum_{j=1}^{N} \tilde{q}_{ji}^{(0)} \left(F_j^{(1)}(t, 0) \right).
\end{aligned}$$

Here, as in the remainder, $F_i^{(k)}(B^{(k)})$ stands for $\lim_{y \uparrow B^{(k)}} F_i^{(k)}(y)$ and $F_i^{(k+1)}(B^{(k)})$ for $\lim_{y \downarrow B^{(k)}} F_i^{(k+1)}(y)$. Assuming stationarity, we now set $F_i^{(k)}(t, y) \equiv$

$F_i^{(k)}(y)$ and $\frac{\partial F_i^{(k)}(t,y)}{\partial t} \equiv 0$ for $i = 1, \ldots, N$, so that we arrive, in matrix form, at

$$\begin{aligned} R^{(k)} \frac{d\mathbf{F}^{(k)}(y)}{dy} &= (Q^{(k)})^T \; \mathbf{F}^{(k)}(y) \\ &+ (\tilde{Q}^{(k-1)} - Q^{(k)})^T \; \mathbf{F}^{(k)}(B^{(k-1)}) \\ &+ (Q^{(k-1)} - \tilde{Q}^{(k-1)})^T \; \mathbf{F}^{(k-1)}(B^{(k-1)}) \\ &\;\; \vdots \\ &+ (Q^{(1)} - \tilde{Q}^{(1)})^T \; \mathbf{F}^{(1)}(B^{(1)}) \\ &+ (\tilde{Q}^{(0)} - Q^{(1)})^T \; \mathbf{F}^{(1)}(0) \qquad k = 1, \ldots, K. \end{aligned} \tag{5}$$

Notice that, unlike in the Markov modulated setting, these equations are inhomogeneous (apart from the one for $k = 1$ if $\tilde{Q}^{(0)} = Q^{(1)}$). Once we solved the K matrix differential equations (5) using some appropriate boundary conditions, we are done in the infinite buffer case. The probability $\mathbf{F}(y)$ that we are looking for is then given by $\mathbf{F}(y) = \mathbf{F}^{(k)}(y)$, where k is such that $B^{(k-1)} \leq y < B^{(k)}$.

In the finite buffer case the same is true for $y < B$. However we now also have to find $p_i \equiv F_i(B), i = 1, \ldots, N$, the stationary probability that the regulating process is in state i. Due to the presence of feedback these probabilities cannot be found beforehand as in the traditional Markov modulated fluid models. However, we can write down balance equations for them in terms of the functions $F_i^{(k)}$ via the forward Kolmogorov equations for the process $(X(t))$. We find in matrix form

$$\begin{aligned} \mathbf{0} &= (\tilde{Q}^{(K)})^T \; \mathbf{p} \\ &+ (Q^{(K)} - \tilde{Q}^{(K)})^T \; \mathbf{F}^{(K)}(B) \\ &+ (\tilde{Q}^{(K-1)} - Q^{(K)})^T \; \mathbf{F}^{(K)}(B^{(K-1)}) \\ &+ (Q^{(K-1)} - \tilde{Q}^{(K-1)})^T \; \mathbf{F}^{(K-1)}(B^{(K-1)}) \\ &\;\; \vdots \\ &+ (Q^{(1)} - \tilde{Q}^{(1)})^T \; \mathbf{F}^{(1)}(B^{(1)}) \\ &+ (\tilde{Q}^{(0)} - Q^{(1)})^T \; \mathbf{F}^{(1)}(0). \end{aligned} \tag{6}$$

4 Solution for the finite buffer

Our next step is to solve the matrix differential equations (5) for $k = 1, \ldots, K$, in the case that $B < \infty$. To deal with the inhomogeneous terms we first differentiate with respect to y, so that we find homogeneous equations for $\mathbf{f}^{(k)}(y) \equiv \frac{d\mathbf{F}^{(k)}(y)}{dy}$. We write down the solution for the resulting system of equations using the spectral method. Therefore we first have a look at the eigensystems

$$(R^{(k)})^{-1}(Q^{(k)})^T\mathbf{v}_j^{(k)} = z_j^{(k)}\mathbf{v}_j^{(k)}, \qquad j = 1, \dots, N,\ k = 1, \dots, K, \tag{7}$$

where $z_j^{(k)}$ and $\mathbf{v}_j^{(k)}, j = 1, \dots, N$ are the j-th eigenvalue and eigenvector corresponding to the matrix $(R^{(k)})^{-1}(Q^{(k)})^T$. Ordering the eigenvalues according to their real values, it is known that

$$Re\left(z_1^{(k)}\right) \leq \dots \leq Re\left(z_{N_+^{(k)}}^{(k)}\right) < 0 = z_{N_+^{(k)}+1}^{(k)} < Re\left(z_{N_+^{(k)}+2}^{(k)}\right) \leq \dots \leq Re\left(z_N^{(k)}\right),$$

see e.g. [7]. In particular we find that 0 is a simple eigenvalue, and that exactly $N_+^{(k)}$ eigenvalues have strictly negative real part.

In the case that all eigenvalues are different, we find the solution to the differentiated system to be of the form

$$\mathbf{f}^{(k)}(y) = \sum_{j=1}^{N} a_j^{(k)} e^{z_j^{(k)} y}\mathbf{v}_j^{(k)},$$

so that integration immediately yields

$$\mathbf{F}^{(k)}(y) = \sum_{j \neq N_+^{(k)}+1} \frac{a_j^{(k)}}{z_j^{(k)}} e^{z_j^{(k)} y}\mathbf{v}_j^{(k)} + a_{N_+^{(k)}+1}^{(k)}\mathbf{v}_{N_+^{(k)}+1}^{(k)} y + \mathbf{w}^{(k)}. \tag{8}$$

In total there are $2KN + N$ unknowns in this expression, namely $a_j^{(k)}, w_i^{(k)}$, and p_i for $i, j = 1, \dots, N$ and $k = 1, \dots, K$.

We now count the number of boundary conditions that we have at our disposal. Substitution of (8) into the original, inhomogeneous differential equations (5) yields KN relations that have to be satisfied. In matrixform these are given as

$$\begin{aligned} a_{N_+^{(k)}+1}^{(k)} R^{(k)} \mathbf{v}_{N_+^{(k)}+1}^{(k)} &= (Q^{(k)})^T\mathbf{w}^{(k)} \\ &+ (\tilde{Q}^{(k-1)} - Q^{(k)})^T\ \mathbf{F}^{(k)}(B^{(k-1)}) \\ &+ (Q^{(k-1)} - \tilde{Q}^{(k-1)})^T\ \mathbf{F}^{(k-1)}(B^{(k-1)}) \\ &\ \vdots \\ &+ (Q^{(1)} - \tilde{Q}^{(1)})^T\ \mathbf{F}^{(1)}(B^{(1)}) \\ &+ (\tilde{Q}^{(0)} - Q^{(1)})^T\ \mathbf{F}^{(1)}(0) \qquad k = 1, \dots, K. \end{aligned} \tag{9}$$

As an aside we mention that when $\tilde{Q}^{(0)} = Q^{(1)}$, it follows that $a^{(1)}_{N^{(1)}_+ +1} = 0$ and that $\mathbf{w}^{(1)}$ is a multiple of $\mathbf{v}^{(1)}_{N^{(1)}_+ +1}$, as should be expected. The remaining boundary conditions for the finite buffer case can be stated as follows.

$$
\begin{aligned}
F_i^{(1)}(0) &= 0, & i \in S_+^{(1)}, && (10)\\
F_i^{(k)}(B^{(k)}) &= F_i^{(k+1)}(B^{(k)}), & i \in S_+^{(k+1)} \cup S_-^{(k)},\ k = 1, \ldots, K-1, && (11)\\
F_i^{(K)}(B) &= p_i, & i \in S_-^{(K)}, && (12)\\
\sum_{i=1}^{N} p_i &= 1 \text{ and balance equations (6).} & && (13)
\end{aligned}
$$

Condition (10) (respectively (12)) states that the buffer is empty (full) with probability 0 when it is filling up (being drained). Condition (11) ensures that the distribution of (X, C) has no probability mass at $(i, B^{(k)})$, unless $i \in S_-^{(k+1)} \cap S_+^{(k)}$, in which case there is a confluence of drifts in $B^{(k)}$.

Denoting the cardinalities of $S_+^{(k)}$ and $S_-^{(k)}$ by $N_+^{(k)}$ and $N_-^{(k)}$, we count the number of conditions in (9) – (13) as

$$
KN + N_+^{(1)} + \sum_{k=1}^{K-1} \left(N_-^{(k)} + N_+^{(k+1)} \right) + N_-^{(K)} + N \;=\; 2KN + N,
$$

where we once again used Assumption 2 in Section 2, as well as the fact that one of the N conditions in (6) is linearly dependent on the others. From these equations all $2KN + N$ unknowns can be determined.

Theorem 4.1 (Finite buffer size B)
If the eigenvalues $z_j^{(k)}, j = 1, \ldots, N$ of the matrices $(R^{(k)})^{-1}(Q^{(k)})^T, k = 1, \ldots, K$ are simple, the stationary distribution of the process $(X(t), C(t))$ is given by (8) where the $2KN + N$ unknowns $a_j^{(k)}, w_i^{(k)}$, and p_i, $i, j = 1, \ldots, N$, $k = 1, \ldots, K$, can be found by the $2KN + N$ boundary conditions (9)–(13).

When not all eigenvalues are simple, the form of the solution is different, but the number of parameters that have to be solved by boundary conditions remains the same.

Once the solution has been obtained, we can easily derive expressions for various performance indicators. We mention a few:

- The probability that the buffer level is at or below y is given by $\sum_{j=1}^{N} F_j^{(k)}(y)$ where k is such that $B^{(k-1)} \leq y < B^{(k)}$.
- The probability of a full buffer is given by $1 - \sum_{j=1}^{N} F_j^{(K)}(B)$.
- The average amount of lost fluid per unit of time can be given as $\sum_{j=1}^{N} r_j^{(K)}(p_j - F_j^{(K)}(B))$.
- The average amount of 'unused capacity' per unit of time can be found as $-\sum_{j=1}^{N} r_j^{(1)} F_j^{(1)}(0)$.
- The probability that the buffer level increases is

$$\sum_{k=1}^{K} \sum_{j \in S_+^{(k)}} (F_j^{(k)}(B^{(k)}) - F_j^{(k)}(B^{(k-1)})).$$

5 Solution for the infinite buffer

In the case $B = B^{(K)} = \infty$ we again find

$$\mathbf{F}^{(k)}(y) = \sum_{j \neq N_+^{(k)}+1} \frac{a_j^{(k)}}{z_j^{(k)}} e^{z_j^{(k)} y} \mathbf{v}_j^{(k)} + a_{N_+^{(k)}+1}^{(k)} \mathbf{v}_{N_+^{(k)}+1}^{(k)} y + \mathbf{w}^{(k)}. \tag{14}$$

The unknown parameters are $a_j^{(k)}$, $w_i^{(k)}$ and p_i, where $i = 1, \ldots, N$, $j = 1, \ldots, N$, and $k = 1, \ldots, K$. However, due to our stability assumption in Section 2 we know that the probabilities in $\mathbf{F}^{(K)}(y)$ must converge to those in $\mathbf{p}$ as $y \to \infty$, so that we have $\mathbf{w}^{(K)} = \mathbf{p}$ and $a_j^{(K)} = 0$ for all j for which $Re(z_j^{(K)}) \geq 0$. This means that we have $N + N_-^{(K)}$ unknowns less than in the finite buffer case. The actual form of the solution in region K becomes

$$\mathbf{F}^{(K)}(y) = \sum_{j=1}^{N_+^{(K)}} \frac{a_j^{(K)}}{z_j^{(K)}} e^{z_j^{(K)} y} \mathbf{v}_j^{(K)} + \mathbf{p}, \tag{15}$$

while for the other regions the solution remains of the form (14), in each of which N eigenvalues play a role.

Next we determine the number of boundary conditions. First we mention that when we substitute (15), into the original, inhomogeneous differential equation (5) for $k = K$, we actually find the N equations for $\mathbf{p}$, that now take the form

$$\begin{aligned}
\mathbf{0} &= (Q^{(K)})^T \ \mathbf{p} \\
&+ (\tilde{Q}^{(K-1)} - Q^{(K)})^T \ \mathbf{F}^{(K)}(B^{(K-1)}) \\
&+ (Q^{(K-1)} - \tilde{Q}^{(K-1)})^T \ \mathbf{F}^{(K-1)}(B^{(K-1)}) \\
& \ \vdots \\
&+ (Q^{(1)} - \tilde{Q}^{(1)})^T \ \mathbf{F}^{(1)}(B^{(1)}) \\
&+ (\tilde{Q}^{(0)} - Q^{(1)})^T \ \mathbf{F}^{(1)}(0).
\end{aligned} \tag{16}$$

This leads to a reduction of N boundary equations. Notice that as in the finite buffer case, (16) has only $N-1$ linearly independent equations, while we also have the normalization condition

$$\sum_{i=1}^{N} p_i = 1. \tag{17}$$

The other boundary conditions remain the same as for the finite buffer case, apart from those in (12) that do not apply here. Hence, we have a total of $2KN - N_-^{(K)}$ unknowns that can be solved by an equal number of boundary conditions. For convenience we summarize this in the following theorem.

Theorem 5.1 (Infinite buffer size)
If the eigenvalues $z_j^{(k)}, j = 1, \ldots, N$ of the matrices $(R^{(k)})^{-1}(Q^{(k)})^T, k = 1, \ldots, K$ are simple, the stationary distribution of the process $(X(t), C(t))$ is given by (14) for $k = 1, \ldots, K-1$ and by (15) for regime K, which holds above the last (finite) threshold $B^{(K-1)}$. The $2KN - N_-^{(K)}$ unknowns in this solution can be found by the $KN + N - N_-^{(K)}$ boundary conditions in (10), (11), (16) and (17), together with the $KN - N$ relations in (9) for $k = 1, \ldots, K-1$.

Again, when not all eigenvalues are simple, the general form of the solution is different, but the number of parameters that have to be solved by boundary conditions, remains the same.

6 Example

As a simple illustration of the method we consider a buffer that is fed by 10 identical and independent on-off sources. Let $X(t)$ denote the number of active sources at time t; we choose the process $(X(t))$ as the regulating process. As before, $C(t)$ will denote the buffer content at time t. The total size of the buffer is $B = 1$ and its output capacity is $c = 11$. The off-times of the sources are exponentially distributed with parameter 3. When a source is in the on-state it sends a job that has an exponentially distributed size with parameter 2. Each source is allowed to send at rate 4 when the buffer content is below $B_1 = 0.8$, while the rate drops to 2 when the content is above 0.8. Hence, given that the buffer content is below 0.8, the on-times are exponentially distributed with parameter 8, while the parameter is 4 at times when the buffer content is above 0.8. Notice also that the content level may stay at 0.8 for a while when 3, 4 of 5 sources are active. In this case the total actual sending rate is 11, i.e. the output capacity, which is equally shared among the active sources. As a consequence the transition intensity of the process $(X(t))$ from state i to $i-1$, given that the buffer process remains at 0.8, is equal to 22. These considerations allow us to write down the diagonal matrices $R^{(1)}$ and $R^{(2)}$, as well as the tri-diagonal matrices $Q^{(1)}$, $Q^{(2)}$ and $\tilde{Q}^{(1)}$. Notice that $\tilde{Q}^{(0)} = Q^{(1)}$ and $\tilde{Q}^{(2)} = Q^{(2)}$. After the numerical determination of the eigensystems of the matrices $(R^{(1)})^{-1}(Q^{(1)})^T$ and $(R^{(2)})^{-1}(Q^{(2)})^T$, we apply the appropriate boundary conditions to find the 55 unknowns (note that $N = 11$ and $K = 2$). After solving this (linear) set of equations, the stationary distribution $\mathbf{F}$ of the joint process $(X(t), C(t))$ can be found numerically. A graphical representation of $Pr[C \leq y] = \sum_{i=0}^{10} F_i(y)$ is given in Figure 1. As an illustration of the different shapes of the functions $F_i(y)$, we also included Figure 2 in which $F_3(y)$, $F_5(y)$ and $F_7(y)$ are plotted, with discontinuities in 0, 0.8 and 1, respectively. Finally, the probability of a full buffer is $1 - \sum_{j=0}^{10} F_j^{(2)}(1) =$

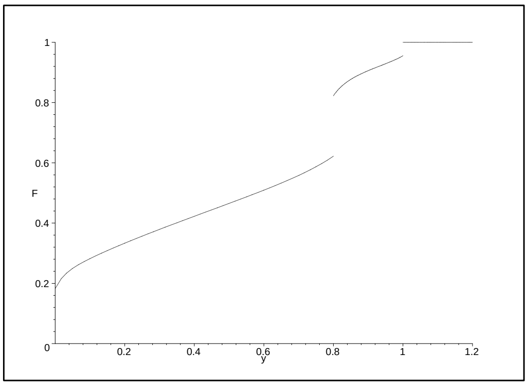

Figure 1: Probability distribution function of C

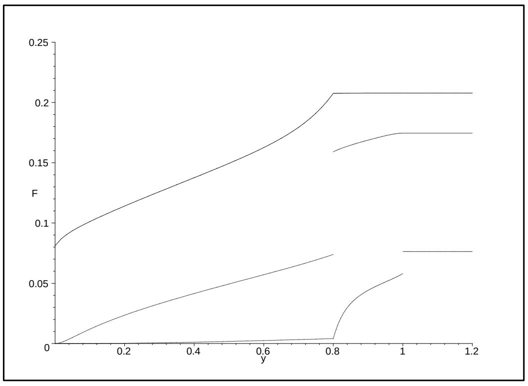

Figure 2: The functions $F_3(y)$, $F_5(y)$ and $F_7(y)$

0.044, while the average amount of lost fluid per unit of time is given by

$$\sum_{j=0}^{10}(2j-11)(p_j - F_j^{(2)}(1)) = 0.118\,.$$

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