Decay of coherent structures in damped Hamiltonian systems

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Abstract

In numerical and physical experiments it is often observed that finite and infinite dimensional systems exhibit a low dimensional behaviour, in the sense that the dynamics looks as if it can be described with a few parameters. Often a spatially coherent structure is characteristic for the phenomenon. A well known example is a solitary wave, characterized by its phase and its amplitude. In this paper we consider a Hamiltonian system (or a Poisson system), that has an additional constant of motion (besides the Hamiltonian). We show coherent structures in such a system, by describing some solutions with 2 parameters, induced by the constant of motion. Further we demonstrate that the coherent structures survive even in cases where a small perturbation, such as dissipation, is present. This is demonstrated in some detail for a spherical pendulum with uniform friction, for the Korteweg-de Vries equation with uniform damping and for the Korteweg-de Vries-Burgers equation.

1 Introduction

We consider a Hamiltonian system, Hamiltonian H, with an additional constant of motion C. First we use this additional constant of motion to obtain low dimensional behaviour. With Noether's theorem it follows that there is a symmetry group, corresponding to the constant of motion C. In our example of the spherical pendulum, the constant of motion is the angular momentum and the group action is an increase in the angle around the vertical axis. Families of certain solutions of the Hamiltonian system, can be described with two parameters. They are the value of the quantity C, denoted by γ and the value of the parameter of the group action, denoted by φ.

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Essential in finding these families is the concept of *relative equilibria*. These equilibria are defined by extending the idea of ordinary equilibria, which are critical points of H. Using the fact that C is conserved for solutions of the system, the relative equilibria are critical points of H on level sets of C. By varying γ and φ we obtain a two dimensional manifold of relative equilibria, the MRE. It is important to observe that the MRE is invariant for the flow of the system; a trajectory starting on the MRE at $\gamma = \gamma_0$ and $\varphi = \varphi_0$ follows an orbit with γ remaining constant, whereas φ increases linearly in time: $\varphi(t) = \varphi_0 + \lambda(\gamma_0)t$.

Next a perturbation, e.g. dissipation, is added to the system. Now the MRE is only invariant for the dynamics if the perturbation has only components tangent to it. The dissipative dynamics on the MRE can be described by making the two parameters time dependent, hence $\gamma(t)$ and $\varphi(t)$. In general the perturbation has also components perpendicular to the MRE. When this component is too large, the solutions can't be expected to stay in a neighbourhood of the MRE. The same is true if the MRE is unstable (meaning that solutions of the unperturbed system, starting just outside the MRE go far away from the MRE). However, if the perpendicular component of the perturbation is small and the MRE is stable, a suitable trajectory of relative equilibria may approximate the solutions quite well. This is true for small times, but may also hold asymptotically, as the examples will show. In those cases the parameters γ and φ are sufficient to describe the system (for some time).

This general consideration is described in more detail in section 2 and is made more specific with two examples in the sections 3 and 4. The first example is a finite dimensional system: a spherical pendulum with uniform friction. Analytically it can be shown that solutions that started in the neighbourhood of the MRE remain there. The second example is infinite dimensional. It is the Korteweg-de Vries (KdV) equation, perturbed with a uniformly damping, or with a viscous dissipation, the KdV-Burgers equation. Here a similar result is obtained numerically.

2 Relative equilibria

Consider a $2N$-dimensional Hamiltonian system

$$\dot{\boldsymbol{u}} = J\nabla H(\boldsymbol{u}), \tag{1}$$

with $\boldsymbol{u} = (q_1, \ldots, q_N\,;\, p_1, \ldots, p_N)$, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I the identity matrix and H the Hamiltonian. It is well-known that the Hamiltonian is a constant of motion. A method to find solutions of system (1) is to look for equilibria of the system. These are points $\boldsymbol{u}_0$ such that $\nabla H(\boldsymbol{u}_0) = 0$, hence critical points of the Hamiltonian, e.g. minima or maxima. It follows immediately that they are solutions of (1).

In this paper it is assumed that the system possesses also another constant of motion $C(\boldsymbol{u})$. The corresponding action of the symmetry group is denoted by $\Phi_\varphi^C(\boldsymbol{u}_0)$. This is the solution of

$$\text{(2)} \qquad \begin{cases} \frac{\partial}{\partial\varphi}\boldsymbol{u} = J\nabla C(\boldsymbol{u}), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases}$$

Because C is conserved, a solution $\boldsymbol{u}(t)$ of (1) satisfies $C(\boldsymbol{u}(t)) = C(\boldsymbol{u}(0))$, $t \geq 0$. With this knowledge more solutions can be characterized. Define a relative equilibrium to be a critical point of H on a level set of C, hence a solution of

$$\text{(3)} \qquad \nabla H(\boldsymbol{U}) = \lambda\nabla C(\boldsymbol{U}),$$

for some constant λ. The Hamiltonian is invariant under the group action, meaning that for all φ and $\boldsymbol{u}$ it holds $H(\boldsymbol{u}) = H(\Phi_\varphi^C(\boldsymbol{u}))$. By varying $\boldsymbol{u}$, this yields that with every relative equilibrium $\boldsymbol{U}$ also the translated point $\Phi_\varphi^C(\boldsymbol{U})$ is a relative equilibrium with the same value of C and for every φ.
From now on, it is assumed that there is a differentiable branch of relative equilibria $\boldsymbol{U}(\gamma)$, with $C(\boldsymbol{U}(\gamma)) = \gamma$. The manifold evolving from this branch by the action of the group Φ_φ^C, is the Manifold of Relative Equilibria (MRE). Every point of the MRE can be given uniquely by $\boldsymbol{U}(\gamma,\varphi) := \Phi_\varphi^C(\boldsymbol{U}(\gamma))$. In this way the manifold is parametrized with two parameters: γ, the value of C and φ, the value of the group action.

It is easy to see that solutions of (1) that start on the MRE, will stay on the MRE, i.e. the MRE is invariant for the unperturbed Hamiltonian system. Indeed, if φ is time dependent (γ has to be fixed , because $C = \gamma$ is a constant of motion), it yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U}(\gamma,\varphi(t)) = \dot{\varphi}\tfrac{\partial}{\partial\varphi}\boldsymbol{U}(\gamma,\varphi).$$

With (2) this gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U}(\gamma,\varphi(t)) = \dot{\varphi}J\nabla C(\boldsymbol{U}(\gamma,\varphi(t))) = \tfrac{\dot{\varphi}}{\lambda}J\nabla H(\boldsymbol{U}(\gamma,\varphi(t))).$$

Hence for $\dot{\varphi} = \lambda$ the function $\boldsymbol{U}(\gamma,\varphi(t))$ is a solution of (1). Solving the equation for φ this gives that $\boldsymbol{U}(\gamma,\lambda t+\varphi_0)$ is the solution of (1), with initial condition $\boldsymbol{U}(\gamma,\varphi_0)$. In order to investigate the relevance of the MRE for the Hamiltonian system with a perturbation added, two vectors that span the tangent space of the MRE at $\boldsymbol{U}(\gamma,\varphi)$ are defined. This definition is based on the two parameters γ and φ. The tangent vectors are $\boldsymbol{e}_1(\gamma,\varphi) = \frac{\partial}{\partial\gamma}\boldsymbol{U}(\gamma,\varphi)$ and $\boldsymbol{e}_2(\gamma,\varphi) = \frac{\partial}{\partial\varphi}\boldsymbol{U}(\gamma,\varphi)$. See also figure 1. (We used implicitly already $\boldsymbol{e}_2$ in proving the invariancy of the MRE.

Next, consider the system to which some perturbation S is added (ε is a parameter)

$$\text{(4)} \qquad \dot{\boldsymbol{u}} = J\nabla H(\boldsymbol{u}) + \varepsilon S(\boldsymbol{u}).$$

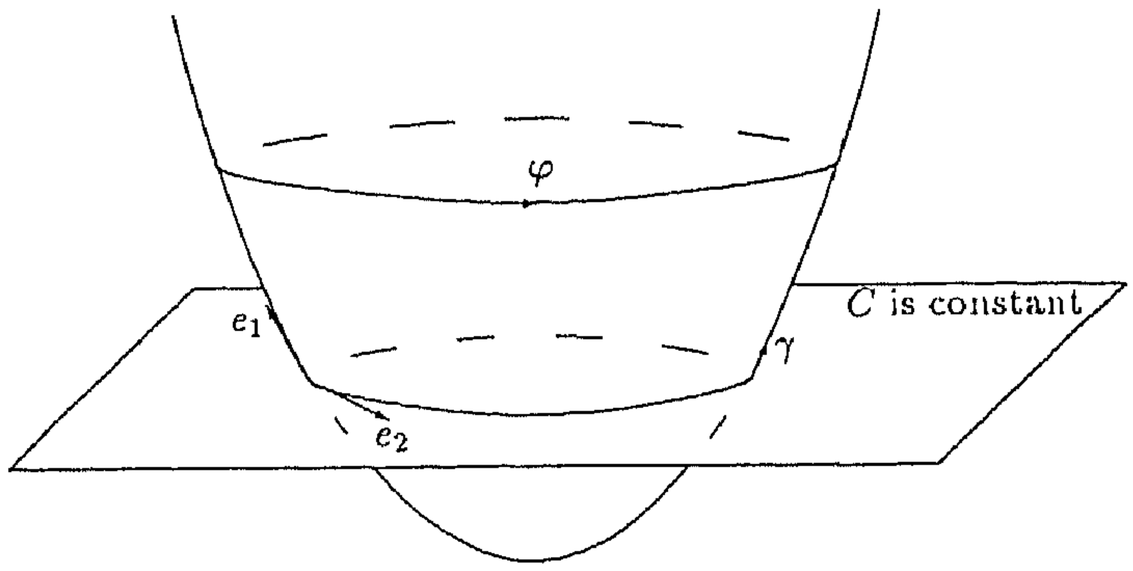

Figure 1: The tangent vectors $e_1(\gamma,\varphi)$ and $e_2(\gamma,\varphi)$ in $\boldsymbol{U}(\gamma,\varphi)$ at the MRE

Intuitivly, if the perturbation on the MRE, $S(\boldsymbol{U}(\gamma,\varphi))$, has only components tangent to the MRE, a solution of (4) that starts on the MRE will stay on it. Then the MRE can be used to describe solutions of (4). But in general $S(\boldsymbol{U}(\gamma,\varphi))$ has also a component transversal to the MRE. To avoid technical details in describing this situation, it is assumed that in the unperturbed situation (1) the constant of motion $C(\boldsymbol{u})$ is one of the momentum coordinates. Then necessarily the corresponding canonical space variable is cyclic, i.e. doesn't enter the Hamiltonian H. The group action Φ_φ^C corresponds to a translation of this cyclic coordinate with φ. In the example of the spherical pendulum, to elaborate in section 3, the constant of motion C is the angular momentum about the vertical axis and φ is the angle describing a rotation around this axis.

Denoting the other coordinates by $\boldsymbol{y} \in \mathbb{R}^{2N-2}$, the Hamiltonian is $H(\gamma,\boldsymbol{y})$. The equation (3) for the MRE reduces to $\frac{\partial}{\partial\gamma}H(\gamma,\boldsymbol{y}) = \lambda\frac{\partial}{\partial\gamma}C(\gamma,\boldsymbol{y}) = \lambda$ and $\nabla_y H(\gamma,\boldsymbol{y}) = \lambda\nabla_y C(\gamma,\boldsymbol{y}) = 0$. On the MRE it holds $\boldsymbol{y} = \boldsymbol{Y}(\gamma)$, where $\boldsymbol{Y}(\gamma)$ are the $\boldsymbol{y}$-coordinates of $\boldsymbol{U}(\gamma,\varphi)$. The dynamics of the perturbed system is given by

$$
(5)\qquad \begin{array}{rcll} \dot\gamma &=& 0 & + \ \varepsilon S_1(\gamma,\varphi,\boldsymbol{y}),\\ \dot\varphi &=& \frac{\partial}{\partial\gamma}H(\gamma,\boldsymbol{y}) & + \ \varepsilon S_2(\gamma,\varphi,\boldsymbol{y}),\\ \dot{\boldsymbol{y}} &=& J\nabla_y H(\gamma,\boldsymbol{y}) & + \ \varepsilon S_Y(\gamma,\varphi,\boldsymbol{y}). \end{array}
$$

Here the perturbation is written $S(\gamma,\varphi,\boldsymbol{y}) = (S_1(\gamma,\varphi,\boldsymbol{y}), S_2(\gamma,\varphi,\boldsymbol{y}), S_Y(\gamma,\varphi,\boldsymbol{y}))$, with S_1 the γ-component of S, hence in the direction $\boldsymbol{e_1}$ if $\boldsymbol{y} = \boldsymbol{Y}(\gamma)$. Similar, S_2 is the φ-component and S_Y the $\boldsymbol{y}$-component of S.

For $\boldsymbol{y} = \boldsymbol{Y}(\gamma)$ (on the MRE) it holds that $\nabla_y H(\gamma,\boldsymbol{Y}(\gamma)) = 0$. This confirms the intuitive idea that a solution of a system with a perturbation for which $S_Y(\gamma,\boldsymbol{Y}(\gamma)) = 0$, remains on the MRE, when it started on the MRE (because $\dot{\boldsymbol{y}} = 0$). If $S_Y(\gamma,\boldsymbol{Y}(\gamma)) \neq 0$, equations (5) show that a solution, that started on the MRE, will leave it and the distance to the MRE will grow in first order with εt. Hence in

the beginning the distance remains small and $\boldsymbol{u} = \boldsymbol{U}(\gamma(t), \varphi(t))$ can be used as an approximation on time scale $\mathcal{O}(1)$, with $\gamma(t)$ and $\varphi(t)$ given by

$$(6) \qquad \begin{array}{rcl} \dot{\gamma} & = & \varepsilon S_1(\gamma, \varphi, \boldsymbol{Y}(\gamma)), \\ \dot{\varphi} & = \lambda(\gamma) + & \varepsilon S_2(\gamma, \varphi, \boldsymbol{Y}(\gamma)). \end{array}$$

In cases that $C(\boldsymbol{u})$ is not one of the momentum coordinates, it is sometimes possible to find a canonical transformation to coordinates such that $C(\boldsymbol{u})$ is one of the momenta. Even if such a transformation can not be found explicitly, new coordinates γ, φ and $\boldsymbol{y}$, for which the equations have the same form as equation (5), with J replaced by some other anti-symmetric operator, still can be defined. This needs some more background in Poisson systems and is not discussed here. Details can be found in [Gro90] and also in [DGV90] and [GBV90], where the following examples are treated.

3 The spherical pendulum

A first example of an application of the theory of section 2 is the damped spherical pendulum. With spherical coordinates φ, θ, p_φ and p_θ defined as in figure 2, Hamiltonian H is given by

$$(7) \qquad H(\boldsymbol{u}) = \tfrac{1}{2}(p_\theta^2 + p_\varphi^2 \frac{1}{\sin^2\theta}) + \omega_0^2(1 - \cos\theta), \text{ with } \omega_0^2 = \frac{g}{l}.$$

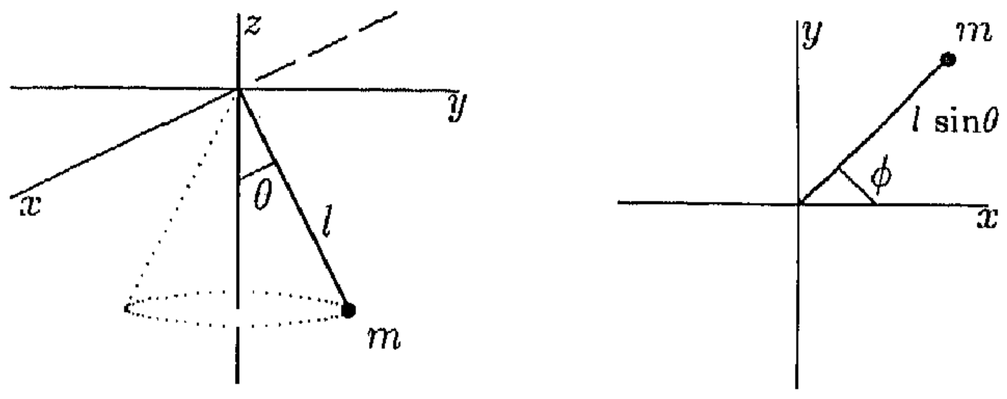

Figure 2: The spherical coordinates φ and θ of the pendulum.

As said before, the constant of motion is the angular momentum, hence $C(\boldsymbol{u}) = p_\varphi$ and φ is the cyclic variable. The group action is an increase in the angle φ. The MRE is found by solving the minimisation problem

$$h(\gamma) = \min_{\boldsymbol{u}} \{\, H(\boldsymbol{u}) \,|\, p_\varphi = \gamma \,\}.$$

Points in which this minimum is obtained are

$$\boldsymbol{U}(\gamma, \varphi) = (\varphi, \hat{\theta}(\gamma)\,;\, \gamma, 0), \quad \text{with} \quad \hat{\theta}(\gamma) = \arccos(\frac{\omega_0^2}{\lambda(\gamma)^2}), \;\; \gamma = \frac{\lambda^4 - \omega_0^4}{\lambda^3}.$$

The solution of this Hamiltonian system on the MRE is the pendulum circling in a horizontal plane, with angular velocity $\lambda(\gamma)$.

Next the system is considered with uniform friction, $S(\boldsymbol{u}) = -(0,0\,;\,p_\varphi, p_\theta)$, added. In [DGV90] this case is treated extensively. The conclusion of this article is stronger than the general result in section 2. It is shown that the approximation by the MRE is valid for all time. This is possible because S doesn't depend on φ and the equation for γ is easy, $\dot{\gamma} = -\varepsilon\gamma$, hence

$$\gamma(t) = \gamma_0 e^{-\varepsilon t},\ t \geq 0.$$

By substituting this in the equation for $\boldsymbol{y} = (\theta, p_\theta)$, which doesn't depend on φ, we get a time dependent equation for $\boldsymbol{y}$. When the pendulum starts in an ε-neighbourhood of the MRE, it can be shown that for ε not too large

$$\boldsymbol{y} = \boldsymbol{Y}(\gamma(t)) + \mathcal{O}(\varepsilon e^{-\frac{1}{2}\varepsilon t}) = (\hat{\theta}(\gamma(t)), 0) + \mathcal{O}(\varepsilon e^{-\frac{1}{2}\varepsilon t}),\ t \geq 0.$$

Since $\hat{\theta}(\gamma(t)) = \mathcal{O}(e^{-\frac{1}{2}\varepsilon t})$, this shows that the distance to the MRE remains relatively small. Solutions that started in the neighbourhood of the MRE can be approximated by $\boldsymbol{U}(\gamma(t), \varphi(t))$ for all time and at the end (when the points on the MRE are small) the relative error is also small. This is expressed in the following proposition.

Proposition 1 *If* $\| \boldsymbol{y}(0) - \boldsymbol{Y}(\gamma(0)) \| = \mathcal{O}(\varepsilon)$ *then*

$$\boldsymbol{u}(t) = \boldsymbol{U}(\gamma(t), \varphi(t))\,[\,1 + \mathcal{O}(\varepsilon)\,],\ t \geq 0,$$

with the evolution of γ *and* φ *given by equation (5). In lowest order these equations are given by (6), in this case*

$$\begin{aligned} \dot{\gamma}(t) &= -\varepsilon\gamma, \\ \dot{\varphi}(t) &= \lambda(\gamma(t)) + \mathcal{O}(\varepsilon). \end{aligned} \tag{8}$$

The results are illustrated by the figures 3 and 4.
Numerically the trajectory of a damped pendulum is calculated with various values of ε with ratio 2, being $\varepsilon = 0.64,\ 0.32,\ 0.16,\ 0.08,\ 0.04$. The pendulum starts on the MRE, $\varphi = 0$, $\theta = \hat{\theta}(\gamma_0) = \frac{\pi}{2}$, $p_\varphi = \gamma_0 = 0.59$ and $p_\theta = 0$. Further we choose $\omega_0 = 1$.

Figure 3a shows the trajectory of the damped pendulum with $\varepsilon = 0.08$, projected on the x, y-plane. The circles of the MRE can be recognized in this trajectory.
Figure 3b is the so called integral-diagram. From the definition (7) of H, it follows that $H - h(\gamma)$ is a measure for $\|(\theta, p_\theta)\|^2 = \|\boldsymbol{y}\|^2$. Further is $C(\boldsymbol{u}) = \gamma = \gamma_0 e^{-\varepsilon t}$ and therefore a measure for time. So an H-C curve can be interpreted as the decay of $\|\boldsymbol{y}\|$ during time.

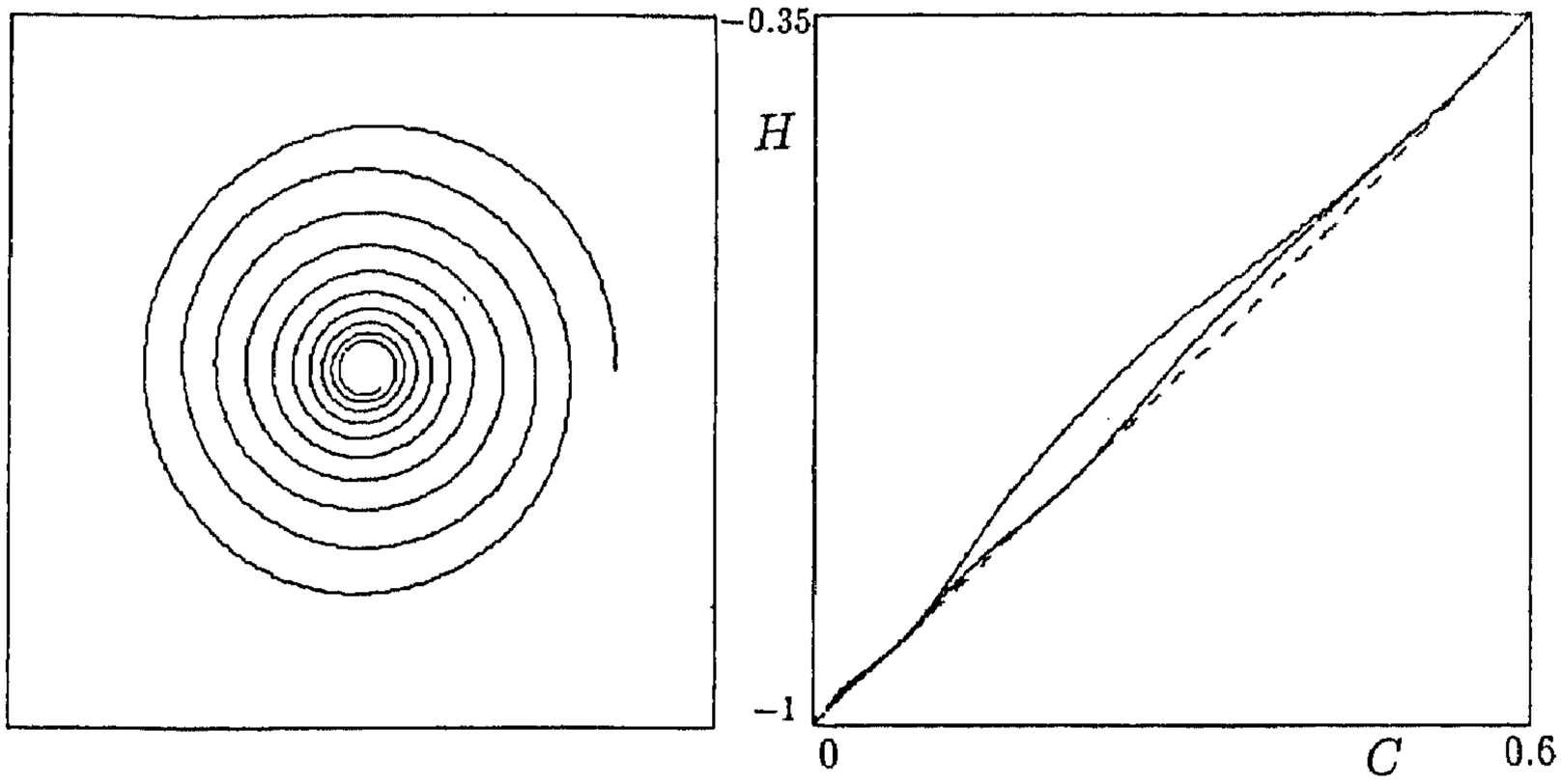

Figure 3a: Projection of a trajectory of a slightly damped pendulum ($\varepsilon = 0.08$) on the horizontal plane. The initial condition is on the MRE, i.e. a purely circular motion of the undamped system. Observe that the orbit remains (nearly) circular.

Figure 3b: Projection in the H-C plane of a trajectory of a damped pendulum, starting on the MRE, with various values of ε: $\varepsilon = 0.64,\ 0.32$. The lowest curve is the projection of the MRE. The curves of the smaller values of ε: $\varepsilon = 0.16,\ 0.08,\ 0.04$ coincides graphically with this curve.

The value-function $h(\gamma)$ of the minimisation problem is differentiable and shown by the dashed curve. This curve can be interpreted as a projection of the MRE on the H-C space. The trajectory of the unperturbed system on the MRE corresponds with a fixed point on this curve, because then H and C are integrals. This doesn't hold anymore in the damped situation and the values of H and C will change in the course of time.

The other curves in figure 3b are the values of H and C of the solutions of the damped system for the different values of ϵ. These curves must lie above the MRE curve, which gives the minimum of H. The dynamics projected on the MRE (hence $\boldsymbol{U}(\gamma(t), \varphi(t))$) is along the lowest curve. The figure shows that the projection of the solution and the projection of the MRE remain near each other, which illustrates that the approximation on the MRE is good, supporting the contents of proposition 1.

Figure 4 is another illustration of the approximation on the MRE.
In this figure $\gamma = C(\boldsymbol{u})$ is plotted along the horizontal axis, where γ decays with $\gamma_0 e^{-\varepsilon t}$, as calculated before. Along the vertical axis is plotted the relative distance to the MRE: $R = \sqrt{\frac{\gamma(t)-\Gamma(t)}{\gamma(t)}}$, where $\Gamma(t) = h^{-1}(H(\boldsymbol{u}(t)))$. This distance R is a measure

for the error of the approximation of the solution by the MRE. The curves are R against γ for the various values of ε. It shows for fixed ε that the maximum error remains almost constant. Further, for different values of ε the ratio between the maximum errors is the same as the ratio between the ε's, hence it can be recognized that the error is of order ε.

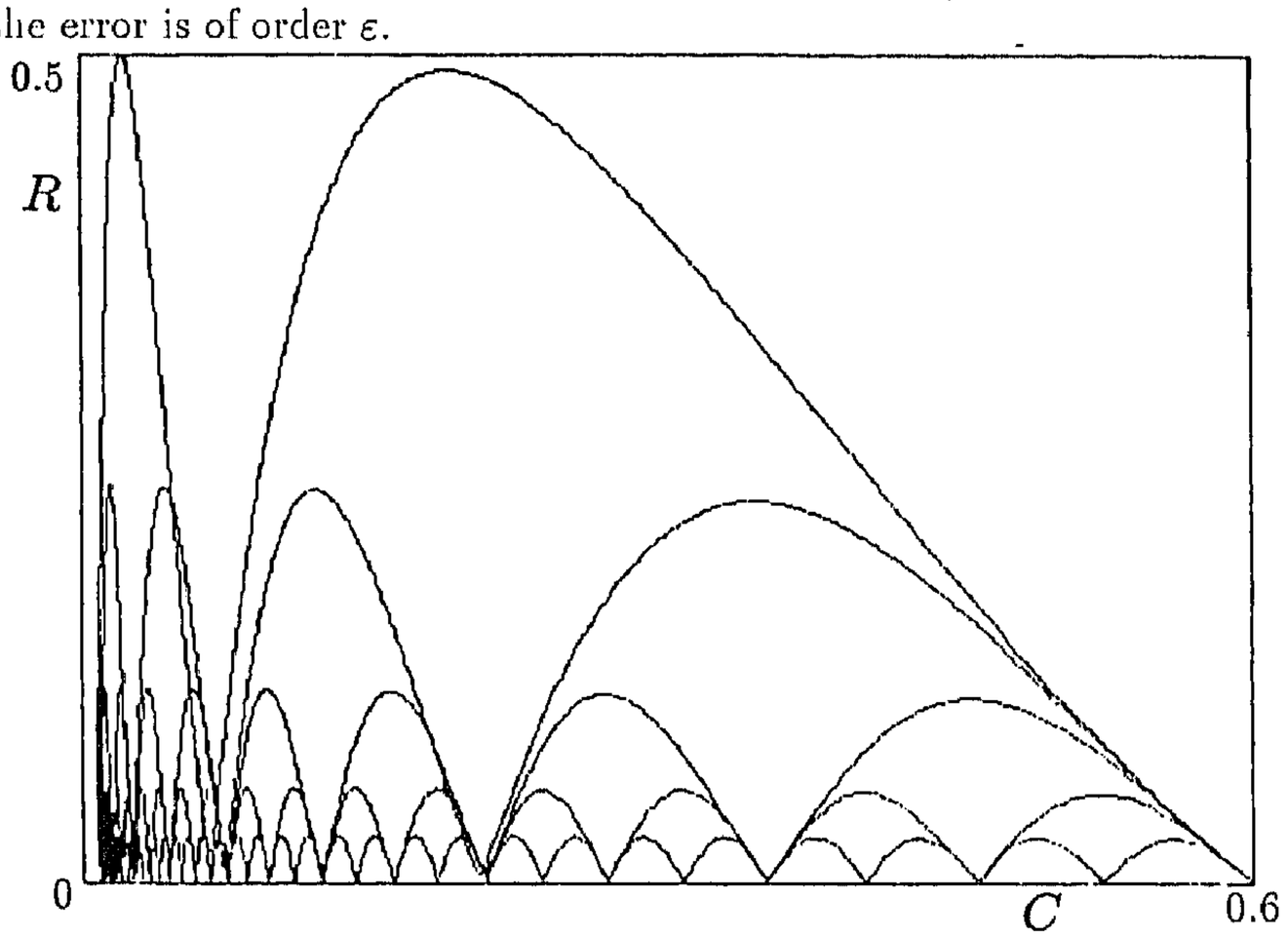

Figure 4: The relative distance to the MRE, R, for the damped pendulum, for various values of ε: $\varepsilon = 0.64,\ 0.32,\ 0.16,\ 0.08,\ 0.04$. The ratio between the distances of the tops of two successive curves is 2, just as the ratio of the ε's. This illustrates that the error is of order ε.

4 The Korteweg-de Vries equation

In this section we consider an infinite dimensional system; the Hamiltonian system of section 2 is generalised to a Poisson system. It turns out that all ideas of section 2 can be extended and equations like (5) and (6) can still be derived.
As a specific example we present the Korteweg-de Vries (KdV) equation with periodic boundary conditions and indicate the differences and similarities with the Hamiltonian structure of section 2. The anti-symmetric matrix J is replaced by the anti-symmetric operator ∂_x and the gradient ∇ is replaced by the variational derivative δ on the linear space

$$\mathsf{M} = \{\, u(x) \,|\, u \text{ is } 2\pi \text{ periodic}, \int u(x)dx = 0 \,\}.$$

The Hamiltonian of the unperturbed KdV-equation is

$$H(u) = \int [\tfrac{1}{2}u_x^2 - u^3]dx,$$

and the variational derivative is

$$\delta H(u) = -u_{xx} - 3u^2.$$

Then the KdV equation can be written like

$$\tfrac{\partial}{\partial t}u = \tfrac{\partial}{\partial x}\delta H(u)$$

and reads more explicit

$$\tfrac{\partial}{\partial t}u + u_{xxx} + 6uu_x = 0.$$

Besides the Hamiltonian that is conserved, the system has another constant of motion $C(u) = \frac{1}{2}\int u^2(x)dx$. Its group action is a translation in x, hence $(\Phi_\varphi^C(u))(x) = u(x+\varphi)$. Relative equilibria of the system can be found by solving the minimisation problem

$$(9)\ h(\gamma) = \min_u \left\{ \int [\tfrac{1}{2}u_x^2 - u^3]dx \,\middle|\, \tfrac{1}{2}\int u^2(x)dx = \gamma,\ \int u(x)dx = 0,\ u \text{ is } 2\pi \text{ periodic} \right\}.$$

The solutions of this problem are the well known cnoïdal-wave profiles with fundamental wavelength 2π. For small values of γ, the shape of this wave is like a sine function. For very large values of γ, the shape is soliton-like, with periodic peaks. The value-function $h(\gamma)$ is differentiable and its graph is given in a "integral-diagram" in figure 5. As mentioned in section 3 this graph can be interpreted as a projection of the MRE on the two-dimensional space, spanned by H and C. $H(u)$ is a norm for u and using this, the integral-diagram will play a key role in the interpretation of the results for the perturbed system.

Next dissipation is added to the system. We consider two kinds of dissipation; in both cases all solutions tend to the trivial state $u \equiv 0$ and the relevance of the MRE for the (asymptotic) behaviour is investigated.
The first choice is uniform damping $S_1(u) = -u$. The dynamical equation becomes

$$(10)\qquad \tfrac{\partial}{\partial t}u + u_{xxx} + 6uu_x = -\varepsilon u.$$

The second dissipation is viscosity and gives the KdV-Burgers equation; $S_2(u) = u_{xx}$ and the dynamics is

$$(11)\qquad \tfrac{\partial}{\partial t}u + u_{xxx} + 6uu_x = \varepsilon u_{xx}.$$

The corresponding approximate equations (6) of section 2 become

$$(12)\qquad \begin{aligned} \dot{\gamma} &= -\varepsilon\langle S_i(U(\gamma,\varphi)), U(\gamma,\varphi)\rangle, \\ \dot{\varphi} &= \lambda(\gamma). \end{aligned}$$

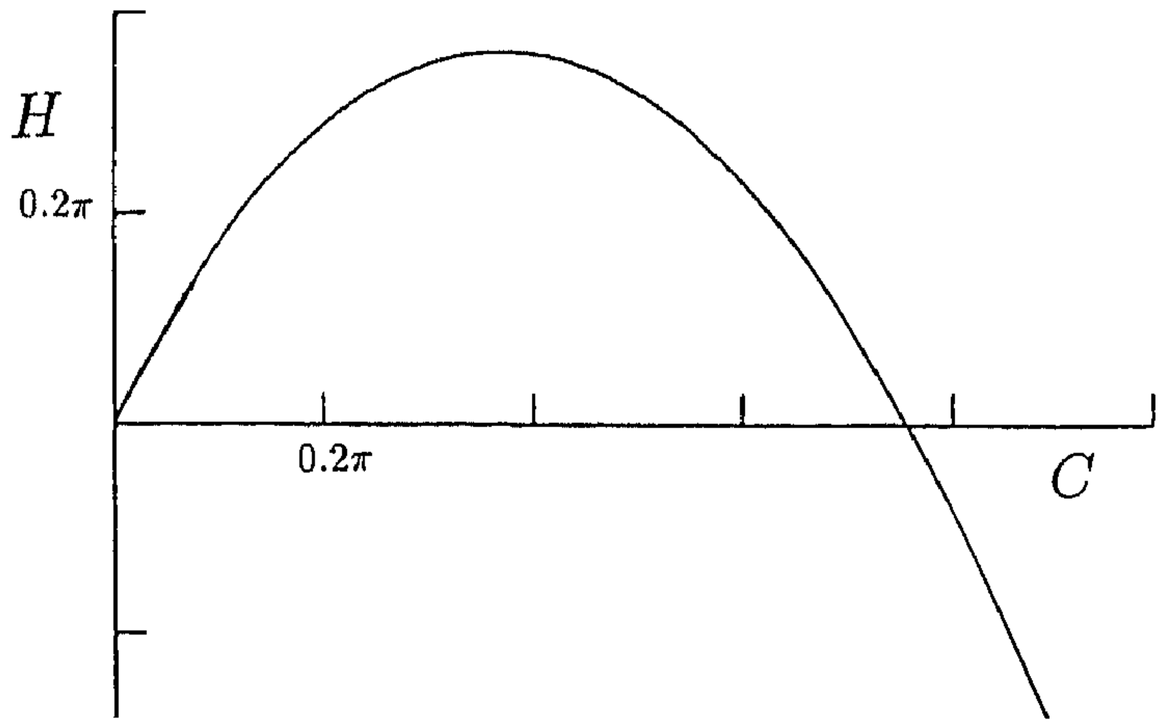

Figure 5: The integral-diagram of the unperturbed KdV-equation

Here $U(\gamma, \varphi)$ is a cnoïdal wave profile on the MRE and $\langle\ ,\ \rangle$ is the L_2-inner product. For the uniform damping the γ-equation becomes $\dot{\gamma} = -\varepsilon\gamma$. This makes clear that the KdV-equation with this dissipation is similar to the pendulum with uniform friction.

We describe numerical results as presented in [GBV90]. By numerical integration the equations (10) and (11) are solved, starting on the MRE, i.e. with an exact cnoïdal wave profile corresponding to $C(u) = \frac{3\pi}{4}$, for various values of ε: $\varepsilon = 0.1,\ 1,\ 10$. Next this solution is compared with the decaying cnoïdal wave profiles $\boldsymbol{U}(\gamma(t), \varphi(t))$, where γ and φ are given by (12).
Figures 6 and 7 show the H-C curves of the solutions, started on the MRE in $C = \frac{3\pi}{4}$, for the various values of ε. In both cases the H-C curve with $\varepsilon = 0.1$ is not drawn, because it was graphically indistinguishable from the dashed line, which is the value-function $h(\gamma)$.
The distance between the H-C curves and the dashed line is a measure for the distance of the solutions to the MRE. Both figures suggest that the approximation by the MRE is rather good, especially for $\varepsilon = 0.1$, for all time. These suggestions have to be verified analytically, but they are encouraging. In particular, figure 7 shows that in case of viscous damping all dissipative solutions are asymptotically tangent to the MRE. This can be explained by a self-organisation of the solutions of the KdV-Burgers equation to the principal sine-mode, see [Has85]. This sine-mode is the solution of (9) when the cubic term is neglected. Hence it determines the MRE near the origin.

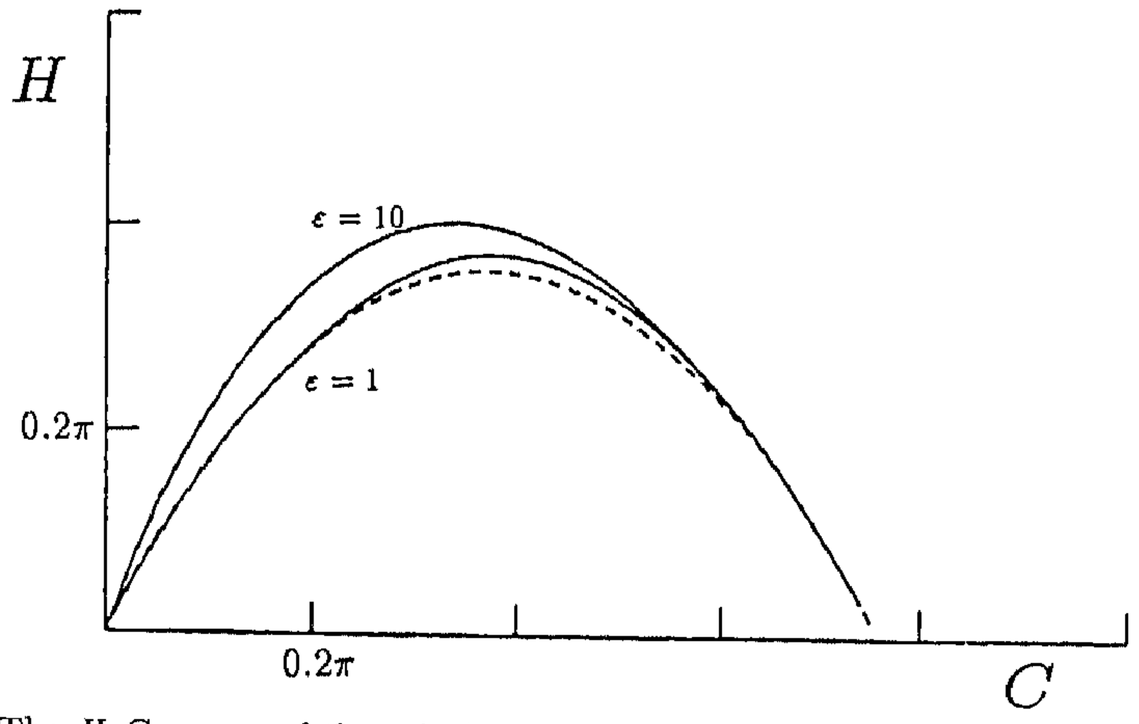

Figure 6: The H-C curves of the solutions of the uniformly damped KdV-equation, with $\varepsilon = 1$ and $\varepsilon = 10$. The curve of the solution with $\varepsilon = 0.1$ is indistinguishable from the dashed line, which is the value of the value-function $h(\gamma)$.

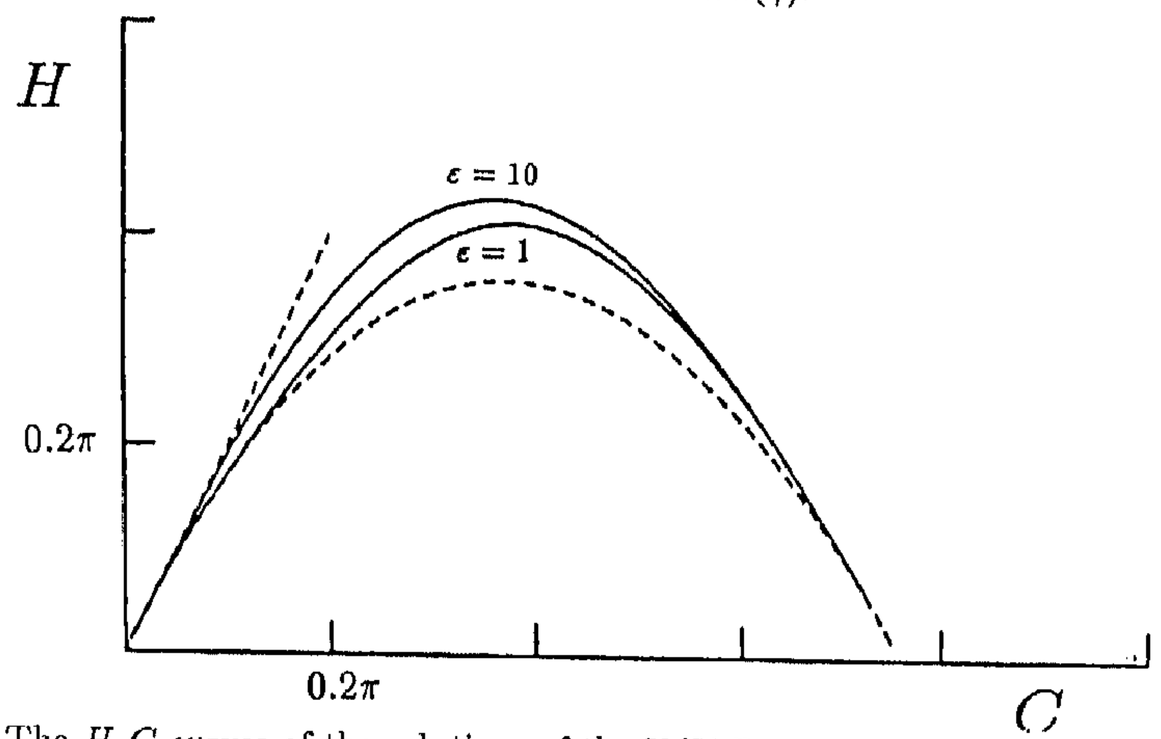

Figure 7: The H-C curves of the solutions of the KdV-Burgers-equation, with $\varepsilon = 1$ and $\varepsilon = 10$. The curve of the solution with $\varepsilon = 0.1$ is indistinguishable from the dashed line, which is the value of the value-function $h(\gamma)$.

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