3D sandbank stability

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6.3.3 3D sandbank stability

Introduction

A new three-dimensional hydrodynamic model was used by UT and UU to study the generation of wave-like bottom patterns produced by tidal currents. Inclusion of the vertical flow structure was necessary to allow for the formation, or absence, of all the existing large-scale regular bottom features observed at Middelkerke. The tide and topography are treated as a coupled system and allowed to interact. The following is based on Hulscher (1996).

Theory

The model consisted of two coupled parts, a flow model and a part describing the sediment dynamics. Water motions driven by tides can be described with the three-dimensional shallow water equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -g\frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial z}\left(A_v \frac{\partial u}{\partial z}\right) \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - fu = -g\frac{\partial \zeta}{\partial y} + \frac{\partial}{\partial z}\left(A_v \frac{\partial v}{\partial z}\right) \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3)$$

Here u,v and w are the velocity components in the x-, y- and z- direction; $z = \zeta$ locates the free surface elevation above a horizontal datum; and h the bottom level with respect to the undisturbed water depth H, see Figure 6.3.3.1. Furthermore f , g and A_v are the Coriolis parameter; the acceleration due to gravity; and the turbulent eddy viscosity, respectively. A derivation of this parametrization can be found in Pedlosky (1987). The parameter A_v is a function of time and all spatial directions. However, in the current model the simplest approximation has been adopted i.e. A_v is a constant (see Pedlosky 1987). Although it may be a very crude assumption, it models the process that horizontal momentum is transferred in the vertical direction and so it enables the study of how this process affects the fluid motion.

The boundaries in the horizontal directions are assumed to be infinitely far away, while the boundary conditions in the vertical direction are given as follows. Assuming a situation in which the wind stress at the surface equals zero, then at the sea surface the conditions are:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \text{ and } w = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} \tag{4}$$

At the seabed (z = -H + h(x, y, t)) a partial slip condition for the horizontal flow components is applied, the vertical velocity component at the seabed is given by the kinematic condition

$$A_v\frac{\partial u}{\partial z} = Su\,,\ A_v\frac{\partial v}{\partial z} = Sv \text{ and } w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \tag{5}$$

The x-axis is chosen in the direction of the depth-averaged current. With help of the partial slip condition at the seabed the near-bed velocity can be modelled without including the complicated thin bed boundary layer processes in which the fluid has to adjust to a no-slip condition. The slip parameter S fixes the relation between the horizontal velocity and the shear stress at the top of the thin bottom boundary layer. In this model it is assumed that the seabed (at z = -K + h) can be interpreted as the physical seabed.

A partial slip boundary condition can be used theoretically to study all situations between perfect slip (S = O) and no slip ($S \to \infty$) at the seabed, see Maas and van Haren (1987), Visser *et al.* (1994). The slip parameter is related to the bed roughness and can be identified with the friction velocity as used in depth-averaged shallow water models. The sediment transport model includes only bed load transport, which is assumed to be dominant in tidal regimes offshore. Considering the flow velocities in the vertical direction explicitly, it is possible to describe the bedload transport directly as a function of the bottom shear stress. This differs from the transport modelling used in depth-averaged flow models, in which the transport is parameterized as a function of depth-averaged velocities, see Huthnance (1982) and Hulscher *et al.* (1993). A general bedload transport formula has been chosen:

$$\vec{S}_b = \alpha|\vec{\tau}|^b\left\{\frac{\vec{\tau}}{|\vec{\tau}|} - \lambda\vec{\nabla}h\right\} \tag{6}$$

in which $\vec{\tau}$ denotes the bed shear stress

$$\vec{\tau} = A_v\left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}\right)\Bigg|_{z=bed} \tag{7}$$

Furthermore $\vec{S}_b$ is the sediment transport vector and α a bed load transport proportionality parameter. The exponent b gives the nonlinearity of transport in relation to the bed shear stress. The sediment transport vector contains a bed-slope correction term which is weighted by λ. Many transport parametrizations are of this type, see the review by van Rijn (1993). The sediment balance (without suspended sediment) is given by the equation:

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot \vec{S}_b = 0 \tag{8}$$

which couples the flow and sediment-transport model. $\vec{\nabla}$ is the spatial gradient operator $(\partial / \partial x, \partial / \partial y)$.

In order to analyse the dynamics of large-scale bedforms described by the shallow water model, the equations of motion are made dimensionless with relevant scales. The tide is assumed to be the main forcing mechanism of these morphological features and gives a time scale σ^{-1} for the flow variations, in which σ is the tidal frequency. The seabed level change is assumed to vary on a much larger time scale denoted as T_{long}, a morphologically relevant time scale. There are also two length scales involved in this model, one (L) is the length on which the tidal wave varies, the other (l_m) is the scale on which the seabed varies. The horizontal velocities are of the order of the depth-averaged velocities, which are scaled by the tidal current amplitude U. The vertical velocity is much smaller and an appropriate scaling factor is obtained by considering the continuity Eq. (3). Observations show that l_m is of the order of the tidal excursion ($U\sigma^{-1}$) or smaller. The scale for the bed shear stress is chosen in order to obtain a dimensionless value as given by Eq. (16). Therefore, the following non-dimensional variables are introduced (denoted by $\breve{}$):

$$(u, v, w) = \left(U\breve{u}, U\breve{v}, U\frac{H}{l_m}\breve{w} \right), \quad (x, y, z) = (l_m\breve{x}, l_m\breve{y}, H\breve{z}) \tag{9}$$

$$\vec{\tau} = l_m H\breve{\vec{\tau}}, \quad t = \frac{\breve{t}}{\sigma}, \quad h = H\breve{h} \text{ and } \zeta = \frac{UL\sigma}{g}\breve{\zeta} \tag{10}$$

Using the dispersion relation for a shallow water wave also gives the equation:

$$L\sigma = 2\pi\sqrt{gH} \tag{11}$$

Therefore, the following nondimensional model is obtained (for convenience the $\breve{}$ is dropped):

$$\delta\left\{ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \hat{f}v - \frac{E_v}{2}\frac{\partial^2 u}{\partial z^2} \right\} = -\frac{\partial \zeta}{\partial x} \tag{12}$$

$$\delta\left\{ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - \hat{f}u - \frac{E_v}{2}\frac{\partial^2 v}{\partial z^2} \right\} = -\frac{\partial \zeta}{\partial y} \tag{13}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{14}$$

$$\frac{\partial h}{\partial t}=-\alpha\vec{\nabla}\cdot\left[|\vec{\tau}|^{b}\left\{\frac{\vec{\tau}}{|\vec{\tau}|}-\hat{\lambda}\vec{\nabla}h\right\}\right] \tag{15}$$

$$\vec{\tau}=\frac{E_v}{2}\left(\frac{\partial u}{\partial z},\frac{\partial v}{\partial z}\right)\Bigg|_{z=-1+h} \tag{16}$$

The boundary conditions at the free surface ($z = \delta\zeta$) becomes:

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=0 \text{ and } w=\delta\left\{\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}\right\} \tag{17}$$

and at the seabed ($z = -1 + h$):

$$\frac{E_v}{2}\frac{\partial u}{\partial z}=\frac{\hat{S}}{2}u,\quad \frac{E_v}{2}\frac{\partial v}{\partial z}=\frac{\hat{S}}{2}v \text{ and } w=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y} \tag{18}$$

Where:

$$\delta=\frac{U}{\sigma L}=\frac{l_m}{L},\ \hat{f}=\frac{f}{\sigma},\ E_v=\frac{2A_v}{\sigma H^2},\ \hat{S}=\frac{2S}{\sigma H},\ \hat{\lambda}=\lambda\frac{H}{l_m} \text{ and} \tag{19}$$

$$\hat{\alpha}=\frac{\alpha(l_m H)^b}{Hl_m\sigma}\equiv\frac{1}{\sigma T_{long}} \tag{20}$$

in which δ gives the ratio between the two length scales and $\hat{f}$ is the non-dimensional Coriolis parameter. The parameter $\sqrt{E_v/2}$ quantifies the part of the water column (the Stokes depth) in which viscous effects influence the tidal motion significantly. The slip parameter $\hat{S}$ gives the amount of slip at the seabed in dimensionless units. Its definition differs from the slip parameter s in Maas and van Haren (1987); their relation being given by $s = \hat{S}/E_v$. The definition in the present model is chosen in order to treat viscous and slip effects separately. The parameter $\hat{\lambda}$ accounts for slope correction on morphological relevant spatial scales and T_{long} follows from a balance of the two terms in the bottom evolution Eq. (8). It has to be interpreted as the time scale on which morphological changes take place.

Typical scales for a representative North Sea location can be readily obtained in order to determine the relevant physical processes in the present model. The maximum tidal velocity is of order 1 ms^{-1}. The frequency of the M_2-tide in the North Sea is $\sigma = 1.4 \times 10^{-4}\ s^{-1}$. This makes the morphological length scale of the order of 7 km. Taking a Coriolis parameter at 52° N, $f \approx 1.15 \times 10^{-4}\ s^{-1}$ which gives $\tilde{f} \approx 0.82$. In the North Sea, the tidal wavelength is the order of hundreds of kilometres, which means that the parameter δ is small, $\delta \approx 0.01$. A typical North Sea depth is H = 30 m. The vertical viscosity coefficient A_v has a wide range of values, see Pedlosky (1987). Using values from 0.01 to 0.5 m^2s^{-1}, gives a Stokes number between 0.2 and 8. For these lattermost values the Stokes depth is of the same order or larger than the local water depth, which is characteristic for most coastal seas. Observations are needed to determine the slip parameter $\hat{S}$. It can, for instance, be calculated from s and the Stokes number, where s is experimentally determined, as $\hat{S} = s\, E_v$. Maas and van Haren (1987) found that s is between 0.1 (nearly free slip) and 10 (almost no slip) in the Southern North Sea. The exponent in the bedload transport formula, b, follows from measurements in combination with physical considerations; typical values being b ~1 - 3. A value of b = 1.5 was used in the present work. Characteristic values for the parameter λ can be estimated theoretically as indicated in Hulscher *et al.* (1993) and are in the range 1-2, which give $\hat{\lambda} \approx 0.01$. The bedload parameter α, for which a value can be estimated from Van Rijn (1993), is of order (O (1)), which yields the morphological time scale

$$T_{long} \approx 70\text{yr}. \qquad \textbf{(21)}$$

The seabed variable h will hardly vary on the tidal time scale. Consequently, it is possible to put $\partial h/\partial t = 0$ in the flow model. Furthermore, the slow evolution of the bed is determined by the tidally-averaged sediment fluxes. Thus the seabed Eq. (15) can be replaced by the equation:

$$\frac{\partial h}{\partial T_m} = -\vec{\nabla}\cdot\left\langle\left\{|\vec{\tau}|^b\left[\frac{\vec{\tau}}{|\vec{\tau}|} - \hat{\lambda}\vec{\nabla}h\right]\right\}\right\rangle \qquad \textbf{(22)}$$

where $T_m \equiv \hat{\alpha}\, t$, a slow time co-ordinate. The brackets denote a tidal-average. More details on this approximation can be found in Hulscher *et al.* (1993). In order to compute the net sediment fluxes, the flow equations need to be solved at a fast time scale, with $\partial h/\partial t = 0$. Therefore, the model equations used in the present work can be summarised as Eq. (12), Eq. (13) and Eq. (14) with $\partial h/\partial t = 0$, supplemented with Eq. (22); the boundary conditions being given by Eq. (17) and Eq. (18).

Linear stability concept

In order to investigate whether the presence of tidal sandbanks and sand waves may be associated with free instabilities in this morphological system, use has been made of linear stability analysis.

This hypothesis can be tested by first defining a basic state which describes the flow of a tidal current over a flat seabed. The basic velocities will have a vertical structure due to the frictional terms in the momentum equations, but Coriolis effects will cause veering of current direction over the flow depth (Ekman spiralling).

The next step is to introduce bed perturbations with arbitrary wavenumbers in the horizontal directions. Subsequently the initial interaction between bedforms and currents is studied to determine whether the perturbations are amplified or reduced. A basic state is said to be stable if arbitrary bedform disturbances with non-zero wavenumbers decay exponentially. This implies that all perturbations have negative growth rates. Conversely, if there is at least one perturbation with a positive growth rate, the basic state is unstable.

From a mathematical point of view this may be formulated as follows. The solution of the problem is symbolically written as the vector equation:

$$\psi = (u, v, w, \zeta, h) \tag{23}$$

Next the response of the system is considered by prescribing a small wave-like perturbation in the seabed as an initial state. In this way the stability of the basic state is investigated. The amplitude of the perturbation with respect to the undisturbed water depth is denoted by γ, where $\gamma << 1$. An approximate solution is then obtained by expanding ψ using two small parameters δ and γ, thus:

$$\begin{aligned}\psi = \;& \gamma^0\{\psi_{00} + \delta\psi_{01} + \delta^2\psi_{02} + \ldots\} \\ & + \gamma^1\{\psi_{10} + \delta\psi_{11} + \delta^2\psi_{12} + \ldots\} \\ & + \gamma^2\{\psi_{20} + \delta\psi_{21} + \delta^2\psi_{22} + \ldots\}\end{aligned} \tag{24}$$

where $h_{0n} = 0$ for n=0,1,... . The matching boundary conditions can be obtained by transforming the boundary conditions at $z = \delta\zeta$, Eq. (17), and $z = -1 + h$, Eq. (18), to the fixed positions $z = 0$ and $z = -1$. This can be done by making a Taylor expansion in the small parameters δ and γ. At $z = 0$ (the transformed surface) the resulting equations can be written in the form:-

$$\frac{\partial u}{\partial z} = -\delta\zeta\frac{\partial^2 u}{\partial z^2} + O(\delta^2), \quad \frac{\partial v}{\partial z} = -\delta\zeta\frac{\partial^2 v}{\partial z^2} + O(\delta^2) \tag{25}$$

$$w = \delta\left\{\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y}\right\} - \delta\zeta\frac{\partial w}{\partial z} + O(\delta^2) \tag{26}$$

and at $z = -1$ (the seabed) the expansion produces the equations:-

$$\frac{\partial u}{\partial z}=\frac{\hat{S}}{E_v}\left\{u+\gamma h\frac{\partial u}{\partial z}\right\}-\gamma h\frac{\partial^2 u}{\partial z^2}+O(\gamma^2) \quad (27)$$

$$\frac{\partial v}{\partial z}=\frac{\hat{S}}{E_v}\left\{v+\gamma h\frac{\partial v}{\partial z}\right\}-\gamma h\frac{\partial^2 v}{\partial z^2}+O(\gamma^2) \quad (28)$$

$$w=\gamma\left\{u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}\right\}-\gamma h\frac{\partial w}{\partial z}+O(\gamma^2) \quad (29)$$

The linear stability of the lowest-order approximation of the basic state solution, ψ_{00} and perturbation ψ_{10}, is formally given by the series $\psi_0=\sum_{n=0}^{\infty}\delta^n\psi_{0n}$. Physically this corresponds to a rigid lid approximation, that is, spatial variations of the tidal wave are assumed to be unimportant for the generation of bedforms. The equations which determine the basic state of lowest order in δ are given as follows:

$$\frac{\partial u_{00}}{\partial t}+w_{00}\frac{\partial u_{00}}{\partial z}-\hat{f}v_{00}-\frac{E_v}{2}\frac{\partial^2 u_{00}}{\partial z^2}=-\frac{\partial\zeta_{10}}{\partial x} \quad (30)$$

$$\frac{\partial v_{00}}{\partial t}+w_{00}\frac{\partial v_{00}}{\partial z}-\hat{f}u_{00}-\frac{E_v}{2}\frac{\partial^2 v_{00}}{\partial z^2}=-\frac{\partial\zeta_{10}}{\partial y} \quad (31)$$

$$\frac{\partial w_{00}}{\partial z}=0 \quad (32)$$

The corresponding boundary conditions follow from Eqs. (25)-(29), at z = 0:

$$\frac{\partial u_{00}}{\partial z}=\frac{\partial v_{00}}{\partial z}=0 \text{ and } w_{00}=0 \quad (33)$$

and at the bed z = -1 :

$$E_v\frac{\partial u_{00}}{\partial z}=\hat{S}u_{00},\ E_v\frac{\partial v_{00}}{\partial z}=\hat{S}v_{00} \text{ and } w_{00}=0 \quad (34)$$

From Eq. (32), together with boundary conditions Eq. (33) and Eq. (34), it can be seen that the vertical velocity in the basic state is zero, that is:

$$w_{00}(z)=0 \quad (35)$$

It is observed that in this model the $O(\delta)$ free surface elevations determine the $O(1)$ velocity field The function $\vec{\nabla}\zeta_{10}$, which represents horizontal variations of the tidal wave, is chosen to be only a function of time. It determines the depth-averaged velocities U_{00} and V_{00} or vice versa. Here the reverse option was chosen in order to compare the results obtained with an earlier depth-averaged model, Hulscher *et al.* (1993). Thus:

$$\left(U_{00}(t), V_{00}(t)\right) \equiv \int_{-1}^{0}\left(u_{00}(z), v_{00}(z)\right)dz = (\sin\ t, \beta \cos\ t) \qquad \textbf{(36)}$$

Here $|\beta| \leq 1$ is assumed, and hence the x-axis is chosen to coincide with the principal direction of the depth-averaged tidal current. Thus $\beta = 0$ describes an oscillating unidirectional tide while $0 < |\beta| < 1$ gives elliptical tides. For $|\beta| = 1$ a circular tide is obtained, owing to the loss of a preferred direction: the choice of the x-axis has no meaning for this lattermost case. Similar models have been considered earlier by Prandle (1982), Maas & van Haren (1987) and Visser *et al.* (1994). As the present approach is slightly different, it is solved as shown in Appendix A of Hulscher (1996). Expressions for the basic velocities u_{00} and v_{00}, which are functions of time and the vertical co-ordinate, are derived below. Since the basic solution can be presented either in tidal components or in terms of elliptical properties, both will be discussed below.

Horizontal basic velocities

The basic velocities can be obtained in two ways. The first one, in tidal components, is also used in the derivation of the linear stability problem discussed in the next section. As can be seen from Eqs. (30) - (36), the solution of the horizontal velocities can be represented by the equations:

$$u_{00}(z,t) = u_s(z)\sin\ t + u_c(z)\cos\ t \qquad \textbf{(37)}$$

$$v_{00}(z,t) = v_s(z)\sin\ t + v_c(z)\cos\ t \qquad \textbf{(38)}$$

The values of $u_s(z)$ etc. can be computed using Eq. (64) in Appendix A of Hulscher (1996). In Figure 6.3.3.2 the vertical structure of the tidal components is shown for a unidirectional tide (β = 0) for characteristic North Sea conditions with the Coriolis parameter $\hat{f} = 0.82$, the Stokes number E_v = 1.0 and a slip parameter $\hat{S} = 2.7$. It can be seen that although the depth-averaged values of $u_c(z)$, $v_s(z)$ and $v_c(z)$ equal zero, internal friction and Coriolis effects cause a vertical structure of these tidal velocity components. Due to the allowance of slip at the seabed, the velocity at the seabed is non-zero.

The second description using elliptical properties is helpful for a physical interpretation of model results and links to existing literature, Prandle (1982); and Visser *et al.* (1994). In Appendix A of Hulscher (1996), the derivation and discussion of the physical quantities which are used for this description are also given. Figure 6.3.3.3 uses the same physical parameters as in Figure 6.3.3.2. In

Figure 6.3.3.3(b) it is observed that the ellipticity appears to be clockwise near the bottom and anti-clockwise at the top of the water column.

Figure 6.3.3.3(c) demonstrates that near the seabed the major axis is rotated clockwise with respect to the depth-averaged flow direction while near the surface the rotation is anti-clockwise. The difference between both axes is in this case 4°.

Due to internal friction the maximum value of the flow velocity is reached at a later stage at the seabed than at the surface, as can be concluded from Figure 6.3.3.3(d). The time difference is in this case 10.5 minutes.

The basic bottom shear stress is completely dominated by the sine-component in the x-direction. The amplitude of this component can be calculated as a function of the Stokes number and the stress parameter. In Figure 6.3.3.4 its (non-dimensional) value is shown as a function of the stress parameter for three values of the Stokes number.

Linear stability analysis

As explained in previous sections the linear stability problem is obtained from the equations describing the $O(\gamma)$ part of the general solution presented in Eq. (24). Here we consider only the ψ_{10} part of the solution. The corresponding equations of motion and boundary conditions can be found from the substitution of the expansion Eq. (24) into Eqs. (12)-(14), Eq. (22) and the transformed boundary conditions Eqs. (25)-(29). The results are the following equations:

$$\frac{\partial u_{10}}{\partial t} + u_{00}\frac{\partial u_{10}}{\partial x} + v_{00}\frac{\partial u_{10}}{\partial y} + w_{10}\frac{\partial u_{00}}{\partial z} + \hat{f}v_{10} - \frac{E_v}{2}\frac{\partial^2 u_{10}}{\partial z^2} = -\frac{\partial \zeta_{11}}{\partial x} \quad \textbf{(39)}$$

$$\frac{\partial v_{10}}{\partial t} + u_{00}\frac{\partial v_{10}}{\partial x} + v_{00}\frac{\partial v_{10}}{\partial y} + w_{10}\frac{\partial v_{00}}{\partial z} + \hat{f}u_{10} - \frac{E_v}{2}\frac{\partial^2 v_{10}}{\partial z^2} = -\frac{\partial \zeta_{11}}{\partial y} \quad \textbf{(40)}$$

$$\frac{\partial u_{10}}{\partial x} + \frac{\partial v_{10}}{\partial y} + \frac{\partial w_{10}}{\partial z} = 0 \quad \textbf{(41)}$$

$$\frac{\partial h_{10}}{\partial T_{ml}} = -\vec{\nabla}\cdot\left(\left|\vec{\tau}_{00}\right|^{b-1}\vec{\tau}_{10} + (b-1)\left|\vec{\tau}_{00}\right|^{b-3}\vec{\tau}_{00}\left(\vec{\tau}_{00}\cdot\vec{\tau}_{10}\right) - \left|\vec{\tau}_{00}\right|^{b}\lambda\vec{\nabla}h_{10}\right) \quad \textbf{(42)}$$

$$\vec{\tau}=\left(\tau_x,\tau_y\right)=\frac{E_v}{2}\left(\frac{\partial u}{\partial z},\frac{\partial v}{\partial z}\right)\Bigg|_{z=-1} \tag{43}$$

The boundary conditions at the free surface, z = 0 are given by the equation:

$$\frac{\partial u_{10}}{\partial z}=\frac{\partial v_{10}}{\partial z}=w_{10}=0 \tag{44}$$

At the seabed, z = -1, the result is the equations:

$$\frac{\partial u_{10}}{\partial z}=\frac{\hat{S}}{E_v}u_{10}+h_{10}\left(-\frac{\partial^2 u_{00}}{\partial z^2}+\hat{S}\frac{\partial u_{00}}{\partial z}\right) \tag{45}$$

$$\frac{\partial v_{10}}{\partial z}=\frac{\hat{S}}{E_v}v_{10}+h_{10}\left(-\frac{\partial^2 v_{00}}{\partial z^2}+\hat{S}\frac{\partial v_{00}}{\partial z}\right) \tag{46}$$

and

$$w_{10}=\left\{u_{00}\frac{\partial h_{10}}{\partial x}+v_{00}\frac{\partial h_{10}}{\partial y}\right\} \tag{47}$$

The bottom perturbation enters this system via the seabed boundary condition. Interactions between the basic solution and the perturbations take place through the oscillatory advection terms in the equations of motion. For a finite slip parameter $\hat{S}$, it is observed that a vertical velocity is generated by form drag, whereas the generation of horizontal velocities at the seabed is due to bed shear stress, which is induced by the basic state. In considering the case where $\hat{S}\rightarrow\infty$, only the latter mechanism is active because in this limit the basic horizontal velocities become zero at the seabed.

In order to determine an equation for seabed perturbation growth, the solution technique as described in Hulscher *et al.* (1993) is used. The problem structure for a wavy seabed perturbation allows a Fourier transformation in horizontal co-ordinates. This implies that solutions have the following structure:

$$\Psi_{10}(\vec{x},z,t)=\iint\tilde{\Psi}\left(\vec{k},z,t\right)e^{-i\vec{k}\cdot\vec{x}}d\vec{k}+cc \tag{48}$$

in which cc stands for the complex conjugate expression and furthermore Ψ_{10} = $(u_{10},v_{10},w_{10},\zeta_{11},h_{10})$. The $\tilde{\Psi}=\left(\tilde{u},\tilde{v},\tilde{w},\tilde{\zeta},\tilde{h}\right)$ as defined in Eq. (48) depends on the wavevector $\vec{k}$ of the seabed pattern, the vertical co-ordinate z and time. The basic state describes an M_2-tide, and due to the structure of the linear problem it can be represented by the solution of the linear problem as a Fourier series in tidal components, M_0, M_2, M_4... Next, a harmonic truncation in time is made, which means that the perturbation is restricted to a finite number of tidal components, for details see Hulscher *et al.* (1993). The lattermost authors have demonstrated that an N = 3 truncation (including M_0, M_2, M_4, M_6) is the most appropriate one for elliptical tides, although an N = 1 truncation (including M_0, M_2) still contains the dominant physics concerning unidirectional tides. As the depth-averaged basic state in all the work for the CSTAB project is a depth-averaged unidirectional tide, an N = 1 truncation will include the dominant physical processes. If the transformed seabed variable $\tilde{h}$ is chosen to be real, the structure of the equations determines that the amplitudes of the various tidal components in the solution vector $\tilde{\Psi}$ are either real or imaginary. This allows the harmonic truncated solution to be written in the form:

$$\tilde{u}_{trunc}(z,t) = \tilde{h}\left[i*a_0(z) + a_s(z)\sin t + a_c(z)\cos t\right] \quad \textbf{(49)}$$

$$\tilde{v}_{trunc}(z,t) = \tilde{h}\left[i*b_0(z) + b_s(z)\sin t + b_c(z)\cos t\right] \quad \textbf{(50)}$$

$$\tilde{w}_{trunc}(z,t) = \tilde{h}\left[c_0(z) + i*c_s(z)\sin t + i*c_c(z)\cos t\right] \quad \textbf{(51)}$$

$$\tilde{\zeta}_{trunc}(z,t) = \tilde{h}\left[d_0(z) + i*d_s(z)\sin t + i*d_c(z)\cos t\right] \quad \textbf{(52)}$$

in which a-$d_{0,s,c}$ are real functions. As all the time-dependencies are now known, tidal averaging in the seabed evolution equation, Eq. (42), can be carried out. The result is the equation:

$$\begin{aligned}
\omega_{gr} &\equiv \frac{1}{\tilde{h}}\frac{\partial\tilde{h}}{\partial T_m} \\
&= \frac{E_v}{2}\left|\vec{\tau}_{00}\right|^{b-1}\left\{-k*a_0'(-1) - lb_0(-1)\right\} - \hat{\lambda}\left|\vec{\tau}_{00}\right|^{b}\left(k^2 + l^2\right) \\
&\quad - \frac{(b-1)E_v^3}{32}\left|\vec{\tau}_{00}\right|^{b-3}\Big\{k\left[\left(u_s'(-1)\right)^2 + \left(u_c'(-1)\right)^2\right]a_0'(-1) \\
&\quad + l\left[\left(v_s'(-1)\right)^2 + \left(v_c'(-1)\right)^2\right]b_0'(-1) + \left(k*a_0'(-1)\right. \\
&\quad \left. + l*b_0'(-1)\right)\left(u_s'(-1)v_s'(-1) + u_c'(-1)v_c'(-1)\right)\Big\}
\end{aligned} \quad \textbf{(53)}$$

in which a prime denotes the vertical derivative of that function. The bottom evolution equation shows that the growth of bedforms is determined by the bed shear stresses induced by the basic velocity field, which are already known (see Section above), and the bed shear stresses induced by the residual horizontal components of the velocities at order ψ_{10} of the solution. To obtain the lattermost ones, the three-dimensional flow field must be solved. To derive the equations and boundary conditions for the real coefficients $(a - d)_{0,s,c}$, it is necessary to substitute Eqs. (49)-(52) into Eq. (48) and consider the Fourier transformation of Eqs. (39)-(41) and boundary conditions Eqs. (44)-(47). The resulting set of differential equations and boundary conditions, which determine the vertical flow structure induced by the perturbations, is given in Appendix B of Hulscher (1996). This set of equations and boundary conditions was solved numerically.

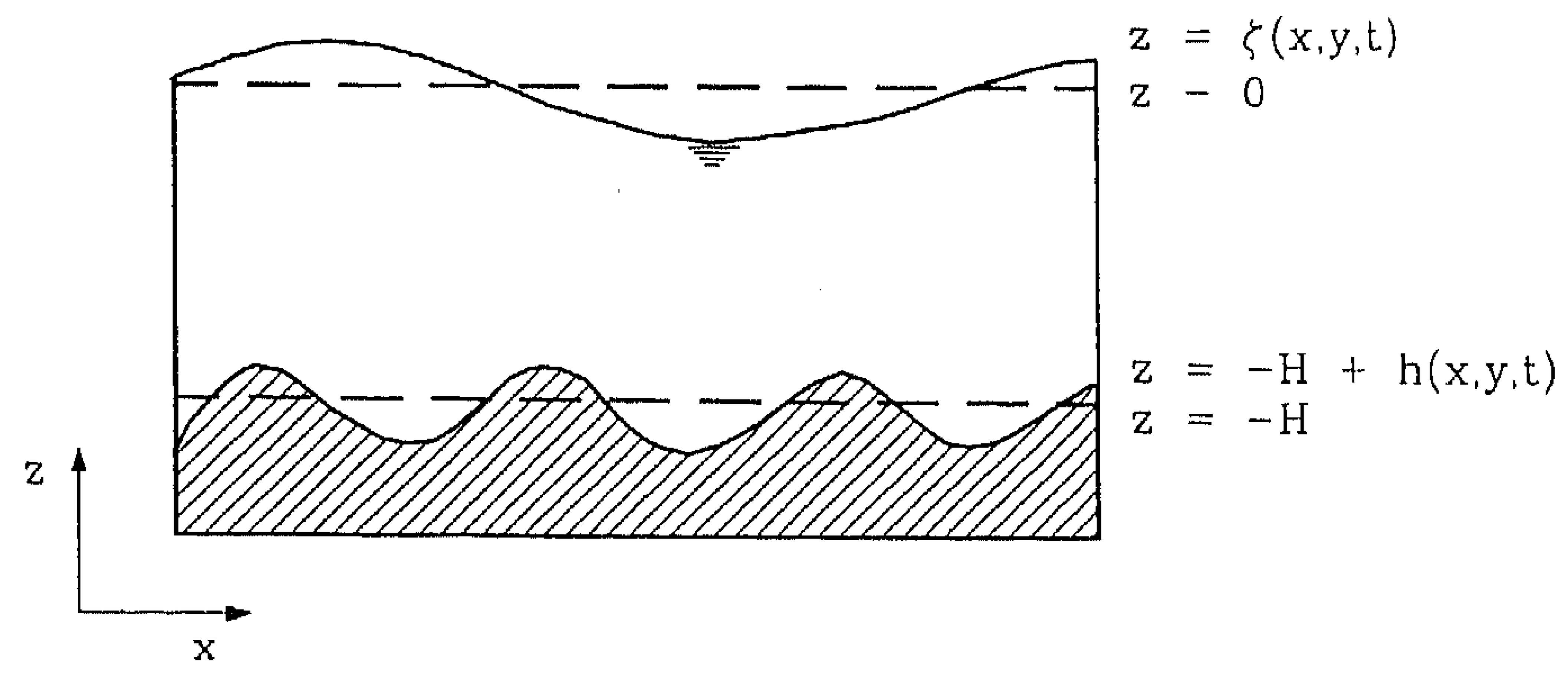

Figure 6.3.3.1: Situation sketch (side view).

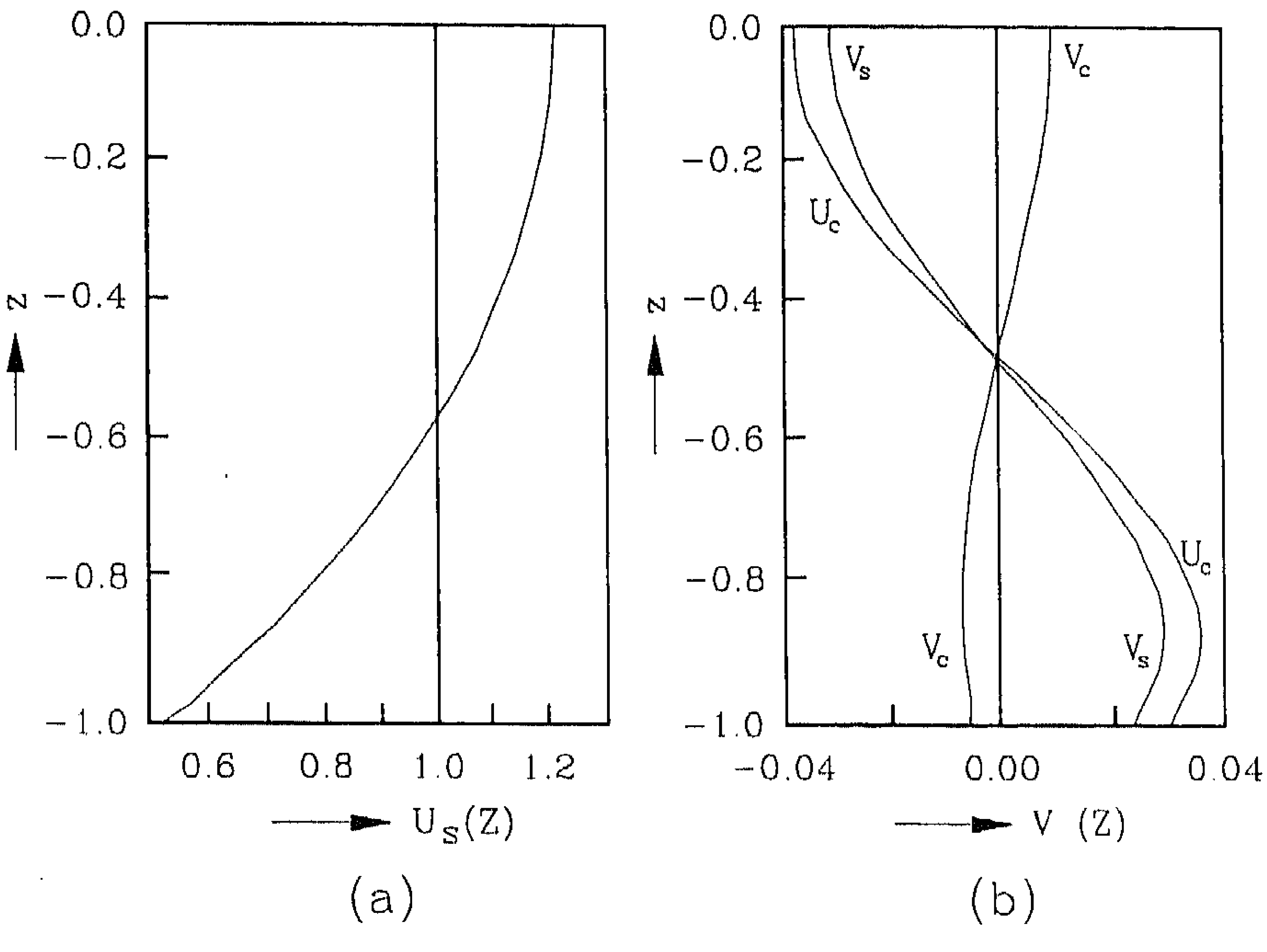

Figure 6.3.3.2: The vertical structure of the tidal components of the basic state as a function of the dimensionless depth. The physical parameters are chosen as follows $\hat{f} = 0.82$, $E_v = 1.0$ and $\hat{S} = 2.7$.

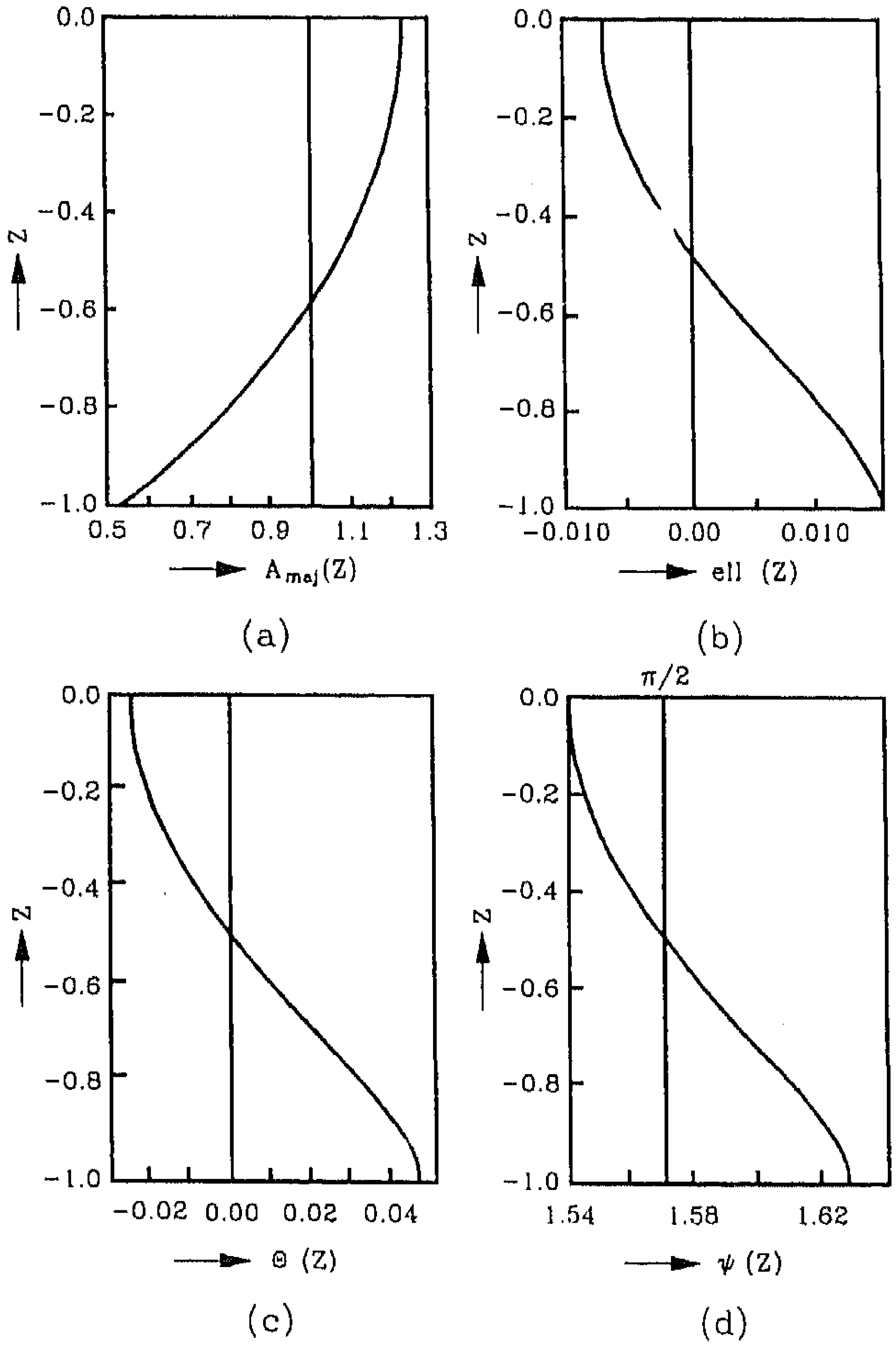

Figure 6.3.3.3: The vertical structure of elliptical properties of the basic state as a function ot the dimensionless depth.

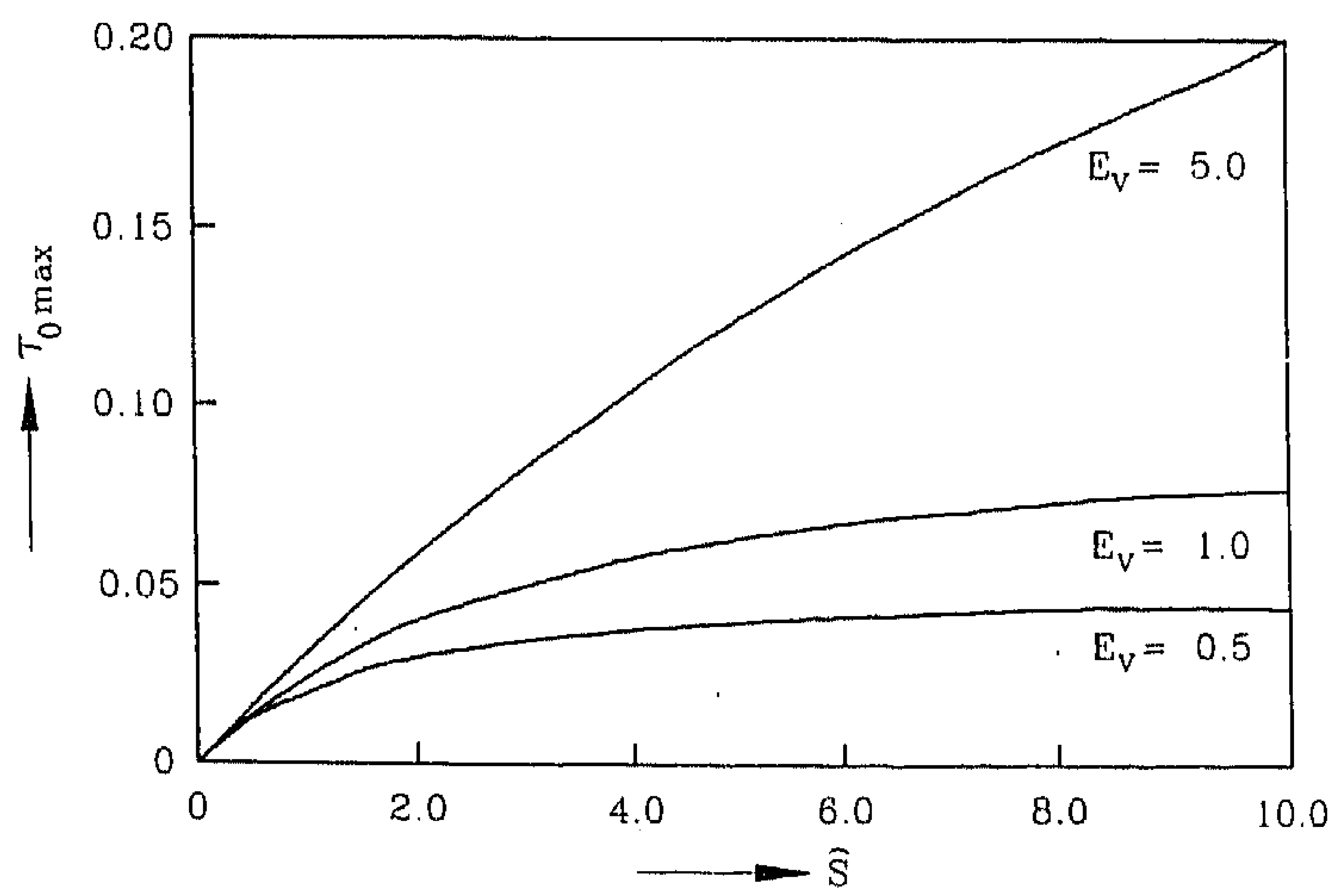

Figure 6.3.3.4: The maximum value of the (dimensionless) bed shear stress as a function of the stress parameter $\hat{S}$. $(\hat{f} = 0.82)$.