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Analysis of random walks using orthogonal polynomials

Pauline Coolen-Schrijner[*,b,1], Erik A. van Doorn[a]

[a] *Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands*
[b] *Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road, Durham, DH1 3LE, UK*

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Abstract

We discuss some aspects of discrete-time birth–death processes or *random walks*, highlighting the role played by orthogonal polynomials. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Orthogonal polynomials play an important role in the analysis of Markov chains (in continuous or discrete time) which are characterized by the fact that direct transitions can occur only between neighbouring states. In particular, Karlin and McGregor [11, 12, 17] showed, in the fifties, that the transition probabilities of such a chain can be represented in terms of the elements of a sequence of orthogonal polynomials and the measure with respect to which these polynomials are orthogonal.

In this paper we consider Markov chains of the above type in discrete time with state space $\{0,1,2,\ldots\}$. In Section 2 we shall briefly introduce the processes at hand, which are also called *discrete-time birth–death processes* or *random walks*, the latter being the term we will use from now on. In Section 3 we shall present Karlin and McGregor's representation formula for the transition probabilities of a random walk, which involves a sequence of orthogonal polynomials called *random-walk polynomials* and the corresponding *random-walk measure*. Some additional properties of sequences of orthogonal polynomials of this type and their orthogonalizing measures are collected

* Corresponding author. E-mail: pauline.schrijner@durham.ac.uk.
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in Section 4. In particular we will show there how one can establish whether a given sequence of orthogonal polynomials is a sequence of random-walk polynomials, and whether a given random-walk measure corresponds to a unique random walk.

We elaborate on the analysis of first passage times of random walks in Section 5. We show in particular that if $i<j$ the probability generating function of the first passage time from state i to state j can be conveniently represented as a ratio of random-walk polynomials. If $i \geqslant j$ the situation is more complicated, but can nevertheless be dealt with using random-walk polynomials and the corresponding random-walk measure.

In Section 6 we look into quasi-stationary distributions for random walks. Specifically, after having explained what quasi-stationary distributions are, we address the problems of establishing whether a given random walk has a quasi-stationary distribution, and, if so, of determining what it looks like. It will be seen that the polynomials from the Karlin–McGregor representation again play a prominent part in the solution of these problems.

We conclude in Section 7 with two examples. Our main goal in this paper is to highlight the role played by orthogonal polynomials in the analysis of random walks. We shall mostly refrain from giving proofs, but refer to the original sources instead.

2. Random walks

A random walk is a stochastic process with discrete time parameter and is therefore represented by a sequence of random variables $X \equiv \{X(n),\ n=0,1,\ldots\}$. It is convenient to think of $X(n)$ as a random variable representing the state of a particle at time n, where $S \equiv \{0,1,\ldots\}$ is the set of possible states. A random walk is a Markov chain, which entails that, for any n, the conditional distribution of $X(n+1)$, the state of the particle at time $n+1$, given the states of the particle at times $0,1,\ldots,n$, is independent of the states of the particle at times $0,1,\ldots,n-1$. In addition, a random walk is characterized by the fact that the particle can jump directly from state i to state j only if $|i-j| \leqslant 1$.

With $P_{ij} \equiv \Pr\{X(n+1)=j \mid X(n)=i\}$ denoting the (stationary) 1-step transition probabilities of the random walk X, we write $p_j = P_{j,j+1}$, $q_{j+1} = P_{j+1,j}$ and $r_j = P_{jj}$, so that

$$P \equiv (P_{ij}) = \begin{pmatrix} r_0 & p_0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}. \tag{2.1}$$

We will assume throughout that $p_j>0$, $q_{j+1}>0$ and $r_j \geqslant 0$ for $j \geqslant 0$. The definition of a random walk implies that $p_j+q_j+r_j=1$ for $j \geqslant 1$. However, we allow $q_0 \equiv 1-p_0-r_0 \geqslant 0$. If $q_0=0$ then the matrix P is stochastic and the random walk is said to have a *reflecting barrier* 0. If $q_0>0$ then the random walk has an (ignored) absorbing state -1 which can be reached through state 0 only. The results of Section 6 on quasi-stationary distributions pertain to random walks with an absorbing state.

The n-step transition probabilities of the random walk X are denoted by $P_{ij}(n) \equiv \Pr\{X(m+n) = j \mid X(m)=i\}$, so that $P_{ij}(1)=P_{ij}$. Defining $P(n) \equiv (P_{ij}(n))$, $i,j \in S$, as the matrix of n-step transition probabilities, a well-known result on Markov chains tells us that

$$P(n) = P(1)^n = P^n, \tag{2.2}$$

so that, in principle, n-step transition probabilities may be obtained from the 1-step transition probabilities by repeated multiplication of the matrix P by itself. In practice, however, this approach is of limited value, particularly since the state space of X is infinitely large. In the next section we will present Karlin and McGregor's alternative representation formula for the n-step transition probabilities.

3. The Karlin–McGregor representation

For a random walk X with matrix of 1-step transition probabilities (2.1), Karlin and McGregor [17] have shown that the n-step transition probabilities may be represented as

$$P_{ij}(n) = \pi_j \int_{-1}^{1} x^n Q_i(x) Q_j(x) \psi(\mathrm{d}x). \tag{3.1}$$

Here π_j, $j \geqslant 0$, are constants given by

$$\pi_0 \equiv 1; \quad \pi_j \equiv \frac{p_0 p_1 \cdots p_{j-1}}{q_1 q_2 \cdots q_j}, \quad j \geqslant 1, \tag{3.2}$$

and $\{Q_j(x)\}_{j=0}^{\infty}$ is a sequence of polynomials defined by the recurrence relations

$$\begin{aligned} &xQ_j(x) = q_j Q_{j-1}(x) + r_j Q_j(x) + p_j Q_{j+1}(x), \quad j \geqslant 1, \\ &Q_0(x) = 1, \qquad p_0 Q_1(x) = x - r_0. \end{aligned} \tag{3.3}$$

Finally, ψ is the (unique) measure of total mass 1 and infinite support in the interval $[-1, 1]$ with respect to which the polynomials $\{Q_j(x)\}_j$ are orthogonal; indeed, substituting $n = 0$ in (3.1) yields

$$\pi_j \int_{-1}^{1} Q_i(x) Q_j(x) \psi(\mathrm{d}x) = \delta_{ij}. \tag{3.4}$$

Because the argument is so nice and simple, we will present the essence of a proof of the representation formula (3.1), taking for granted the stated properties of the polynomials $\{Q_j(x)\}$ and the associated measure ψ; see [17, 23, 6] for proofs of the latter results. Namely, we observe from (2.1) and the recurrence formula (3.3) that

$$xQ_i(x) = \sum_{j=0}^{\infty} P_{ij}(1) Q_j(x).$$

But in view of (2.2) one readily establishes by induction that, actually,

$$x^n Q_i(x) = \sum_{k=0}^{\infty} P_{ik}(n) Q_k(x), \quad n = 0, 1, \ldots . \tag{3.5}$$

As a consequence,

$$\pi_j \int_{-1}^{1} x^n Q_i(x) Q_j(x) \psi(\mathrm{d}x) = \sum_{k=0}^{\infty} P_{ik}(n) \pi_j \int_{-1}^{1} Q_k(x) Q_j(x) \psi(\mathrm{d}x),$$

where the interchange of summation and integration is justified, since we are dealing essentially with a finite number of terms. But assuming that (3.4) holds true, the representation formula (3.1) follows.

We note that (3.1) immediately leads to some conclusions concerning the limiting behaviour of the n-step transition probabilities as n goes to infinity. Indeed, we see that

$$\lim_{n\to\infty} P_{ij}(n) \text{ exists} \Leftrightarrow \psi(\{-1\}) = 0, \tag{3.6}$$

and

$$\lim_{n\to\infty} P_{ij}(n) > 0 \Leftrightarrow \psi(\{-1\}) = 0 \quad \text{and} \quad \psi(\{1\}) > 0. \tag{3.7}$$

Concluding, to obtain information about a random walk in terms of the 1-step transition probabilities, one might try to obtain information about the associated polynomials and their orthogonalizing measure in terms of these probabilities. Some information of the latter type will be collected in the next section.

4. Random-walk polynomials and measures

In what follows any sequence of polynomials which can be normalized to satisfy a recurrence relation of the type (3.3), where $p_j > 0$, $q_{j+1} > 0$ and $r_j \geqslant 0$ for $j \geqslant 0$, $q_0 \equiv 1 - p_0 - r_0 \geqslant 0$ and $p_j + q_j + r_j = 1$ for $j \geqslant 1$, will be called a sequence of *random-walk polynomials*; the corresponding measure will be called a *random-walk measure*.

An interesting problem is now to establish whether a given sequence of orthogonal polynomials $\{R_j(x)\}$ is a sequence of random-walk polynomials. The answer to this problem may be obtained with the help of the next theorem, which collects results in [23, 6]. As preliminary result we note that if $\{R_j(x)\}$ is an orthogonal polynomial sequence, then there are unique constants c_j, α_j and β_{j+1}, $j \geqslant 0$, such that the polynomials $P_j(x) \equiv c_j R_j(x)$ satisfy the recurrence relations

$$\begin{aligned} &P_{j+1}(x) = (x - \alpha_j) P_j(x) - \beta_j P_{j-1}(x), \quad j \geqslant 1, \\ &P_0(x) = 1, \qquad P_1(x) = x - \alpha_0, \end{aligned} \tag{4.1}$$

where $\beta_j > 0$ for $j \geqslant 1$, see, e.g., [2]. The next theorem is formulated in terms of the monic sequence $\{P_j(x)\}$.

Theorem 4.1. *The following statements are equivalent:*

(i) *The sequence $\{P_j(x)\}$ is a random-walk polynomial sequence.*

(ii) *There are numbers $p_j > 0$, $q_{j+1} > 0$ and $r_j \geqslant 0$, $j \geqslant 0$, satisfying $p_0 + r_0 \leqslant 1$ and $p_j + q_j + r_j = 1$, $j \geqslant 1$, such that $\alpha_j = r_j$ and $\beta_{j+1} = p_j q_{j+1}$, $j \geqslant 0$.*

(iii) *The normalized sequence* $\{Q_j(x)\}$, *where* $Q_j(x) = P_j(x)/P_j(1)$, *satisfies a recurrence of the type* (3.3) (*with* $p_0 + r_0 = 1$ *and* $p_j + q_j + r_j = 1$, $j \geqslant 1$, *since* $Q_j(1) = 1$ *for all* j), *where* $p_j > 0$, $q_{j+1} > 0$ *and* $r_j \geqslant 0$, *for* $j \geqslant 0$.

(iv) *The sequence* $\{P_j(x)\}$ *is orthogonal with respect to a measure with support in* $[-1,1]$ *and* $\alpha_j \geqslant 0$, $j \geqslant 0$.

We remark that any sequence of polynomials which is orthogonal with respect to a measure on $[-1,1]$ can be normalized to satisfy a recurrence of the type (3.3) with $p_j > 0$ and $q_{j+1} > 0$. It is the fact that $r_j \geqslant 0$ for all j, which distinguishes random-walk polynomial sequences from the others. In general it will be difficult to establish whether the given measure ψ on $[-1,1]$ is a random-walk measure, since one has to check, for all j, the non-negativity of

$$\int_{-1}^{1} x P_j^2(x)\psi(\mathrm{d}x),$$

where $\{P_j(x)\}$ is the orthogonal polynomial sequence for ψ, see [23] or [6].

Once it has been established that a measure is a random-walk measure, the question arises whether it determines a unique random walk, that is, a unique sequence of random-walk polynomials satisfying (3.3). To settle this question we first note that any random-walk measure corresponds to a unique set of random-walk polynomials satisfying (3.3) with $q_0 \equiv 1 - p_0 - q_0 = 0$, as can easily be observed from Theorem 4.1. The following theorem is the analogue for random walks of Lemma 1 in Karlin and McGregor [12].

Theorem 4.2. *Let* ψ *be a random-walk measure and* $\{Q_j(x)\}_{j=0}^{\infty}$ *the corresponding sequence of random-walk polynomials satisfying* (3.3) *with* $q_0 \equiv 1 - p_0 - r_0 = 0$. *Then there is a sequence of positive constants* $\{\gamma_j\}_{j=0}^{\infty}$ *such that the polynomials* $Q_j^*(x) \equiv \gamma_j Q_j(x)$, $j \geqslant 0$, *satisfy*

$$xQ_j^*(x) = q_j^* Q_{j-1}^*(x) + r_j^* Q_j^*(x) + p_j^* Q_{j+1}^*(x), \quad j \geqslant 1$$
$$Q_0^*(x) = 1, \quad p_0^* Q_1^*(x) = x - r_0^*,$$

with $q_0^* \equiv 1 - p_0^* - r_0^* > 0$ *if and only if* $q_0^* \int_{-1}^{1} (1-x)^{-1}\psi(\mathrm{d}x) \leqslant 1$.

As a direct consequence of this theorem we have the following.

Corollary 4.3. (i) *The sequence* $\{Q_j(x)\}_{j=0}^{\infty}$ *satisfying* (3.3) *with* $q_0 = 0$ *is the unique sequence of random-walk polynomials for* ψ *satisfying* (3.3) *if and only if the integral*

$$I \equiv \int_{-1}^{1} (1-x)^{-1}\psi(\mathrm{d}x) \tag{4.2}$$

diverges (*that is, see* [17] *or* [7], *the corresponding random walk is recurrent*).

(ii) *If the integral* (4.2) *converges then there is an infinite family of sequences of random-walk polynomials for* ψ *satisfying* (3.3), *that is, an infinite family of random walks corresponding to* ψ, *parametrized by* q_0 *which can be any number in the interval* $[0, I^{-1}]$.

The above results were mentioned in [8], where they were subsequently used to obtain conditions for Jacobi polynomials to be random-walk polynomials. Part (i) of the corollary has recently been rederived by different means by Dette and Studden [5].

Before continuing with some properties of the random-walk polynomials $\{Q_j(x)\}$ of (3.3), we must introduce some terminology. From, e.g., Chihara's book [2] we know that $Q_j(x)$ has j real distinct zeros $x_{j1} < x_{j2} < \cdots < x_{jj}$. These zeros have the separation property

$$x_{j+1,i} < x_{ji} < x_{j+1,i+1}, \quad i = 1, 2, \ldots, j, \ j \geqslant 1, \tag{4.3}$$

whence

$$\xi_i \equiv \lim_{k\to\infty} x_{ki} \quad \text{and} \quad \eta_j \equiv \lim_{k\to\infty} x_{k,k-j+1}, \quad i, j \geqslant 1, \tag{4.4}$$

exist in the extended real number system; also

$$-\infty \leqslant \xi_i \leqslant \xi_{i+1} < \eta_{j+1} \leqslant \eta_j \leqslant \infty, \quad i, j \geqslant 1. \tag{4.5}$$

As is well known there is a close relation between the quantities ξ_i and η_j and the support

$$\mathrm{supp}(\psi) \equiv \{x \mid \psi((x-\varepsilon, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}$$

of the measure ψ on the interval $[-1, 1]$ with respect to which the random-walk polynomials $\{Q_j(x)\}$ are orthogonal. Indeed, defining

$$\sigma \equiv \lim_{i\to\infty} \xi_i \quad \text{and} \quad \tau \equiv \lim_{j\to\infty} \eta_j \tag{4.6}$$

and the (possibly finite) sets

$$\Xi \equiv \{\xi_1, \xi_2, \ldots\} \quad \text{and} \quad H \equiv \{\eta_1, \eta_2, \ldots\},$$

Chihara [2, pp. 61–63] tells us the following.

Theorem 4.4. *One has supp*$(\psi) = \bar{\Xi} \cup S_l \cup \bar{H}$ *(a bar denoting closure), where* S_l *is a subset of* (σ, τ)*; also,* σ *is the smallest and* τ *is the largest limit point of supp*(ψ).

Since $\mathrm{supp}(\psi) \subset [-1, 1]$, it follows in particular that

$$-1 \leqslant \xi_1 = \inf \mathrm{supp}(\psi) \leqslant \sigma \leqslant \tau \leqslant \sup \mathrm{supp}(\psi) = \eta_1 \leqslant 1. \tag{4.7}$$

Interestingly, one can prove, see [7],

$$\xi_1 + \eta_1 \geqslant 0 \quad \text{and} \quad \sigma + \tau \geqslant 0, \tag{4.8}$$

that is, the smallest (limit) point of $\mathrm{supp}(\psi)$ is at least as close to 0 as the largest (limit) point of $\mathrm{supp}(\psi)$. The representation formula (3.1) now shows that the transition probabilities $P_{ij}(n)$ go to zero geometrically fast as $n \to \infty$ if $\eta_1 < 1$; it can be shown [7] that this condition is also necessary. We shall encounter the same condition again in Section 6.

5. First passage times

We denote by T_{ij} the random variable which represents the number of transitions for the random walk X to move from state i to state j, and let

$$F_{ij}(n) \equiv \Pr\{T_{ij} = n\}, \quad n \geqslant 1,$$

and

$$\tilde{F}_{ij}(z) \equiv \sum_{n=1}^{\infty} F_{ij}(n) z^n.$$

With $\tilde{P}_{ij}(z)$ denoting the probability generating function

$$\tilde{P}_{ij}(z) \equiv \sum_{n=0}^{\infty} P_{ij}(n) z^n,$$

it is well known, see, e.g., Kijima [19], that

$$\tilde{P}_{ij}(z) = \delta_{ij} + \tilde{F}_{ij}(z)\tilde{P}_{jj}(z), \tag{5.1}$$

where δ_{ij} is Kronecker's delta.

As noted by Karlin and McGregor [17], relation (5.1) and the representation (3.1) allow us to express the generating functions $\tilde{F}_{ij}(z)$ in terms of integrals involving the random-walk measure ψ of X. Indeed, it follows directly from (3.1) that

$$\tilde{P}_{ij}(z) = \pi_j \int_{-1}^{1} \frac{Q_i(x)Q_j(x)}{1 - xz} \psi(\mathrm{d}x), \tag{5.2}$$

which may be substituted in (5.1) to obtain an expression for $\tilde{F}_{ij}(z)$.

An alternative representation may be obtained by parallelling the argument of [12, pp. 522–523] in a discrete-time setting, which yields

$$\tilde{P}_{ij}(z) = Q_i^{(j)}(z^{-1}) + Q_i(z^{-1})Q_j^{(0)}(z^{-1})\pi_j + Q_i(z^{-1})Q_j(z^{-1})\tilde{P}_{00}(z), \tag{5.3}$$

where

$$Q_k^{(\ell)}(y) \equiv 0, \quad 0 \leqslant k \leqslant \ell,$$

and, for $k > \ell$, $Q_k^{(\ell)}(y)$ is a polynomial of degree $k - \ell - 1$ satisfying

$$Q_k^{(\ell)}(y) \equiv \pi_\ell \int_{-1}^{1} Q_\ell(x) \frac{Q_k(x) - Q_k(y)}{1 - xy^{-1}} \psi(\mathrm{d}x). \tag{5.4}$$

It follows in particular that

$$\tilde{P}_{ij}(z) = Q_i(z^{-1})\{Q_j^{(0)}(z^{-1})\pi_j + Q_j(z^{-1})\tilde{P}_{00}(z)\}, \quad i \leqslant j, \tag{5.5}$$

which, in combination with (5.1), gives us

$$\tilde{F}_{ij}(z) = \frac{Q_i(z^{-1})}{Q_j(z^{-1})}, \quad i < j. \tag{5.6}$$

This result, which is the discrete-time analogue of a result of Karlin and McGregor's [15, p. 378; 16, p. 1131] for birth–death processes, has been derived in the literature in various ways, see [22, 4]; it is also implicitly given in [18].

Restricting attention to the case $j=i+1$ we note that (5.6) in combination with the recurrence relation (3.3) for the polynomials $\{Q_j(x)\}$ leads to the recurrence relation

$$\tilde{F}_{i,i+1}(z)=\frac{p_i z}{1-r_i z-q_i z\tilde{F}_{i-1,i}(z)}, \quad i\geqslant 1, \tag{5.7}$$

which, together with the obvious result

$$\tilde{F}_{0,1}(z)=\frac{p_0 z}{1-r_0 z}, \tag{5.8}$$

gives a means of obtaining $\tilde{F}_{i,i+1}(z)$ for all i. The recurrence relation (5.7) was observed already by Harris [10] (in the case $r_i\equiv 0$), see also [18, 22, 4].

The result (5.6) also provides a basis for dealing with the problem of characterizing first passage time distributions when $i<j$. An interesting result in this vein is the following.

Theorem 5.1. *If $q_0=0$, then the following statements are equivalent*:
(i) $\xi_1\geqslant 0$.
(ii) *For all $i>0$ the distribution of $T_{i-1,i}$ is a mixture of i distinct geometric distributions.*
(iii) *For all $i>0$ the distribution of T_{0i} is a convolution of i distinct geometric distributions.*

The theorem is readily obtained from Sumita and Masuda [22, Theorems 2.2 and 2.8] and the fact that

$$x\leqslant\xi_1 \Leftrightarrow (-1)^j Q_j(x)>0 \quad \text{for all} \quad j\geqslant 0, \tag{5.9}$$

which is obvious from (3.3) and the definition of ξ_1, see [23] for related results. For more information on the problem of characterizing the distribution of T_{ij} when $i<j$ we refer to [18, 22, 21].

The representation (5.3) for $\tilde{P}_{ij}(z)$ suggests that the analysis of first passage times T_{ij} in the case $i\geqslant j$ is considerably more involved than in the case $i<j$. However, Karlin and McGregor [17] have proposed a method for computing *moments* of first passage time distributions, which is based on (5.2) rather than (5.3). The method amounts to using tools of the theory of orthogonal polynomials to evaluate integrals of the type

$$\int_{-1}^{1}\frac{Q_i(x)Q_j(x)}{(1-x)^n}\psi(\mathrm{d}x),$$

which arise in the computation of such moments, in terms of the parameters of the process. Similar calculations have been carried out for birth–death processes in [13], and the results may be used in the present setting by an appropriate transformation, as described in [17].

By way of illustration we will consider $T_{i,-1}$, the time it takes for absorption at -1 to occur when the initial state is i and $q_0>0$. Evidently, $F_{i,-1}(n)=q_0P_{i0}(n-1)$, so (5.2) gives us

$$\tilde{F}_{i,-1}(z)=q_0 z\int_{-1}^{1}\frac{Q_i(x)}{1-xz}\psi(\mathrm{d}x). \tag{5.10}$$

The probability of eventual absorption is given by $\Pr\{T_{i,-1}<\infty\}=\tilde{F}_{i,-1}(1)$. This can be evaluated in the manner indicated in [17] as

$$\Pr\{T_{i,-1}<\infty\}=q_0\int_{-1}^{1}\frac{Q_i(x)}{1-x}\psi(\mathrm{d}x)=\frac{q_0\sum_{k=i}^{\infty}(p_k\pi_k)^{-1}}{1+q_0\sum_{k=0}^{\infty}(p_k\pi_k)^{-1}}, \tag{5.11}$$

which should be interpreted as 1, that is, absorption is certain, if

$$\sum_{k=0}^{\infty}(p_k\pi_k)^{-1}=\infty. \tag{5.12}$$

If absorption is certain the expected time to absorption is given by $ET_{i,-1}=\tilde{F}'_{i,-1}(1)$, which can be evaluated as

$$ET_{i,-1}=q_0\int_{-1}^{1}\frac{Q_i(x)}{(1-x)^2}\psi(\mathrm{d}x)=\sum_{j=0}^{i}(q_j\pi_j)^{-1}\sum_{k=j}^{\infty}\pi_k. \tag{5.13}$$

The results (5.11) and (5.13) constitute the discrete-time counterpart of [13, Theorem 10].

6. Quasi-stationary distributions

In this section we consider a random walk X with the property $q_0\equiv 1-p_0-r_0>0$, so that -1 is an absorbing state. We will assume that the probability of eventual absorption at -1 is 1, that is, (5.12) holds true.

An (honest) probability distribution $\{\theta_j\}_{j=0}^{\infty}$ on the nonnegative integers is called a *quasi-stationary distribution* for X if the state probabilities $p_j(n)\equiv\Pr\{X(n)=j\}$ of the random walk with initial distribution $p_j(0)=\theta_j$, $j\geqslant 0$, satisfy

$$\frac{p_j(n)}{1-p_{-1}(n)}=\theta_j,\quad j\geqslant 0,\ n\geqslant 0. \tag{6.1}$$

In other words, a quasi-stationary distribution is an initial distribution such that the conditional probability of the process being in state j at time n, given that no absorption has occurred by that time, is independent of n for all j.

Before we can identify the quasi-stationary distributions for X we must quote a result in [7] which tells us that for $\{\theta_j\}$ to constitute a quasi-stationary distribution it is necessary and sufficient that $\{\theta_j\}$ satisfies the system of equations

$$\begin{aligned}(1-q_0\theta_0)\theta_j&=p_{j-1}\theta_{j-1}+r_j\theta_j+q_{j+1}\theta_{j+1},\quad j\geqslant 1,\\(1-q_0\theta_0)\theta_0&=r_0\theta_0+q_1\theta_1.\end{aligned} \tag{6.2}$$

From this result and the recurrence relations (3.3) it is now easily seen that any quasi-stationary distribution $\{\theta_j\}$ must satisfy $\theta_j=\theta_0\pi_jQ_j(1-q_0\theta_0)$, $j\geqslant 0$, that is,

$$\theta_j=q_0^{-1}\pi_j(1-x)Q_j(x),\quad j\geqslant 0, \tag{6.3}$$

for some x. Evidently, for $\{\theta_j\}$ to constitute a probability distribution the quantities in (6.3) must be nonnegative, and hence we must have $\eta_1 \leqslant x \leqslant 1$, since

$$x \geqslant \eta_1 \Leftrightarrow Q_j(x) > 0 \quad \text{for all } j \geqslant 0, \tag{6.4}$$

by definition of η_1. Finally, for the θ_j's in (6.3) to constitute an honest probability distribution we must have

$$q_0^{-1}(1-x)\sum_{j=0}^{\infty} \pi_j Q_j(x) = 1. \tag{6.5}$$

Obviously, this is not true when $x=1$, but, by applying the transformation $P_j(x) = Q_j(1-x)$ and invoking [20, Theorem 4.3], it is readily verified that (6.5) does hold for x when $\eta_1 \leqslant x < 1$.

The conclusions from the preceding analysis are collected in the following theorem, see also [7].

Theorem 6.1. *Quasi-stationary distributions for X exist if and only if $\eta_1 < 1$, in which case there is a one-parameter family $\{\{\sigma_j(x)\}_{j=0}^{\infty}, \eta_1 \leqslant x < 1\}$ of quasi-stationary distributions, where $\sigma_j(x) = q_0^{-1}\pi_j(1-x)Q_j(x)$.*

7. Examples

7.1. Random walks related to an orthogonal polynomial sequence studied by Chihara and Ismail

We consider the orthogonal polynomial sequence $\{R_j(x)\}_j$ defined by the recurrence relations

$$\begin{aligned} &R_0(x) = 1, \qquad R_1(x) = x - d/a, \\ &R_{j+1}(x) = \left(x - \frac{d}{j+1}\right) R_j(x) - \frac{jc}{(j+a)(j+a-1)} R_{j-1}(x), \quad j \geqslant 1, \end{aligned} \tag{7.1}$$

where $a>0, c>0$. The polynomials $\{R_j(x)\}_j$ are related to the polynomials $\{P_j(x,a,b)\}_j$ studied in Chihara and Ismail [3] by

$$R_j(x) = \frac{j!c^{j/2}}{(a)_j} P_j\left(\frac{x}{\sqrt{c}}, a, -\frac{d}{\sqrt{c}}\right),$$

where $(a)_0 = 1, (a)_j = a(a+1)\cdots(a+j-1),\ j \geqslant 1$. The special case $d=0, c=a$ gives

$$R_j(x) = (a^j/(a)_j) r_j(x,a),$$

where $\{r_j(x,a)\}_j$ are the random-walk polynomials associated with the $M/M/\infty$ system studied by Karlin and McGregor [14], see also Charris and Ismail [1] and the references mentioned there.

We will investigate under which conditions on the parameters the sequence $\{R_j(x)\}_j$ constitutes a sequence of random-walk polynomials. From Chihara and Ismail [3] we readily obtain

$$\begin{aligned} \xi_1 &= \frac{1}{2a}(d - \sqrt{d^2 + 4ac}), \\ \eta_1 &= \frac{1}{2a}(d + \sqrt{d^2 + 4ac}), \end{aligned} \tag{7.2}$$

so that

$$-1 \leqslant \xi_1 < \eta_1 \leqslant 1 \Leftrightarrow c - a \leqslant d \leqslant a - c.$$

Theorems 4.1 and 4.4 subsequently tell us the following.

Theorem 7.1. *The orthogonal polynomial sequence* $\{R_j(x)\}_j$ *defined by* (7.1) *constitutes a sequence of random-walk polynomials if and only if* $d \geqslant 0$ *and* $0 < c \leqslant a - d$.

Once q_0 has been fixed, the random walk parameters can be solved from the recurrence relations

$$\begin{aligned} p_j + q_j &= 1 - r_j = \frac{j+a-d}{j+a}, \quad j \geqslant 0, \\ p_{j-1}q_j &= \frac{jc}{(j+a)(j+a-1)}, \quad j \geqslant 1, \end{aligned} \tag{7.3}$$

cf. Theorem 4.1. Restricting ourselves to the extremal case $c = a - d$, it follows from (7.2) that $\eta_1 = 1$ and $\xi_1 = -1 + d/a$. Moreover, Chihara and Ismail [3] have shown that the associated measure is discrete with 0 as single limit point of its support. Since $\eta_1 = 1$ is an isolated mass point of the orthogonality measure the integral (4.2) diverges. Hence, Corollary 4.3 implies that the recurrence relations (7.3) have a feasible solution if and only if $q_0 = 0$, in which case the random walk parameters are given by

$$p_j = \frac{a-d}{j+a}, \quad q_j = \frac{j}{j+a}, \quad r_j = \frac{d}{j+a}, \quad j \geqslant 0. \tag{7.4}$$

If $d \geqslant 0$ but $0 < c < a - d$, then $\eta_1 < 1$, so that the integral (4.2) converges. As a consequence, q_0 may be positive.

7.2. The random walk with constant parameters

We consider the random walk with parameters

$$p_j = p, \quad q_j = q \equiv 1 - p, \quad r_j = 0, \quad j \geqslant 0. \tag{7.5}$$

Karlin and McGregor [17] have shown that the associated random-walk polynomials can be expressed as

$$Q_j(x) = (q/p)^{j/2} U_j(x/\sqrt{4pq}), \tag{7.6}$$

where $U_j(\cdot)$ is a Tchebichef polynomial of the second kind (see, e.g., [2]) while the corresponding measure consists of the density

$$\psi'(x) = \frac{\sqrt{4pq - x^2}}{2\pi pq} \tag{7.7}$$

over the interval $-\sqrt{4pq} < x < \sqrt{4pq}$, so that $\eta_1 = \sqrt{4pq}$.

The generating function (5.10) can now be evaluated explicitly. Indeed, restricting attention to the case $i=0$, we get

$$\tilde{F}_{0,-1}(z)=\frac{z}{2\pi p}\int_{-\sqrt{4pq}}^{\sqrt{4pq}}\frac{\sqrt{4pq-x^2}}{1-xz}\mathrm{d}x=\frac{1-\sqrt{1-4pqz^2}}{2pz},$$

which can be inverted to yield the well-known results $F_{0,-1}(n)=0$ if $n=2m$, and

$$F_{0,-1}(n)=\frac{q}{2m+1}\binom{2m+1}{m}(pq)^m, \quad n=2m+1,\ m=0,1,\ldots. \tag{7.8}$$

We refer to Gulati and Hill [9] for more general results in this vein.

Assuming $p<\frac{1}{2}$, condition (5.12) is easily seen to be satisfied, while $\eta_1<1$. So, by Theorem 6.1, there exists a family of quasi-stationary distributions $\{\{\sigma_j(x)\}_j,\ \sqrt{4pq}\leqslant x<1\}$, where

$$\sigma_j(x)=q^{-1}(p/q)^{j/2}(1-x)U_j(x/\sqrt{4pq}), \quad j\geqslant 0.$$

In particular, since $U_j(1)=j+1$, the quasi-stationary distribution corresponding to $x=\eta_1=\sqrt{4pq}$ is given by

$$\sigma_j(\eta_1)=(1-\sqrt{p/q})^2(j+1)(p/q)^{j/2}, \quad j\geqslant 0, \tag{7.9}$$

a negative binomial (or *Pascal*) distribution.

We refer to [7, 8] for generalizations of this example.

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