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Energy propagation in dissipative systems. Part II: Centrovelocity for nonlinear wave equations

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We consider nonlinear wave equations, first order in time, of a specific form. In the absence of dissipation, these equations are given by a Poisson system, with a Hamiltonian that is the integral of some density. The functional I, defined to be the integral of the square of the waveform, is a constant of motion of the unperturbed system. It will be shown that Z, the center of gravity of this density, is canonically conjugate to I and can be used as a measure to locate the position of the waveform. By introducing new coordinates based on Z, we get expressions for the centrovelocity, $\dot{Z}$, and the decay of I, which compare to the ones of part I, in both the conservative and the dissipative case. With the derived expressions, we investigate the decay of solitary waves of the Korteweg–de Vries equation, when different kinds of dissipation are added.

1. Introduction

In part I of this series of papers [5], the propagation of energy in dissipative systems is investigated. To that end a balance equation for the energy density $\mathscr{E}(u)$ is used to derive an expression for the centrovelocity of energy (the velocity of the center of gravity of energy). This balance equation is

$$\partial_t \mathscr{E}(u) + \partial_x \mathscr{F}(u) = -\mathscr{S}(u), \tag{1.1}$$

with $\mathscr{F}(u)$ the flux density corresponding to $\mathscr{E}(u)$ and $\mathscr{S}(u)$ the loss density. Introducing the decomposition

$$\mathscr{S} = \frac{\langle \mathscr{S} \rangle}{\langle \mathscr{E} \rangle} \mathscr{E} + \partial_x \Phi,$$

for some loss-flux density Φ, it was shown that this velocity differs from the usual energy-flux velocity $\langle \mathscr{F} \rangle / \langle \mathscr{E} \rangle$ by a loss-flux term $-\langle \Phi \rangle / \langle \mathscr{E} \rangle$, due to the presence of dissipation. Furthermore, the derived expression for the centrovelocity is investigated for linear wave equations, viz. first order hyperbolic systems.

In this part II, we shall concentrate on two aspects. First we will provide another interpretation of the center of gravity and its velocity, as a measure to locate the position of a wave. In fact, we show that for any functional Z that is the center of gravity of some density, the value of Z decreases by an amount of φ if the waveform u is translated over a distance φ. This property will also be stated in terms which relate to notions of classical mechanics. This is possible since the (nonlinear) wave equations that will be considered form a class of equations which, in the absence of dissipation, have a (generalised) Hamiltonian structure. For such equations the invariance under translation implies the existence of a first integral (constant of motion), say I. Then a center of gravity of any density can be interpreted as a coordinate functional canonically conjugate to I.

Secondly, we will explicitly reformulate the wave equation using the functionals I and Z as a kind of pair of coordinates and emphasize the presence of dissipation in this formulation. Approximating the whole wave equation by only dynamical equations for I and Z, explicit expressions for the time asymptotic wave velocity will be derived, for solutions in the neighbourhood of a family of special solutions describing the decay of solitary waves. The dependence of the asymptotic results on the kind of dissipation will become clear.

The wave equation that will be considered in this paper can be described by the following partial differential equation for the wave elevation $u(x, t): \mathbb{R} \times [0, \infty) \to \mathbb{R}$

$$\dot{u} = \partial_x[\delta H(u)] - S(u). \tag{1.2}$$

This is a Poisson (generalised Hamiltonian) system, with Hamiltonian $H(u)$, to which a dissipation $S(u)$ is added. It is assumed that both the Poisson system and the dissipation are translationally invariant. The invariancy of the Poisson system makes the integral

$$I(u) = \frac{1}{2} \int_{-\infty}^{\infty} u^2(x)\,\mathrm{d}x \tag{1.3}$$

a constant of motion of the nondissipative system and the corresponding conservation is written as

$$\partial_t \mathscr{E}(u) + \partial_x \mathscr{F}(u) = 0. \tag{1.4}$$

The density $\mathscr{E}(u) = \frac{1}{2}u^2$ will be called the energy density and the flux density $\mathscr{F}(u)$ is some density depending on the Hamiltonian H.

In Section 2 the dynamical equations for the nondissipative case ($S=0$) are written in a moving coordinate frame, in which the center of gravity of energy $Z(u) =: \varphi$ is a new coordinate. The set of new coordinates is completed with a function v, defined by $v = \Phi_\varphi(u)$. This reduces the dynamics for v to the codimension 2 space with $I(v) = \gamma$ and $Z(v) = 0$. Further, because φ is the center of gravity of energy, $\dot{\varphi}$ is the centrovelocity and the equation for $\dot{\varphi}$ is an expression for this velocity. To compare this with the one in part I, we rewrite the expression, using the flux density $\mathscr{F}$ from the conservation law (1.4) and the equivalence of the two expressions is shown.

Next we use in Section 3 the previous variables γ, φ and v to rewrite the dynamics for the dissipative case. Again the resemblance with part I is shown.

In Section 4 and 5 the equations in the variables introduced in Section 3 are used to describe the decay of solitary waves of the 1-dimensional Korteweg–de Vries (KdV) equation, with a small dissipation added. The dissipations considered here are uniform damping, viscosity (KdV–Burgers) and a damping of the form $(-\partial_x^2)^\nu$. We truncate the full equations and compare the time-asymptotic behaviour of the centrovelocity with results of a two-timescale method by Ott and Sudan [9].

2. Dynamics of the Poisson system

As stated in the introduction, we first consider nondissipative dynamical equations of the form

$$\dot{u} = \partial_x \delta H(u), \tag{2.1}$$

with u a function in the space $\mathscr{U}$ of smooth functions defined on $\mathbb{R}$ and vanishing, together with all their derivatives, sufficiently fast at infinity. Furthermore, δH means the variational derivative of H, with respect to u in $\mathcal{M}$. It can be proved that there is a Poisson structure on $\mathcal{M}$, such that this system can be written as a Poisson or Hamiltonian system. The bracket which defines this structure is

$$\{F, G\}(u)=\int_{-\infty}^{\infty} \delta F(u)\, \partial_x[\delta G(u)]\, \mathrm{d}x=\langle\delta F(u), \partial_x[\delta G(u)]\rangle, \tag{2.2}$$

with F and G functionals on $\mathcal{M}$ and the equation $\dot{F}=\{F, H\}$ for any functional F, leads directly to (2.1). The functional H is called the Hamiltonian of the system. Since ∂_x is anti-symmetric, it holds that $\{H, H\}=0$ and so it follows immediately that the Hamiltonian is conserved.

From now on, it is assumed that $H(u)$ is the integration over $\mathbb{R}$ of some density $\mathscr{H}(u)$ on $\mathcal{M}$; $\mathscr{H}(u)$ at x depends on $u(x)$ and a finite number of its derivatives at x, but not explicitly on the point x itself. Hence $H(u)$ is a functional of the form

$$H(u)=\int_{-\infty}^{\infty} \mathscr{H}(u)\, \mathrm{d}x. \tag{2.3}$$

This implies that H is translationally invariant. Namely, define $\Phi_\varphi: \mathcal{M} \rightarrow \mathcal{M}$ to be the translation operator, meaning for any φ it holds

$$[\Phi_\varphi(u)](x)=u(x+\varphi), \quad x \in \mathbb{R}. \tag{2.4}$$

Since $\mathscr{H}(u)$ does not depend explicitly on x, it follows that

$$H(\Phi_\varphi(u))=H(u), \tag{2.5}$$

and therefore H is translationally invariant. This invariancy of H under any translation gives that $I(u)=\frac{1}{2}\int u^2$ is a constant of motion. In agreement with the use in part 1, we will call it the energy. The constancy of I follows with partial integration as

$$\frac{\mathrm{d}}{\mathrm{d}t} I(u)=-\int_{-\infty}^{\infty} \delta H(u)\, \partial_x u\, \mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}\varphi} H(\Phi_\varphi(u))|_{\varphi=0}=\frac{\mathrm{d}}{\mathrm{d}\varphi} H(u)|_{\varphi=0}=0. \tag{2.6}$$

(In fact it holds that $\{I, H\}=0$, for any translation invariant functional H on $\mathcal{M}$.)

Because I is conserved, the dynamics trivially reduces with one dimension. Also a reduction with a second dimension can be made. The system (2.1) is translationally invariant, which makes it natural to define a coordinate transformation to a moving frame, with new coordinates (φ, v) instead of u, defined by

$$u=\Phi_{-\varphi} v. \tag{2.7}$$

To make this definition unique we look for a functional Z, such that $Z(\Phi_\varphi(v))=Z(v)-\varphi$, for all φ and v. If we consider v to be the waveform, then φ is the position at which the waveform is located. Hence by this definition of Z, we get a measure to locate the waveform.

Proposition 1. *For any density $\mathscr{E}(u)$ that depends on u and a finite number of its derivatives, but not explicitly on x, the functional Z defined by*

$$Z(u)=\frac{\int_{-\infty}^{\infty} x\mathscr{E}(u)\,\mathrm{d}x}{\int_{-\infty}^{\infty} \mathscr{E}(u)\,\mathrm{d}x} \tag{2.8}$$

satisfies for all φ and for all v, for which $Z(v)$ is defined, the equation

$$Z(\Phi_\varphi(v))=Z(v)-\varphi. \tag{2.9}$$

For such a functional Z, the transformation from u to (φ, v) defined by $\varphi=Z(u)$, $v=\Phi_\varphi(u)$ is unique.

Proof. For a functional Z as defined above it holds for any φ and v that

$$Z(\Phi_\varphi(v))=\frac{\int_{-\infty}^{\infty} (x-\varphi)\mathscr{E}(v)\,\mathrm{d}x}{\int_{-\infty}^{\infty} \mathscr{E}(v)\,\mathrm{d}x}=Z(v)-\varphi. \tag{2.10}$$

□

Remark. The definition of φ and v implies that $Z(v)=0$. Furthermore Z is canonically conjugate to I, meaning that $\{Z, I\}=-1$. This follows by differentiating eq. (2.9) with respect to φ and evaluating this expression at $\varphi=0$:

$$-1=\left\langle \delta Z(\Phi_\varphi(v)), \frac{\mathrm{d}}{\mathrm{d}\varphi}[\Phi_\varphi(v)]\right\rangle\Big|_{\varphi=0}=\langle \delta Z(\Phi_\varphi(v)), \partial_x \delta I(\Phi_\varphi v)\rangle|_{\varphi=0}=\{Z, I\}(v). \tag{2.11}$$

From now on we will choose $\mathscr{E}(u)$ to be $\frac{1}{2}u^2$, hence

$$Z(u)=\frac{\int_{-\infty}^{\infty} xu^2(x)\,\mathrm{d}x}{\int_{-\infty}^{\infty} u^2(x)\,\mathrm{d}x}, \tag{2.12}$$

the center of gravity of energy as defined in part I.

Now everything is prepared to define at any solution u of the system (2.1), two canonically conjugate variables γ and φ and a function v, that stays in a codimension 2 space of $\mathcal{M}$. This transformation is depicted in Fig. 1.

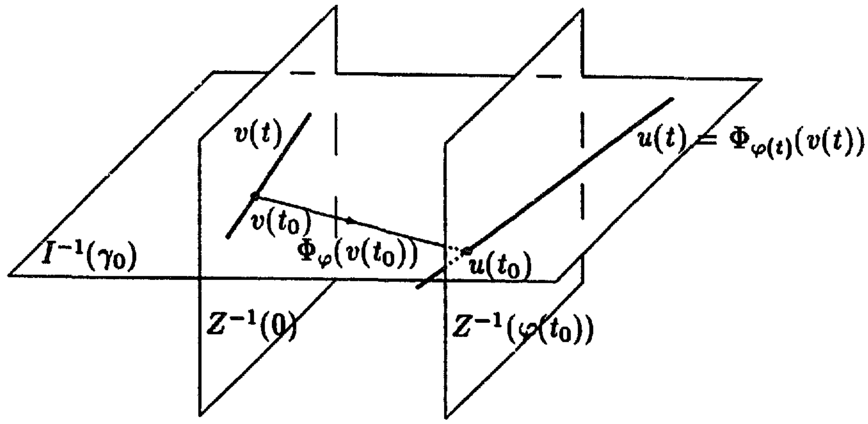

Fig. 1. Transformation from u to the variables (γ, φ, v). Here we see that $Z^{-1}(0) \cap I^{-1}(\gamma_0)$ is the space of the reduced dynamics.

Proposition 2. *Define for given u the scalar variables γ and φ by*

$$\gamma = I(u), \qquad \varphi = Z(u), \tag{2.13}$$

and the function v by

$$v = \Phi_\varphi(u). \tag{2.14}$$

Then $Z(v) = 0$ and $I(v) = \gamma$. The dynamical equation for u in these variables becomes

$$\begin{aligned} \dot{\gamma} &= 0, \\ \dot{\varphi} &= \{Z, H\}(v), \\ \dot{v} &= \partial_x[\delta H(v) + \{Z, H\}(v)\delta I(v)]. \end{aligned} \tag{2.15}$$

Remark. The function v satisfies the two constraints $I(v)$ is constant and $Z(v) = 0$ for all time. This means that v is from a space of codimension 2 and the dynamical equation for v is called the *reduced dynamics*. (Solving v from this equation, a direct integration gives φ and then $u = \Phi_{-\varphi}(v)$.) Stated differently the vectorfield for the v-equation is tangent to the levelsets of Z and I. Indeed, writing

$$\delta H(v) = \alpha \delta I(v) + \beta \delta Z(v) + \xi, \tag{2.16}$$

implies

$$\{H, Z\}(v) = \alpha \{I, Z\}(v) + \langle \xi, \partial_x \delta Z(v) \rangle, \tag{2.17}$$

$$\{H, I\}(v) = \beta \{I, Z\}(v) + \langle \xi, \partial_x \delta I(v) \rangle. \tag{2.18}$$

Since $\{Z, I\} = -1$ and $\{H, I\} = 0$, it follows that

$$\langle \xi, \partial_x \delta I(v) \rangle = 0 = \langle \xi, \partial_x \delta Z(v) \rangle, \tag{2.19}$$

if α and β are taken to be

$$\alpha = \{H, Z\}(v), \qquad \beta = 0. \tag{2.20}$$

Hence the vectorfield $\partial_x[\delta H(v)+\{Z, H\}(v)\delta I(v)]=\partial_x\xi$ is tangent to the levelsets of I and Z by (2.19). This is in line with the reduction in presence of symmetry which is well known in literature and elaborated in e.g. [1, 7, 8].

Proof of Proposition 2. The first equation trivially follows from I being a constant of motion. Using $u=\Phi_{-\varphi}(v)$ in the differential equation (2.1) gives

$$-\dot{\varphi}\frac{\mathrm{d}}{\mathrm{d}\varphi}[\Phi_{-\varphi}(v)]+D\Phi_{-\varphi}(v)\dot{v}=\partial_x\delta H(\Phi_{-\varphi}(v)). \tag{2.21}$$

By definition it holds that $Z(v)=0$, taking the time derivative of this equality gives that $\langle D\Phi_{-\varphi}^{-1*}(v)\delta Z(v), D\Phi_{-\varphi}(v)\dot{v}\rangle=0$. Hence taking the inner product of (2.21) with $D\Phi_{-\varphi}^{-1*}(v)\delta Z(v)$ and using that $\mathrm{d}/\mathrm{d}\varphi\Phi_{-\varphi}(v)=-\partial_x\delta I(\Phi_{-\varphi}(v))$ gives

$$\dot{\varphi}=\langle\delta Z(v), D\Phi_{-\varphi}^{-1}(v)\partial_x\delta H(\Phi_{-\varphi}(v))\rangle. \tag{2.22}$$

Denote the solution of the Poisson system (2.1) with initial value u_0 by $\Phi_t^H(u_0)$. From $\{H, I\}=0$, it follows that $\Phi_t^H(\Phi_\varphi(u_0))=\Phi_\varphi(\Phi_t^H(u_0))$, for every t, φ and u_0 (see e.g. [8]). By differentiating this with respect to t and then evaluating the expression in $t=0$, it follows that

$$D\Phi_{-\varphi}(v)\partial_x\delta H(v)=\partial_x\delta H(\Phi_{-\varphi}(v)). \tag{2.23}$$

Using eq. (2.23) in eq. (2.22) gives the expression for $\dot{\varphi}$. The equation for $\dot{v}$ follows from (2.21), by substitution of the expression for $\dot{\varphi}$ and eq. (2.23) in it. □

By definition it holds that $\varphi=Z(u)$ is the center of gravity of energy of a solution. Hence $\dot{\varphi}$ is the centrovelocity and the second equation in (2.15) is an expression for the centrovelocity of a translationally invariant Hamiltonian system. The translation invariancy gives a local conservation law, which can be written as

$$\partial_t\mathscr{E}(u)+\partial_x\mathscr{F}(u)=0, \tag{2.24}$$

where the energy density is $\mathscr{E}(u)=\frac{1}{2}u^2$ and the flux density $\mathscr{F}(u)$ is some density depending on the Hamiltonian H, by the relation

$$-\partial_x\mathscr{F}(u)=u\,\partial_x\delta H(u). \tag{2.25}$$

Now we can compare the expression for $\dot{\varphi}$ in eq. (2.15) with the one in part I. From (2.9) it follows that $D\Phi_\varphi(v)\delta Z(\Phi_\varphi(v))=\delta Z(v)$ and with (2.23) this gives that $\{Z, H\}(v)=\{Z, H\}(u)$. Using the definition of this bracket and the fact that $\{I, H\}=0$, we see that

$$\{Z, H\}(u)=\frac{1}{I(u)}\int_{-\infty}^{\infty}x\,xu\,\partial_x\delta H(u)\,\mathrm{d}x=-\frac{1}{I(u)}\int_{-\infty}^{\infty}x\,\partial_x\mathscr{F}(u)\,\mathrm{d}x=\frac{\displaystyle\int_{-\infty}^{\infty}\mathscr{F}(u)\,\mathrm{d}x}{I(u)}=\frac{\langle\mathscr{F}(u)\rangle}{\langle\mathscr{E}(u)\rangle}. \tag{2.26}$$

Hence $\dot{\varphi}=\langle\mathscr{F}(u)\rangle/\langle\mathscr{E}(u)\rangle$ is the energy-flux velocity, in accordance to part I, in the case in which no dissipation is present.

3. The effect of dissipation

Next we consider the Hamiltonian system with a dissipation S added

$$\dot{u}=\partial_x\delta H(u)-S(u). \tag{3.1}$$

As said in the introduction, we will assume that this dissipation is translationally invariant, hence $S(\Phi_\varphi(v))=D\Phi_\varphi(v)S(v)$.

In general the energy will not be conserved anymore. We will investigate the evolutions of the variables (γ,φ,v), defined in (2.13). To that end, the dissipation S is decomposed in a component S_3, tangent to the I-levelset, and a transversal component. Writing

$$S(v)=-S_2(v)\,\partial_x\delta I(v)+S_3(v) \tag{3.2}$$

it holds that

$$\langle S_3(v),\delta I(v)\rangle=0, \tag{3.3}$$

for $S_2(v)=\langle S(v),\delta Z(v)\rangle$. Further define $S_1(v)=\langle S(v),\delta I(v)\rangle$.

Proposition 3. *The dynamical equations for the system with dissipation in the variables defined by eq.* (2.13) *are*

$$\begin{aligned}\dot{\gamma}&=-S_1(v),\\ \dot{\varphi}&=\{Z,H\}(v)-S_2(v),\\ \dot{v}&=\partial_x[\delta H(v)+\{Z,H\}(v)\delta I(v)]-S_3(v),\end{aligned} \tag{3.4}$$

where S_1, S_2 and S_3 are as above.

Proof. We only prove the first equation. The second and the third are derived in the same way as in the conservative case. By definition it holds that $\gamma=I(u)$, hence $\dot{\gamma}=\langle\delta I(u),\dot{u}\rangle$. Using (3.1) and the fact that $\{I,H\}(u)=\int u\,\partial_x\delta H(u)\,dx=0$, gives $\dot{\gamma}=-\langle S(u),\delta I(u)\rangle=-S_1(u)$. Using the translationally invariancy of I and S we see that $\delta I(\Phi_{-\varphi}(v))=D\Phi_{-\varphi}^{-1*}(v)\delta I(v)$ and $S(\Phi_{-\varphi}(v))=D\Phi_{-\varphi}(v)S(v)$. Hence $S_1(u)=S_1(v)$ and by putting this in the equation for $\dot{\gamma}$, we get the first equation. □

Remark. Unlike the conservative case, the function v is not in a space of codimension 2 now, since γ is not constant in general. It is possible however, to define a coordinate transformation $u\to(\gamma,\varphi,w)$, in which w is in a codimension 2 space. This transformation uses the Hamiltonian flow of the functional Z. In Appendix A we will describe this transformation and provide the explicit expression for the Z-flow in this case.

Now we compare the equations for $\dot{\gamma}$ and $\dot{\varphi}$ of (3.4) in Proposition 3 with the ones in part I. Defining a loss density $\mathscr{S}$, such that

$$\mathscr{S}(u)=uS(u), \tag{3.5}$$

the local conservation law of the unperturbed case becomes a balance law, when a perturbation is added to the system:

$$\partial_t\mathscr{E}(u)+\partial_x\mathscr{F}(u)=-\mathscr{S}(u). \tag{3.6}$$

By definition it holds that $\gamma = I(u)$ is the energy, hence $\dot{\gamma} = \partial_t\langle\mathscr{E}\rangle$, with the $\mathscr{E}(u) = \frac{1}{2}u^2$, just as in part I. Furthermore, it holds that $-\langle S(v), \delta I(v)\rangle = -\langle S(u), \delta I(u)\rangle$, hence

$$\partial_t\langle\mathscr{E}\rangle = \dot{\gamma} = \int_{-\infty}^{\infty} uS(u)\,\mathrm{d}x = \langle\mathscr{S}\rangle \tag{3.7}$$

similar to eq. (3.14) in part I.

Next the equation for $\dot{\varphi}$. We saw already that $\{Z, I\}(v)$ is the energy-flux velocity. Hence $(S(v), \delta Z(v))$ has to be the loss-flux term introduced in part I. Calculations give

$$(S(v), \delta Z(v)) = (S(u), \delta Z(u)) = \frac{1}{I(u)}\int_{-\infty}^{\infty} xuS(u)\,\mathrm{d}x - \frac{Z(u)}{I(u)}\int_{-\infty}^{\infty} uS(u)\,\mathrm{d}x$$

$$= \frac{\langle x\mathscr{S}(u)\rangle}{I(u)} - \frac{\langle\mathscr{S}(u)\rangle Z(u)}{I(u)}. \tag{3.8}$$

In part I it is proved that

$$\mathscr{S}(u) = \frac{\langle\mathscr{S}(u)\rangle}{\langle\mathscr{E}(u)\rangle}\,\mathscr{E}(u) + \partial_x\Phi(u) \tag{3.9}$$

for some loss-flux Φ. Hence by partial integration it follows

$$\frac{\langle x\mathscr{S}(u)\rangle}{I(u)} = \frac{\langle\mathscr{S}(u)\rangle Z(u)}{I(u)} - \frac{\langle\Phi(u)\rangle}{I(u)}, \tag{3.10}$$

and therefore

$$(S(v), \delta Z(v)) = -\frac{\langle\Phi(u)\rangle}{\langle\mathscr{E}(u)\rangle}, \tag{3.11}$$

the loss-flux term of part I.

4. The Korteweg–de Vries equation and its solitary waves

A typical nonlinear example, which we now use to illustrate the general equations of the previous section, is the unidirectional Korteweg–de Vries (KdV) equation

$$\dot{u} = -u_{xxx} - 6uu_x. \tag{4.1}$$

Various physical situations in which nonlinearity and dispersion are important (e.g. magnetosonic wave propagation perpendicular to an applied magnetic field [2], ion sound waves with $T_{\text{ion}} \cong 0$ [10] and shallow short water waves [6]) lead to this equation.

With the following Hamiltonian and its variational derivative

$$H(u) = \int_{-\infty}^{\infty} [\tfrac{1}{2}u_x^2 - u^3]\,\mathrm{d}x, \tag{4.2}$$

$$\delta H(u) = -u_{xx} - 3u^2, \tag{4.3}$$

it can be seen that this equation is a Poisson system given by

$$\dot{u} = \partial_x \delta H(u). \tag{4.4}$$

As treated in part I, this system obeys an energy balance

$$\partial_t \mathscr{E} + \partial_x \mathscr{F} = 0, \tag{4.5}$$

with the following flux and energy density

$$\mathscr{F}(u) = 2u^3 + u_{xx}u - \tfrac{1}{2}u_x^2, \tag{4.6}$$

$$\mathscr{E}(u) = \tfrac{1}{2}u^2. \tag{4.7}$$

Hence the flux density $\mathscr{F}$ is such that $\partial_x \mathscr{F} = u\, \partial_x \delta H(u)$.

With the transformation presented in Section 2, it is very easy to find special travelling wave solutions of the KdV equation. Every equilibrium v of the reduced dynamics, i.e. a function v, which satisfies

$$\delta H(v) + \lambda \delta I(v) = 0, \quad \text{with } \lambda = -\{Z, H\}(v), \tag{4.8}$$

gives a solution of the KdV equation, which is of the form $u(t) = \Phi_{\lambda t}(v)$. Therefore we call u a relative equilibrium. This is in line with the definition of a relative equilibrium in [7] and [1], where the notation of a relative equilibrium is used in the finite dimensional case.

To find these travelling wave solutions, we have to solve (4.8). This problem is equivalent to looking for functions v that are critical points of H, under the constraint that I is constant, say γ.

In case of the KdV-equation this means that we have to find functions, that are solutions of the following ordinary differential equation with λ a parameter

$$v_{xx} + 3v^2 = \lambda v. \tag{4.9}$$

A solution v of this equation, that is normalised such that $I(v) = \gamma$, can be written like

$$V_\gamma(x) = \tfrac{1}{2}\lambda \operatorname{sech}^2[\tfrac{1}{2}\sqrt{\lambda}\, x], \quad \text{with } \lambda = (3\gamma)^{2/3}. \tag{4.10}$$

According to eq. (2.15) for the non dissipative dynamical system, this gives a solution of the unperturbed KdV-equation, with $\varphi = -\lambda t$, hence expressed in the original coordinate u

$$U_\gamma(x, t) = \tfrac{1}{2}\lambda \operatorname{sech}^2[\tfrac{1}{2}\sqrt{\lambda}\, x + \tfrac{1}{2}\lambda^{3/2} t] = \Phi_{\lambda t}[V_\gamma](x). \tag{4.11}$$

This family of solutions is the well-known family of solitary-like wave-solutions of the KdV-equation, which are parametrised here by γ.

5. Decaying solitary waves

When a small dissipation (εS, with ε a small scaling parameter) is added to the system, in general, the solitary waves are no solutions anymore but the shape will remain in a decaying form. This decay of the solitary waves is analysed by writing

$$v(x, t) = V_{\gamma(t)}(x) + z(x, t). \tag{5.1}$$

The equations for γ, φ and z can be written down, using the eqs (3.4) of Section 3. The equation for z implies that, for initial data with $z(0) = 0$ (i.e. the initial data is a solitary wave), it follows that $z = O(\varepsilon)$, for small ε. For the dissipations considered below, numerical calculations have shown that z remains of order ε for all t. Here we simply take the lowest order analysis: assuming $z=0$, the equations for γ and φ, given by (3.4), are evaluated at a purely solitary wave:

$$\dot{\gamma}=-\varepsilon S_1(V_\gamma), \qquad \dot{\varphi}=-\lambda-\varepsilon S_2(V_\gamma). \tag{5.2}$$

These equations will be analysed and compared by the results of Ott and Sudan [9], in case S is given by
(i) $S(u)=u$, uniform damping,
(ii) $S(u)=-u_{xx}$, viscosity, giving the KdV-Burgers equation,
(iii) $S(u)=(-\partial_x^2)^\nu u$, $0<\nu<1$, fractional damping. The limit $\nu\to 0$ gives (i) and the limit $\nu\to 1$ gives (ii).
The operator $(-\partial_x^2)^\nu$ is a pseudo-differential operator, defined formally by Fourier transformation. If $\hat{u}$ denotes the Fourier transform of u:

$$\hat{u}(k)=\mathbb{F}(u)(k)=\int_{-\infty}^{\infty} e^{-2\pi i x k}u(x)\,dx, \tag{5.3}$$

then the Fourier transform of $(-\partial_x^2)^\nu u$ is simply $\hat{u}$ multiplied by the symbol of $(-\partial_x^2)^\nu$, which is $k^{2\nu}$:

$$\mathbb{F}((-\partial_x^2)^\nu u)(k)=k^{2\nu}\hat{u}(k). \tag{5.4}$$

For $\nu=n\in\mathbb{N}_0$, we see that this is indeed equivalent with the Fourier transform of $(-1)^n\partial_x^{2n}u$. Hence $\nu\to 0$ gives (i) and $\nu\to 1$ gives (ii) and we can work out together the approximation for the three kinds of dissipation.

To analyse eq. (5.2), first note that

$$S_2(V_\gamma)=0. \tag{5.5}$$

Indeed, in Appendix B it is proved that $(-\partial_x^2)^\nu V_\gamma$ is an even function, hence by symmetry it follows from the definition of $S_2(v)$ that $S_2(V_\gamma)=0$. Hence φ can be found from $\dot{\varphi}=-\lambda(\gamma)$, once the decay of γ is found from (5.2).

To investigate the equation for γ in (5.2), we recall the definition of the decay rate α from part I, eq. (3.14):

$$\alpha(t)=-\frac{\dot{I}(t)}{I(t)}; \tag{5.6}$$

here this means

$$\dot{\gamma}=-\alpha\gamma, \quad \text{i.e. } \alpha=\frac{\varepsilon S_1(V_\gamma)}{\gamma}. \tag{5.7}$$

This definition gives α its dynamical interpretation.

Another interpretation of α is as a power of the averaged wave number. By using (5.7) and the definition of $S_1(v)$ we can define an averaged wavenumber $\bar{k}$ by

$$(\bar{k})^{2\nu}=\frac{\int |\hat{u}(k)|^2 k^{2\nu}}{\int |\hat{u}(k)|^2}. \tag{5.8}$$

Then $\alpha=2\varepsilon\bar{k}^{2\nu}$. (Note that $\bar{k}$ is defined if $\nu>0$.)

With the explicit expression for V_γ, S_1 can be calculated. The result is derived in Appendix B and reads

$$S_1(V_\gamma)=C(\nu)\gamma^{1+\frac{2}{3}\nu}, \tag{5.9}$$

where $C(\nu)$ is a parameter depending on ν. Explicitly $C(0)=2$ and $C(1)=\frac{2}{5}3^{2/3}$ for the cases (i) and (ii).

Comparing with (5.7), it follows that the averaged wavenumber is related to the value of γ according to

$$\bar{k}=\left(\frac{C(\nu)}{2}\right)^{1/(2\nu)}\gamma^{1/3}. \tag{5.10}$$

This shows that the multiplier λ is proportional to $\bar{k}^2$, for $\nu>0$:

$$\lambda=3^{2/3}\left(\frac{C(\nu)}{2}\right)^{-1/\nu}\bar{k}^2. \tag{5.11}$$

Of course, with (5.2) and (5.7) the precise decay of γ follows. Two cases have to be distinguished.

If $\nu=0$ (uniform damping), then $\alpha=2\varepsilon$ and γ decays exponentially

$$\gamma(t)=\gamma_0\,\mathrm{e}^{-2\varepsilon t}, \tag{5.12}$$

while if $\nu>0$ then $\alpha=\varepsilon C(\nu)\gamma^{2\nu/3}$ and the decay of γ is algebraic

$$\gamma(t)=\left(\gamma_0^{-2\nu/3}+\frac{2\nu}{3}C(\nu)\varepsilon t\right)^{-3/2\nu}. \tag{5.13}$$

The corresponding exponential respectively algebraic decay of the averaged wavenumber $\bar{k}$ gives a measure of the different dynamic spatial "spreading" of the waveform.

The value of the centrovelocity $\dot{\varphi}=-\lambda(\gamma)$ is proportional to the square of the averaged wavenumber

$$\dot{\varphi}=-3^{2/3}\left(\frac{C(\nu)}{2}\right)^{-1/\nu}\bar{k}^2. \tag{5.14}$$

With (5.10) and (5.13) the position of the centre of gravity of the density $\int u^2$ for the decaying solitary waves follows as an explicit function of time, for $\nu>0$:

$$\dot{\varphi}=3^{\frac{2}{3}}\left(\gamma_0^{-2\nu/3}+\frac{2\nu}{3}C(\nu)\varepsilon t\right)^{-1/\nu}. \tag{5.15}$$

For $\nu=0$ it holds that

$$\dot{\varphi}=-\lambda(\gamma(t))=(3\gamma_0)^{2/3}\,\mathrm{e}^{-\frac{4}{3}\varepsilon t}. \tag{5.16}$$

Remarks

- In [4] the same was done for a cnoidal wave on a periodic interval. Then (for $\nu=0$ and $\nu=1$) α is bounded from below (if $\nu=0$, it holds $\alpha=2\varepsilon$; if $\nu=1$, then α is larger or equal to the smallest eigenvalue of minus epsilon times the Laplacian: $-\varepsilon\Delta$). This implies exponential decay in both cases. The fact that α decays to its lowest value for $\nu=1$ implies a self organisation to the largest wavelength. In this paper, the spreading of the solitary wave on $\mathbb{R}$ slows down the decrease of γ ($\alpha\to 0$ if $\nu>0$ and $\gamma\to 0$) and affects consequently $\dot{\varphi}$ (and $\bar{k}$).
- In [9] Ott and Sudan consider also the decay of the amplitude of solitary waves with the dissipations (i) and (ii) using a two time scale mechanism. The time-behaviour derived with our method is similar to the behaviour they have derived in their article.

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Appendix A. Coordinate transformation to a codimension 2 space

In this appendix we will provide a coordinate transformation from u to the coordinates (γ, φ, w), by using the so-called Z-flow.

We have seen already that

$$\frac{\partial}{\partial\varphi}\Phi_\varphi(u_0) = \partial_x \delta I(\Phi_\varphi(u_0)), \qquad \Phi_0(u_0) = u_0. \tag{A.1}$$

To stress the dependence on the functional I, we denote Φ_φ also by Φ^I_φ and call Φ^I_φ the I-flow. In the same way we can define the Z-flow Φ^Z_γ, by $\Phi^Z_\gamma : \mathcal{M} \to \mathcal{M}$ which satisfies

$$\frac{\partial}{\partial\gamma}\Phi^Z_\gamma(u_0) = \partial_x \delta Z(\Phi^Z_\gamma(u_0)), \qquad \Phi^Z_0(u_0) = u_0. \tag{A.2}$$

Now we are able to define a transformation to a coordinate frame, moving with the Φ^I_φ- and Φ^Z_γ-flow by

$$u(t) = \Phi^I_{-\varphi(t)}(\Phi^Z_{\gamma(t)}(w(t))). \tag{A.3}$$

The Φ^I_φ- and Φ^Z_γ-flow commute, because $\{Z, I\} = -1$. See e.g. [8]. Hence the transformation (A.3) is the same as $\Phi^Z_{\gamma(t)}(\Phi^I_{-\varphi(t)}(w(t)))$.

From $\{Z, I\} = -1$ it also follows that

$$Z(\Phi^I_\varphi(u_0)) = Z(u_0) - \varphi, \qquad Z(\Phi^Z_\gamma(u_0)) = Z(u_0), \qquad I(\Phi^Z_\gamma(u_0)) = I(u_0) + \gamma, \qquad I(\Phi^I_\varphi(u_0)) = I(u_0). \tag{A.4}$$

Hence the transformation (A.3) implies that w is in the codimension 2 space $I^{-1}(\gamma_0) \cap Z^{-1}(0)$ and the change $\dot{w}$ is tangent to this space. Furthermore, γ is an independent variable, hence we have a genuine coordinate transformation from u to (γ, φ, w).

Lemma 4. *Define the following coordinate transformation from u to (γ, φ, w) by*

$$\gamma = I(u), \qquad \varphi = Z(u), \qquad w = \Phi^Z_{-\gamma}(\Phi^I_\varphi(u)). \tag{A.5}$$

The dynamical equations of the system (2.1) *in these new coordinates become*

$$\begin{aligned}
\dot{\gamma} &= \langle \delta I(u), \dot{u} \rangle = -\varepsilon S_1(\Phi^Z_\gamma(w)), \\
\dot{\varphi} &= \langle \delta Z(u), \dot{u} \rangle = \{Z, H\}(\Phi^Z_\gamma(w)) - \varepsilon S_2(\Phi^Z_\gamma(w)), \\
\dot{w} &= (D\Phi^Z_\gamma(w))^{-1}[\partial_x[\delta H(\Phi^Z_\gamma(w)) + \{Z, H\}(\Phi^Z_\gamma(w))\delta I(\Phi^Z_\gamma(w))] - \varepsilon S_3(\Phi^Z_\gamma(w))],
\end{aligned} \tag{A.6}$$

with

$$S_1(v) = \langle S(v), \delta I(v)\rangle, \qquad S_2(v) = \langle S(v), \delta Z(v)\rangle,$$
$$S_3(v) = S(v) - S_1(v)\,\partial_x\delta Z(v) + S_2(v)\,\partial_x\delta I(v). \tag{A.7}$$

Note that this last definition is different from the one in Section 3.

The equations for $\dot\gamma$ and $\dot\varphi$ give the change of I and Z. Hence $S_1(\Phi^Z_\gamma(w))$ and $S_2(\Phi^Z_\gamma(w))$ are the components of S in the $\partial_x\delta Z(w)$- and $\partial_x\delta I(w)$-direction. $(D\Phi^Z_\gamma(w))^{-1}S_3(\Phi^Z_\gamma(w))$ is the component of S tangent to the plane $I^{-1}(\gamma_0)\cap Z^{-1}(0)$, which gives the change of w caused by the perturbation.

With this transformation we get a very nice description of the dynamics with three independent variables. Often it is difficult to find Φ^Z_γ explicit, which makes the method not so useful for practical applications. However, in this case Φ^Z_γ can be found explicit. Namely

$$\Phi^Z_\gamma(u_0) = \frac{\gamma+\gamma_0}{\gamma_0}\,u_0\left(\frac{\gamma+\gamma_0}{\gamma_0}(x-\varphi_0)+\varphi_0\right), \quad \text{with } \gamma_0 = I(u_0), \quad \varphi_0 = Z(u_0). \tag{A.8}$$

This expression can be verified by substituting it in the definition of the Z-flow.

Appendix B. Calculations

In this appendix we perform the calculations needed to show that $S_2(V_\gamma)=0$ and to derive the expression (5.9) for $S_1(V_\gamma)$. By definition it holds

$$((-\partial_x^2)^\nu V_\gamma)(x) = (\mathbb{F}^{-1}(\mathbb{F}((-\partial_x^2)^\nu V_\gamma)))(x) = \int_{-\infty}^{\infty} dk\, e^{2\pi i x k}((2\pi k)^2)^\nu \int_{-\infty}^{\infty} d\xi\, e^{2\pi i \xi k} V_\gamma(\xi). \tag{B.1}$$

First we calculate $(-\partial_x^2)^\nu V_\gamma$. Using the fact that $V_\gamma = \frac{1}{2}\lambda\,\mathrm{sech}^2[\frac{1}{2}\sqrt{\lambda}\,x]$ is even, $\sin(2\pi\xi k)$ is odd and using the relation (3.982) from [3]:

$$\int_0^\infty \cos(ax)\,\mathrm{sech}^2(\beta x)\,dx = \frac{a\pi}{2\beta^2}\,\mathrm{csch}\left(\frac{a\pi}{2\beta}\right), \quad \mathrm{Re}(\beta)>0,\ a>0, \tag{B.2}$$

we see

$$((-\partial_x^2)^\nu V_\gamma)(x) = 2(\lambda)^{\nu+1}\pi^{-2-2\nu}\int_0^\infty \xi^{2\nu+1}\cos\left(\frac{\sqrt{\lambda}}{\pi}x\xi\right)\mathrm{csch}(\xi)\,d\xi. \tag{B.3}$$

From this we see that $(-\partial_x^2)^\nu V_\gamma$ is even. By definition of Z it holds that $\delta Z(V_\gamma) = \frac{1}{\gamma}xV_\gamma$, an odd function, hence the inner product $\langle(-\partial_x^2)^\nu V_\gamma, \delta Z(V_\gamma)\rangle = 0$. And therefore $S_2(V_\gamma)=0$. The calculation of the other inner product takes some more effort. With (B.3) and $\delta I(V_\gamma) = V_\gamma$ we see that

$$S_1(V_\gamma) = \langle(-\partial_x^2)^\nu V_\gamma, \delta I(V_\gamma)\rangle$$
$$= 2(\lambda)^{\nu+\frac{3}{2}}\pi^{-2-2\nu}\int_0^\infty d\xi\,\xi^{2\nu+1}\,\mathrm{csch}(\xi)\int_0^\infty dx\,\cos\left(\frac{2}{\pi}x\xi\right)\mathrm{sech}^2(x). \tag{B.4}$$

Now we use twice [3], again relation (3.982) and relation (3.527):

$$\int_0^\infty x^{\mu-1}\operatorname{csch}^2(ax)\,\mathrm{d}x=\frac{4}{(2a)^\mu}\Gamma(\mu)\zeta(\mu-1),\quad \mathrm{Re}(a)>0,\ \mathrm{Re}(\mu)>2, \tag{B.5}$$

with $\zeta(z)$ the Riemann zeta function.

This gives in (B.4) for $S_1(V_\gamma)$

$$\begin{aligned} S_1(V_\gamma)&=\langle(-\partial_x^2)^\nu V_\gamma,\delta I(V_\gamma)\rangle=2^{1-2\nu}\pi^{-2-2\nu}\Gamma(2\nu+3)\zeta(2\nu+2)(\lambda)^{\nu+\frac{3}{2}}\\ &=2^{1-2\nu}\pi^{-2-2\nu}3^{1+\frac{2}{3}\nu}\Gamma(2\nu+3)\zeta(2\nu+2)\gamma^{1+\frac{2}{3}\nu}\\ &=C(\nu)\gamma^{1+\frac{2}{3}\nu}. \end{aligned} \tag{B.6}$$

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