The Non-Self-Embedding Property for Generalized Fuzzy Context-Free Grammars

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Abstract. A fuzzy context-free K-grammar is a fuzzy context-free grammar with a countable rather than a finite number of rules satisfying the following condition: for each symbol α, the set containing all right-hand sides of rules with left-hand side equal to α forms a fuzzy language that belongs to a given family K of fuzzy languages. In this paper we study the effect of the non-self-embedding restriction on the generating power of fuzzy context-free K-grammars. Our main result shows that under weak assumptions on the family K, a fuzzy language is generated by a non-self-embedding fuzzy context-free K-grammar if and only if either it is a fuzzy regular language or it belongs to the substitution closure K_∞ of the family K. The proof heavily relies on the closure properties of the families K and K_∞.

1 Introduction

The notion of context-free grammar plays a central part in computer science and it has been modified and generalized in numerous ways for as many reasons. The motivations for these modifications range from pure mathematical interest to meeting particular demands in modeling certain linguistic phenomena. Of these many grammatical devices based on context-free production rules we will consider but a few in this paper.

In [21] van Leeuwen introduced context-free grammars with a countable rather than a finite number of productions. This model is due to the following restriction: for each symbol α in V —where V is the alphabet of the grammar— the set $P(\alpha)$ consisting of all right-hand sides of rules with left-hand side equal to α forms a language over V that belongs to a given family K of languages. So the family K serves the role of parameter in this grammatical model which is also reflected in its name: context-free K-grammar. Under minor assumptions on this family K some interesting properties of context-free grammars can be extended to this model [21, 22, 2, 3, 9].

Lee and Zadeh studied fuzzy context-free grammars in [19]: they established a Chomsky and a Greibach Normal Form for these grammars. In a fuzzy context-free grammar G the degree of membership of a production is allowed to be in between 1 (i.e., fully belonging to the set P of productions) and 0 (i.e., not belonging tot P at all). Consequently, derived strings fully belong to the language $L(G)$, or they do not belong to $L(G)$ at all, or their degree of membership with respect to $L(G)$ strictly lies in between 0 and 1. In [4, 5] it was argued that fuzzy context-free grammars are suitable for modeling situations in which only finitely many grammatical errors occur.

Combining the approaches of [21] and [19], as carried out in [4, 5, 6], results in the notion of fuzzy context-free K-grammar, being a fuzzy context-free grammar G with a countable number of productions satisfying the condition: for each symbol α of the grammar, the set

$P(\alpha)$ of all right-hand sides of productions with left-hand side equal to α constitutes a fuzzy language —i.e., a fuzzy subset of $V^\star$, where V is the alphabet of G— that belongs to a given family K of fuzzy languages.

This grammar type provides a general framework in which grammatical errors can be modeled as follows; cf. [4, 5, 6, 7]. For each ordinary rule $\alpha \to \omega$, the string ω fully belongs to $P(\alpha)$, i.e., its degree of membership with respect to $P(\alpha)$ equals 1: $\mu(\omega; P(\alpha)) = 1$. For each grammatical error —i.e., replacing α by an incorrect string ω' — the string ω' does not fully belong to $P(\alpha)$: $0 < \mu(\omega'; P(\alpha)) < 1$. Since there is an infinite number of ways to err in general, $P(\alpha)$ should be an infinite fuzzy set. But to obtain a sensible theory, we have to restrict the fuzzy sets $P(\alpha)$ in some way: the restriction that each $P(\alpha)$ should be a member of a given family K of fuzzy languages is a natural one. In a slightly further generalization a finite number of countable sets of grammatical productions is allowed rather than a single set; cf. Definition 4.1 for details.

In this paper we study the effect of the non-self-embedding restriction on the generating power of fuzzy context-free K-grammars. The generalization of the non-self-embedding property from ordinary context-free grammars [10, 11] to fuzzy context-free K-grammars is straightforward; see Definition 5.1 below. Then as our main result (Theorem 5.6) we have: under weak assumptions on the family K of fuzzy languages, a fuzzy language L is generated by a non-self-embedding fuzzy context-free K-grammar if and only if either L is a fuzzy regular language or L belongs to the substitution closure of the family K. As we will see in Section 5, closure properties of K and of its substitution closure K_∞ play a dominant role in the proof of this result.

This paper is organized as follows. Section 2 is devoted to fuzzy languages and some operations on fuzzy languages. In Section 3 we consider families of fuzzy languages and the result of substituting languages from one family of fuzzy languages into another one. The concept of fuzzy context-free K-grammar is formally defined in Section 4 where we also survey some properties of the family of fuzzy languages generated by these grammars. Section 5 is devoted to not-self-embedding fuzzy context-free K-grammars: it contains the main result of this paper together with its proof. Finally, Section 6 consists of some corollaries of our main result and a few remarks.

2 Fuzzy Languages

The reader is assumed to be familiar with standard definitions and results of formal language theory; cf. [17, 18, 20] for basic texts on formal languages and [13] for closure properties of language families. We also assume familiarity with the rudiments of lattice theory which can be found in many books on algebra; a summary of the relevant concepts is included in [3].

Originally, fuzzy languages and fuzzy grammars have been introduced in [19]. However, we will mainly cite from more recent contributions to the fuzzy counterpart of formal language theory, viz. from [4, 5, 6, 7]. This applies particularly to the membership function μ of a fuzzy language. In the early days of fuzzy language theory μ was a mapping of type $\mu : \Sigma^\star \to [0, 1]$. Nowadays, the real interval $[0, 1]$ has been replaced by a much more general, lattice-ordered structure. The following concepts are modifications of a structure introduced in [15].

Definition 2.1. An algebraic structure $\mathcal{L}$ or $(\mathcal{L}, \wedge, \vee, 0, 1, \star)$ is a *type-00 lattice* if

- $(\mathcal{L}, \wedge, \vee, 0, 1)$ is a completely distributive complete lattice. So $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$ and $(\bigvee a_i) \wedge b = \bigvee_i (a_i \wedge b)$ hold for all a_i, a, b_i and b in $\mathcal{L}$. And 0 and 1 are the smallest and the greatest element of $\mathcal{L}$, respectively: $0 = \bigwedge \mathcal{L}$ and $1 = \bigvee \mathcal{L}$.

– $(\mathcal{L}, \star)$ is a commutative semigroup.
– The following identities hold

$$a \star \bigvee_i b_i = \bigvee_i (a \star b_i) \ ,$$
$$(\bigvee_i a_i) \star b = \bigvee_i (a_i \star b) \ ,$$
$$0 \wedge a = 0 \star a = a \star 0 = 0 \ ,$$
$$1 \wedge a = 1 \star a = a \star 1 = a \ ,$$

for all a_i's, b_i's, a and b in $\mathcal{L}$.

A type-00 lattice in which the operation $\star$ coincides with the operation $\wedge$ is called a *type-01 lattice*: in fact it is a completely distributive complete lattice. A *type-10 lattice* is a type-00 lattice in which $(\mathcal{L}, \wedge, \vee, 0, 1)$ is a totally ordered set or chain, i.e., for all a and b in $\mathcal{L}$, we have $a \wedge b = a$ or $a \wedge b = b$. In a type-10 lattice the operations $\vee$ and $\wedge$ are usually denoted by max and min, respectively. Finally, when $\mathcal{L}$ is both a type-01 lattice and a type-10 lattice, $\mathcal{L}$ is called a *type-11 lattice*. □

Remark that in each type-00 lattice $\mathcal{L}$, we have $a \star b \leq a \wedge b$ for all $a, b \in \mathcal{L}$. Indeed by the distributivity of $\star$ over $\vee$, $a \star (1 \vee b) = a \star 1 \vee a \star b$ holds. Since $1 \vee b = 1$ and $a \star 1 = a$, this reduces to $a = a \vee a \star b$; so $a \star b \leq a$. Similarly, we have $a \star b \leq b$, and hence $a \star b \leq a \wedge b$.

Example 2.2. Let $[0, 1]$ be the interval of real numbers in between 0 and 1.
(1) Then $([0, 1] \times [0, 1], \wedge, \vee, (0, 0), (1, 1), \star)$ with operations defined by $(x_1, y_1) \wedge (x_2, y_2) = (\min\{x_1, x_2\}, \min\{y_1, y_2\})$, $(x_1, y_1) \vee (x_2, y_2) = (\max\{x_1, x_2\}, \max\{y_1, y_2\})$ and $(x_1, y_1) \star (x_2, y_2) = (x_1 x_2, y_1 y_2)$ for all x_1, x_2, y_1 and y_2 in $[0, 1]$ is a type-00 lattice.
(2) Similarly, $([0, 1] \times [0, 1], \wedge, \vee, (0, 0), (1, 1), \star)$ where the operations $\wedge$ and $\vee$ are defined as in (1) and $(x_1, y_1) \star (x_2, y_2) = (\min\{x_1, x_2\}, \min\{y_1, y_2\})$ for all x_1, x_2, y_1 and y_2 in $[0, 1]$, is a type-01 lattice.
(3) The structure $([0, 1], \min, \max, 0, 1, \star)$ with $x_1 \star x_2 = x_1 x_2$ for all x_1 and x_2 in $[0, 1]$ is a type-10 lattice.
(4) Taking $\star$ equal to min in (3) yields a type-11 lattice. □

Definition 2.3. Let $\mathcal{L}$ be a type-00 lattice and let Σ be an alphabet. A $\mathcal{L}$-*fuzzy language* over Σ is a $\mathcal{L}$-fuzzy subset of $\Sigma^\star$, i.e., it is a triple (Σ, μ_{L_0}, L_0) where μ_{L_0} is a function $\mu_{L_0} : \Sigma^\star \to \mathcal{L}$, the *degree of membership function*, and L_0 is the support of μ_{L_0}; so we have $L_0 = \{w \in \Sigma^\star \mid \mu_{L_0}(w) > 0\}$. Usually, we write L_0 instead of (Σ, μ_{L_0}, L_0).

Henceforth, when $\mathcal{L}$ is clear from the context, we use "fuzzy language" instead of "$\mathcal{L}$-fuzzy language". Often we will also write $\mu(x; L_0)$ rather than $\mu_{L_0}(x)$.

For each fuzzy language L_0 over Σ, the *crisp language* $c(L_0)$ induced by L_0 —also known as the *crisp part* of L_0— is the subset $c(L_0) = \{w \in \Sigma^\star \mid \mu(w; L_0) = 1\}$ of $\Sigma^\star$. Each ordinary (non-fuzzy) language L_0 coincides with its crisp part $c(L_0)$. So an ordinary language will also be called a *crisp* language. □

Example 2.4. Let $\mathcal{L}$ be the type-00 lattice of Example 2.2.(1). Consider the $\mathcal{L}$-fuzzy language L_0 over $\Sigma = \{a, b\}$ defined by

$$\mu(a^m b^n; L_0) = \left(\tfrac{m}{\max\{m,n\}}, \tfrac{n}{\max\{m,n\}}\right) \text{ if } m, n \geq 1.$$

In definitions of this type, we always tacitly assume that $\mu(x; L_0) = (0, 0)$ in all other, unmentioned cases for x in $\Sigma^\star$. So, e.g., $\mu(ba^2; L_0) = \mu(a^3 b^2 a; L_0) = \mu(b^2 a b^4; L_0) = (0, 0)$.

Then the crisp part of L_0 equals $c(L_0) = \{a^m b^m \mid m \geq 1\}$. Note that for each x in $c(L_0)$, we have $\mu(x; L_0) = (1, 1)$. □

In many practical examples we will restrict ourselves to the computable or even to the rational elements in $[0,1]$. Cf. [12] for the impact of computability constraints in fuzzy formal languages.

Operations on fuzzy languages play an important part in the present paper. First, we consider the operations union, intersection and concatenation which have originally been introduced in [19] for the type-11 lattice $[0,1]$. Here we will define these operations for arbitrary type-00 lattices. So let $(\Sigma_1, \mu_{L_1}, L_1)$ and $(\Sigma_2, \mu_{L_2}, L_2)$ be fuzzy languages, then the *union* of the fuzzy languages L_1 and L_2, denoted by $(\Sigma_1 \cup \Sigma_2, \mu_{L_1 \cup L_2}, L_1 \cup L_2)$ or $L_1 \cup L_2$ for short, is defined by

$$\mu(x; L_1 \cup L_2) = \mu(x; L_1) \vee \mu(x; L_2) \ ,$$

for all x in $(\Sigma_1 \cup \Sigma_2)^\star$. Analogously, for the *intersection* of fuzzy languages L_1 and L_2, denoted by $(\Sigma_1 \cap \Sigma_2, \mu_{L_1 \cap L_2}, L_1 \cap L_2)$ or abbreviated by $L_1 \cap L_2$, the equality

$$\mu(x; L_1 \cap L_2) = \mu(x; L_1) \wedge \mu(x; L_2) \ ,$$

holds for all x in $(\Sigma_1 \cap \Sigma_2)^\star$. And for the *concatenation* of fuzzy languages L_1 and L_2, denoted by $(\Sigma_1 \cup \Sigma_2, \mu_{L_1 L_2}, L_1 L_2)$ or $L_1 L_2$ for short, we have

$$\mu(x; L_1 L_2) = \bigvee \{\mu(y; L_1) \star \mu(z; L_2) \mid x = yz\}$$

for all x in $(\Sigma_1 \cup \Sigma_2)^\star$.

The operations of *Kleene* $+$ and *Kleene* $\star$ for a fuzzy language L are of course defined in terms of concatenation and infinite union:

$$L^+ = L \cup LL \cup LLL \cup \ldots = \bigcup \{L^i \mid i \geq 1\} \ ,$$

$$L^\star = \{\lambda\} \cup L \cup LL \cup LLL \cup \ldots = \bigcup \{L^i \mid i \geq 0\} \ ,$$

respectively, where $L^0 = \{\lambda\}$, λ is the empty word, and $L^{n+1} = L^n L$ with $n \geq 0$. In defining $L^\star$ we demand that $\mu(\lambda; L^\star) = 1$. Consequently, $L^\star = L^+ \cup \{\lambda\}$ where the latter set in this union is a crisp set.

Some operations on fuzzy languages, like homomorphisms and substitutions, are defined as fuzzy functions on fuzzy languages. A fuzzy function is a fuzzy relation that satisfies some additional properties. In its turn a *fuzzy relation* R between crisp sets X and Y is a fuzzy subset of $X \times Y$. If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are fuzzy relations, then their composition $R \circ S$ is defined by

$$\mu((x,z); R \circ S) = \bigvee \{\mu((x,y); R) \star \mu((y,z); S) \mid y \in Y\}. \tag{1}$$

Now a *fuzzy function* $f : X \to Y$ is just a fuzzy relation $f \subseteq X \times Y$, satisfying the restriction that for all x in X: if $\mu((x,y); f) > 0$ and $\mu((x,z); f) > 0$ hold, then $y = z$ and hence $\mu((x,y); f) = \mu((x,z); f)$. For fuzzy functions (1) holds as well, but we write the composition of two functions $f : X \to Y$ and $g : Y \to Z$ as $g \circ f : X \to Z$ rather than as $f \circ g$.

Let $\mathcal{P}(X)$ denote the power set of the set X. In Sections 3 and 4 we will encounter functions $f : V^\star \to \mathcal{P}(V^\star)$ that will be extended to $f : \mathcal{P}(V^\star) \to \mathcal{P}(V^\star)$ by $f(L) = \bigcup \{f(x) \mid x \in L\}$ and for each subset L of $V^\star$,

$$\mu(y; f(L)) = \bigvee \{\mu(x; L) \star \mu((x,y); f) \mid x \in V^\star\}. \tag{2}$$

Consequently, by (1) and (2) fuzzy functions like $f \circ f$, $f \circ f \circ f$, and so on, which are obtained by iterating the function f, are now defined. Clearly, each of these functions $f^{(n)}$ is of type $f^{(n)} : \mathcal{P}(V^\star) \to \mathcal{P}(V^\star)$.

3 Families of Fuzzy Languages

Apart from individual fuzzy languages we also need families of fuzzy languages. The next few definitions are straightforward generalizations based on well-known concepts for families of crisp languages.

Throughout this paper Σ_ω denotes a countably infinite set of symbols. All families of languages that we will consider in the sequel only use symbols from this set. Henceforth, $\mathcal{L}$ is a type-00 lattice.

Definition 3.1. A *family of fuzzy languages* K is a set of fuzzy languages (Σ_L, μ_L, L) such that each Σ_L is a finite subset of Σ_ω. Of course, we assume that for each fuzzy language (Σ_L, μ_L, L) in the family K, the alphabet Σ_L is minimal with respect to μ_L, i.e., a symbol α belongs to Σ_L if and only if there exists a word w in which α occurs and for which $\mu_L(w) > 0$ or, equivalently, for which $w \in L$ holds.

A family K of fuzzy languages is called *nontrivial* if K contains a language (Σ_L, μ_L, L) with $L \cap \Sigma_L^+ \neq \varnothing$, i.e., (Σ_L, μ_L, L) satisfies $\mu(x; L) > 0$ for some $x \in \Sigma_L^+$.

For each family K of fuzzy languages, the *crisp part* of K, denoted by $c(K)$, is defined by $c(K) = \{c(L) \mid L \in K\}$. □

Remember that we write L rather than (Σ_L, μ_L, L) for members of a family of fuzzy languages. And henceforth, we also assume that each family of fuzzy languages is closed under isomorphism ("renaming of symbols"): so for each family K we assume that for each language L in K over some alphabet Σ_L and for each bijective non-fuzzy mapping $i : \Sigma_L \to \Sigma'_L$ —extended to words and to languages in the usual way— we have that the language $i(L)$ also belongs to K. Note that $\mu(x; L) = \mu(i(x); i(L))$ holds for all x in $\Sigma_L^\star$.

We need a few simple, nontrivial families of crisp languages in the sequel: they are the family FIN of finite languages

$$\mathrm{FIN} = \{\{w_1, w_2, \ldots, w_n\} \mid w_i \in \Sigma_\omega^\star,\ 1 \leq i \leq n;\ n \geq 0\}\ ,$$

the family ONE of singleton languages

$$\mathrm{ONE} = \{\{w\} \mid w \in \Sigma_\omega^\star\}\ ,$$

the family ALPHA of alphabets

$$\mathrm{ALPHA} = \{\Sigma \mid \Sigma \subset \Sigma_\omega,\quad \Sigma \text{ is finite}\}\ ,$$

and the family SYMBOL of singleton alphabets

$$\mathrm{SYMBOL} = \{\{\alpha\} \mid \alpha \in \Sigma_\omega\}\ .$$

The fuzzy counterparts of these language families are denoted by FIN_f, ONE_f, ALPHA_f, and SYMBOL_f, respectively. Clearly, the equality $c(\mathrm{FIN}_f) = \mathrm{FIN}$ holds, as well as similar statements for the other families of languages.

The family of fuzzy regular languages is denoted by REG_f; it is defined in a way very similar to its crisp counterpart $\mathrm{REG} = c(\mathrm{REG}_f)$.

Definition 3.2. The *family of fuzzy regular languages* REG_f is the smallest family of fuzzy languages satisfying:

- The fuzzy subsets $\varnothing$ and $\{\lambda\}$ of $\varnothing^\star$ belong to REG_f.
- For each σ in Σ_ω, the fuzzy subset $\{\sigma\}$ of $\{\sigma\}^\star$ belongs to REG_f.
- If R_1 and R_2 are in REG_f, then so are $R_1 \cup R_2$, $R_1 R_2$, and $R_1^\star$. □

The concept of fuzzy substitution plays an important role in the remainder of this paper. Its definition is an obvious extension of the notion of substitution for crisp languages.

Definition 3.3. Let K be a family of fuzzy languages and let V be an alphabet. A mapping $\tau : V \to K$ is called a *fuzzy K-substitution* τ on V; it is extended to words over V by $\tau(\lambda) = \{\lambda\}$ with $\mu(\lambda; \tau(\lambda)) = 1$, and $\tau(\alpha_1 \ldots \alpha_n) = \tau(\alpha_1) \ldots \tau(\alpha_n)$ where $\alpha_i \in V$ $(1 \leq i \leq n)$, and to languages L over V by $\tau(L) = \bigcup\{\tau(w) \mid w \in L\}$.

If for each $\alpha \in V$, $\tau(\alpha) \subseteq V^\star$, then $\tau : V \to K$ is called a *fuzzy K-substitution over V*. If K equals FIN_f or REG_f, τ is called a *fuzzy finite* or a *fuzzy regular substitution*, respectively.

Given families K and K' of fuzzy languages, let $\hat{\mathrm{Sub}}(K, K') = \{\tau(L) \mid L \in K; \tau$ is a fuzzy K'-substitution$\}$. A family K is *closed* under fuzzy K'-substitution if $\hat{\mathrm{Sub}}(K, K') \subseteq K$, and K is *closed under fuzzy substitution*, if K is closed under fuzzy K-substitution. □

Taking K and K' equal to families of crisp languages yields the well-known definition of (ordinary, non-fuzzy) substitution. Then a ONE-substitution is just a homomorphism and an isomorphism ("renaming of symbols") is a one-to-one SYMBOL-substitution.

A nontrivial family K of fuzzy languages is called a *fuzzy prequasoid* if K is closed under fuzzy finite substitution (i.e., $\hat{\mathrm{Sub}}(K, \mathrm{FIN}_f) \subseteq K$) and under intersection with fuzzy regular languages. A *fuzzy quasoid* is a fuzzy prequasoid that contains an infinite fuzzy language. It is easy to show that each fuzzy [pre]quasoid includes the smallest fuzzy [pre]quasoid REG_f [FIN_f, respectively], whereas FIN_f is the only fuzzy prequasoid that is not a fuzzy quasoid. The reader is referred to [1, 2] where similar statements for the corresponding families of crisp languages are established.

A *full Abstract Family of Fuzzy Languages* or *full AFFL* is a nontrivial family of fuzzy languages closed under union, concatenation, Kleene $\star$, (possibly erasing) fuzzy homomorphism, inverse fuzzy homomorphism, and intersection with fuzzy regular languages. Equivalently, K is a full AFFL if and only if K is a fuzzy prequasoid closed under fuzzy regular substitution (i.e., $\hat{\mathrm{Sub}}(K, \mathrm{REG}_f) \subseteq K$), and under substitution in the fuzzy regular languages (i.e., $\hat{\mathrm{Sub}}(\mathrm{REG}_f, K) \subseteq K$); cf. [8] for a proof of this equivalence. A *full substitution-closed AFFL* is a full AFFL closed under fuzzy substitution.

Let K_∞ denote the smallest family closed under fuzzy substitution that includes a given family K. In [14] Ginsburg and Spanier proved for crisp language families that closure under substitution can be obtained by iterating the $\hat{\mathrm{Sub}}$-operator. A proof of the following fuzzy counterpart of their result can be found in [8].

Proposition 3.4. [8]

(1) If SYMBOL $\subseteq K$*, then*

$$K_\infty = \bigcup_{n=0}^{\infty} \mathrm{SUB}^n(K) \ , \qquad \textit{with}$$

$$\mathrm{SUB}^0(K) = K \ , \qquad \textit{and}$$

$$\mathrm{SUB}^{n+1}(K) = \hat{\mathrm{Sub}}(\bigcup_{i=0}^{n} \mathrm{SUB}^i(K), K) \ , \qquad \textit{for each } n \geq 0.$$

(2) If K is a quasoid, then K_∞ is a full substitution-closed AFFL. □

In the proof of this statement we use the fact that K is closed under isomorphism; cf. the general assumption made earlier.

4 Fuzzy Context-Free *K*-grammars and Their Languages

The concept of fuzzy context-free K grammar has been introduced in [4, 5] in order to provide a framework for dealing with grammatical errors. In essence it is an obvious generalization of both the fuzzy context-free grammar of [19] and the context-free K-grammar of [21, 22]. As in the case of context-free K-grammars, the grammatical model of Definition 4.1 is based on a countable rather than a finite number of production rules.

Definition 4.1. Let K be a family of fuzzy languages. A fuzzy K-substitution $\tau : V \to K$ is *nested* [16], if $\mu(\alpha; \tau(\alpha)) = 1$ for each α in V.

A *fuzzy context-free K-grammar* G is a four-tuple $G = (V, \Sigma, U, S)$ where
- V is an alphabet (the *alphabet* of G);
- Σ is an alphabet with $\Sigma \subseteq V$ (the *terminal alphabet* of G);
- S is a symbol in V (the *initial symbol* of G);
- U is a finite set of nested fuzzy K-substitutions over V.

The fuzzy language $L(G)$ generated by G is defined by

$$L(G) = U^\star(S) \cap \Sigma^\star = \bigcup \{\tau_p(\ldots(\tau_1(S))\ldots) \mid \ p \geq 0; \ \ \tau_i \in U, \ \ 1 \leq i \leq p\}.$$

The family of fuzzy languages generated by fuzzy context-free K-grammars is denoted by $A_f(K)$. For each $m \geq 1$, $A_{f,m}(K)$ is the family of fuzzy languages generated by fuzzy context-free K-grammars that contain at most m nested fuzzy K-substitutions in U. □

Note that $A_f(K) = \bigcup \{A_{f,m}(K) \mid m \geq 1\}$ for each family K of fuzzy languages.

Example 4.2. Let $\mathcal{L}$ be the type-10 lattice of Example 2.2.(3).
(1) Consider the fuzzy context-free REG$_f$-grammar $G = (V, \Sigma, U, S)$ defined by $\Sigma = \{a, b\}$, $V = \Sigma \cup \{S\}$, $U = \{\tau\}$ with $\tau(S) = \{\lambda, S\} \cup \{a^m S b^n \mid 0 \leq m \leq n\}$, $\tau(a) = \{a\}$, and $\tau(b) = \{b\}$. The degrees of membership are as follows: $\mu(a; \tau(a)) = \mu(b; \tau(b)) = \mu(\lambda; \tau(S)) = \mu(S; \tau(S)) = 1$ and $\mu(a^m S a^n; \tau(S)) = (1 + n - m)^{-1}$ with $0 \leq m \leq n$.

Then for the $\mathcal{L}$-fuzzy language $L(G)$ we have $L(G) = \{a^m b^n \mid 0 \leq m \leq n\}$ and $\mu(a^m b^n; L(G)) = (1 + n - m)^{-1}$. Note that $\mu(a^m b^m; L(G)) = 1$ for all $m \geq 0$.

The "grammatically correct" strings are those in the crisp language $c(L(G)) = \{a^m b^m \mid m \geq 0\}$. "Grammatically incorrect" strings arise due to grammatical errors, i.e., applying rules for S with right-hand sides from $\{a^m S b^n \mid 0 \leq m < n\}$.
(2) Define the fuzzy context-free REG$_f$-grammar $G = (V, \Sigma, U, S)$ by $\Sigma = \{a, b\}$, $V = \Sigma \cup \{S\}$, $U = \{\tau\}$ where $\tau(S) = \{S, \lambda\} \cup \{a^{2m} S \mid m \geq 0\} \cup \{b^n S \mid n \geq 1\}$, $\tau(a) = \{a\}$ and $\tau(b) = \{b\}$. For the membership functions we have: $\mu(a; \tau(a)) = \mu(b; \tau(b)) = \mu(S; \tau(S)) = \mu(\lambda; \tau(S)) = \mu(a^{2m}; \tau(S)) = 1$ with $m \geq 0$, and $\tau(b^n S; \tau(S)) = 2^{-n}$ for $n \geq 1$.

Then $L(G) = \{w \in \Sigma^\star \mid \#_a(w) \ \text{ is even}\}$ where $\#_\sigma(w)$ denotes the number of times that the symbol σ occurs in the word w. Now $c(L(G)) = \{a^{2n} \mid n \geq 0\}$ and $\mu(w; L(G)) = 2^{-\#_b(w)}$ for each w for which $\#_a(w)$ is even. □

We call a family K of fuzzy languages *simple* if $K \supseteq$ SYMBOL and if K is closed under union with SYMBOL-languages and under isomorphism (as already assumed). If K is simple, then any number of fuzzy K-substitutions in a fuzzy context-free K-grammars can be reduced to a single, equivalent nested fuzzy K-substitution.

Theorem 4.3. *If K is a simple family of fuzzy languages, then for each $m \geq 1$, $A_{f,1}(K) = A_{f,m}(K) = A_f(K)$.*

Proof. We have to show that for each $m \geq 1$, $A_f(K) \subseteq A_{f,m}(K) \subseteq A_{f,1}(K)$, since the converse inclusions follow from the definitions.

Let $G = (V, \Sigma, U, S)$ be a fuzzy context-free K-grammar that contains m nested fuzzy K-substitutions $\tau_1, \ldots, \tau_m$ for some $m \geq 2$. Define for each k $(1 \leq k \leq m)$ an isomorphism $\varphi_k(\alpha) = \alpha_k$ ($\alpha \in V$; all α_k's are new mutually distinct symbols) and extend these isomorphisms in the usual way to words and to languages. Define a new alphabet $V_0 = V \cup \{\varphi_k(\alpha) \mid \alpha \in V;\ 1 \leq k \leq m\}$.

Consider the fuzzy context-free K-grammar $G_0 = (V_0, \Sigma, \{\tau\}, S)$ where the nested fuzzy K-substitution τ over V_0 is defined by

$$\tau(\alpha) = \{\alpha, \alpha_1, \ldots, \alpha_m\} \qquad \alpha \text{ in } V\ ,$$
$$\tau(\alpha_k) = \{\alpha_k\} \cup \tau_k(\alpha) \qquad \text{with } 1 \leq k \leq m.$$

Note that since K is a simple family of fuzzy languages we have that $K \supseteq$ ALPHA, and hence $\tau(\alpha)$ is a crisp language in K for each α in V.

The basic idea of the simulation of G by G_0 is the following: each occurrence of any symbol β in V_0 may be object to the following replacements in an "asynchronous" way:
(i) changing into α_k if $\beta = \alpha$ ($\alpha \in V$), i.e., from α we can reach α_k for each k ($1 \leq k \leq m$),
(ii) substituting $\tau_k(\alpha)$ into that particular instance of $\beta = \alpha_k$, i.e., simulating the application of τ_k on that occurrence of α_k while the subscript k is removed, and
(iii) the identity replacement.

By this construction we obtain $L(G_0) = L(G)$, i.e., $\mu(x; L(G_0)) = \mu(x; L(G))$ holds for all x in $\Sigma^\star$. Hence the inclusions $A_f(K) \subseteq A_{f,m}(K) \subseteq A_{f,1}(K)$ hold for each $m \geq 1$. □

Note that this result is much easier to prove whenever K satisfies the much stronger condition that K is closed under union.

Using Theorem 4.3 it is straightforward to show that the generating power of fuzzy context-free FIN_f-grammars coincides with the one of the fuzzy context-free grammars of [19]: it yields the family CF_f of fuzzy context-free languages. The proof is therefore omitted.

Corollary 4.4. *For each $m \geq 1$, $A_{f,1}(\mathrm{FIN}_f) = A_{f,m}(\mathrm{FIN}_f) = A_f(\mathrm{FIN}_f) = \mathrm{CF}_f$.* □

A family K is closed under *nested iterated fuzzy substitution* if for each fuzzy language L in K over some alphabet V ($L \subseteq V^\star$), and each finite set U of nested fuzzy K-substitutions over V, the language $U^\star(L)$ defined by

$$U^\star(L) = \bigcup \{\tau_p(\ldots(\tau_1(L))\ldots) \mid\ p \geq 0;\ \ \tau_i \in U,\ \ 1 \leq i \leq p\}$$

is in K.

A *full super-AFFL* is a full AFFL closed under nested iterated fuzzy substitution. Each full super-AFFL is a full substitution-closed AFFL [6], but the converse implication does not hold.

The concept of fuzzy context-free K-grammar is a major tool in studying nested iterated fuzzy substitution. Using results in [6] the proof of the following characterization is straightforward and so it is left to the reader.

Proposition 4.5. *A family K is a full super-AFFL if and only if K is a fuzzy prequasoid and $A_f(K) = K$.* □

For an arbitrary family K of fuzzy languages, let $\Pi_f(K)$ denote the smallest fuzzy prequasoid that includes K.

Theorem 4.6. [6]
(1) If K is a fuzzy prequasoid, then $A_f(K)$ is a full super-AFFL.
(2) For each arbitrary family K of fuzzy languages, $A_f\Pi_f(K)$ is the smallest full super-AFFL that includes K. □

Corollary 4.7. [6] CF_f *is the smallest full super-AFFL.* □

5 Non-Self-Embedding Fuzzy Context-Free K-grammars

This section is devoted to the effect of the non-self-embedding restriction on the generative capacity of fuzzy context-free K-grammars. The generalization of the non-self-embedding property to fuzzy context-free K-grammars is straightforward; cf. [10, 11, 9]. Viz.

Definition 5.1. Let K be a family of fuzzy languages and let U be a finite set of nested fuzzy K-substitutions over an alphabet V. Then U is called *not self-embedding* if for all $u \in U^\star$ and for all α in V, the implication

$$\mu(w_1\alpha w_2; u(\alpha)) > 0 \qquad \Rightarrow \qquad (w_1 = \lambda \text{ or } w_2 = \lambda)$$

holds for all w_1 and w_2 in $V^\star$.

A *fuzzy regular K-grammar* G is a fuzzy context-free K-grammar (V, Σ, U, S) such that U is not self-embedding.

The family of languages generated by fuzzy regular K-grammars is denoted by $R_f(K)$. Similarly, for each $m \geq 1$, $R_{f,m}(K)$ is the subfamily of languages generated by fuzzy regular K-grammars that contain at most m nested fuzzy K-substitutions. □

Clearly, for each family K of fuzzy languages, we have $R_f(K) = \bigcup\{R_{f,m}(K) \mid m \geq 1\}$.

Example 5.2. It is straightforward to check that the fuzzy context-free REG_f-grammar of Example 4.2.(2) is not self-embedding, whereas the grammar of Example 4.2.(1) is, since we have, e.g., $\mu(a^3Sb^7; \tau(aSb^2)) > 0$ and $a^2 \neq \lambda$ and $b^5 \neq \lambda$. □

It is easy to verify that the construction in the proof of Theorem 4.3 preserves the non-self-embedding property, and so any number of fuzzy substitutions in a fuzzy regular K-grammar may be reduced to a single, equivalent, not self-embedding fuzzy substitution.

Theorem 5.3. *If K is a simple family of fuzzy languages, then for each $m \geq 1$, $R_{f,1}(K) = R_{f,m}(K) = R_f(K)$.* □

The main result in this section (Theorem 5.6) establishes the equivalence of $R_f(K)$ and the substitution closure K_∞ [or $K_\infty \cup \mathrm{REG}_f$] of K, provided K is a fuzzy quasoid [fuzzy prequasoid, respectively]. The proof is based on the following two lemmas.

Lemma 5.4. *If K is a simple family of fuzzy languages, then $K_\infty \subseteq R_f(K)$.*

Proof. Following Proposition 3.4.(1) it is sufficient to show that for all natural numbers n ($n \geq 0$), $\mathrm{SUB}^n(K) \subseteq R_f(K)$. We prove this inclusion by induction on n.
Initial step: ($n = 0$). By definition $\mathrm{SUB}^0(K) = K$. Let $L_0 \subseteq \Sigma^\star$ be a fuzzy language in K and consider the fuzzy context-free K-grammar $G = (V, \Sigma, \{\tau\}, S)$ where $V = \Sigma \cup \{S\}$, $S \notin \Sigma$, $\tau(S) = \{S\} \cup L_0$ with $\mu(S; \tau(S)) = 1$, and $\tau(a) = \{a\}$ with $\mu(a; \tau(a)) = 1$ for each a in Σ. Clearly, G is not self-embedding, hence $L(G)$ is in $R_f(K)$, and $L_0 = L(G)$ or, equivalently, $\mu(x; L_0) = \mu(x; L(G))$ for all x in $\Sigma^\star$.
Induction step: Suppose $\mathrm{SUB}^i(K) \subseteq R_f(K)$ holds for all $i \leq n$. Then we have to show that $\mathrm{SUB}^{n+1}(K) \subseteq R_f(K)$.

By the induction hypothesis and the monotinicity of the Sûb-operator we obtain

$\mathrm{SUB}^{n+1}(K) = \hat{\mathrm{Sub}}(\bigcup_{i=0}^n \mathrm{SUB}^i(K), K) \subseteq \hat{\mathrm{Sub}}(R_f(K), K)$.

Thus it only remains to prove that $\hat{\mathrm{Sub}}(R_f(K), K) \subseteq R_f(K)$.

Let $L' \subseteq \Sigma^\star$ be a fuzzy language in $R_f(K)$ generated by the fuzzy regular K-grammar $G = (V, \Sigma, \{\tau\}, S)$, and let g be a fuzzy K-substitution on Σ with $\bigcup\{g(\alpha) \mid \alpha \in \Sigma\} \subseteq \Sigma_0^\star$

for some alphabet Σ_0. Without loss of generality we may assume that $V \cap \Sigma_0 = \varnothing$. Consider the fuzzy context-free K-grammar $G_0 = (V_0, \Sigma_0, U_0, S)$ where $V_0 = V \cup \Sigma_0$, $U_0 = \{\tau_0, \tau_1\}$ with

$$\begin{array}{lll} \tau_0(\alpha) = \{\alpha\}\ , & \mu(\alpha;\tau_0(\alpha)) = 1 & \text{for } \alpha \text{ in } V_0 - \Sigma, \\ \tau_0(\alpha) = \{\alpha\} \cup g(\alpha)\ , & \mu(\alpha;\tau_0(\alpha)) = 1 & \text{for } \alpha \text{ in } \Sigma, \\ \tau_1(\alpha) = \tau(\alpha)\ , & \mu(\alpha;\tau_1(\alpha)) = 1 & \text{for } \alpha \text{ in } V, \\ \tau_1(\alpha) = \{\alpha\}\ , & \mu(\alpha;\tau_1(\alpha)) = 1 & \text{for } \alpha \text{ in } \Sigma_0. \end{array}$$

Clearly, G_0 is a not self-embedding fuzzy context-free K-grammar, $L(G_0) = g(L(G)) = g(L')$, i.e., $\mu(x; L(G_0)) = \mu(x; g(L'))$ for all x in $\Sigma_0^\star$. Therefore we have $g(L') \in R_f(K)$.

This completes the induction and establishes the inclusion $K_\infty \subseteq R_f(K)$. □

Lemma 5.5. *If K is a fuzzy prequasoid, then $R_f(K) \subseteq K_\infty \cup \mathrm{REG}_f$.*

Proof. First we assume that K is a fuzzy quasoid and finally we consider the case that K is a fuzzy prequasoid but not a fuzzy quasoid.

So let K be a fuzzy quasoid, then $\mathrm{REG}_f \subseteq K \subseteq K_\infty$. By Theorem 5.3 we only have to prove that $R_{f,1}(K) \subseteq K_\infty$. However, we will show that $R_{f,1}(K_\infty) \subseteq K_\infty$. Since $K \subseteq K_\infty$, this inclusion yields $R_{f,1}(K) \subseteq R_{f,1}(K_\infty) \subseteq K_\infty$ by the monotonicity of $R_{f,1}$.

The proof of the inclusion $R_{f,1}(K_\infty) \subseteq K_\infty$ is a generalization of the argument that each non-self-embedding context-free grammar generates a regular language; cf. e.g. [20, 17].

Consider the fuzzy regular K_∞-grammar $G = (V, \Sigma, \{\tau\}, S)$. Without loss of generality we may assume that $\tau(\sigma) = \{\sigma\}$ and $\mu(\sigma; \tau(\sigma)) = 1$ for each σ in Σ. Otherwise, we introduce for each σ in Σ a new symbol A_σ, we replace σ by A_σ everywhere in G, and finally we change the modified fuzzy K-substitution τ' into $\tau(A_\sigma) = \{A_\sigma, \sigma\} \cup \tau'(A_\sigma)$, and $\tau(\sigma) = \{\sigma\}$ with $\mu(A_\sigma; \tau(A_\sigma)) = \mu(\sigma; \tau(A_\sigma)) = \mu(\sigma; \tau(\sigma)) = 1$.

Moreover we assume that for each α in V there is a sequence u in $\tau^\star$ such that there is a word ω with $\mu(\omega; u(S)) > 0$ and ω contains an occurrence of α. Otherwise we can remove α from V and intersect all languages involved in the definition of τ with the crisp language $(V - \{\alpha\})^\star$, without affecting $L(G)$. Note that by Proposition 3.4.(2), K_∞ is a full AFFL and so it is closed under intersection with crisp regular languages. So each symbol α in V is "reachable" from S.

We distinguish the following two cases:

Case 1: For each α in $V - \Sigma$ there is a sequence u in $\tau^\star$ such that there exists a word w in $V^\star$ such that $\mu(w; u(\alpha)) > 0$ and w contains an occurrence of the symbol S.

If $w \in \tau(A)$ is an arbitrary word containing a nonterminal symbol, say B, then it is of one of the four forms: (i) $w = \varphi B \psi$, (ii) $w = \varphi B$, (iii) $w = B\psi$, or (iv) $w = B$, where φ and ψ are nonempty words over V. If w satisfies (i) we must have by the assumption of Case 1: there exist sequences u_1 and u_2 in $\tau^\star$ such that

$$\mu(\varphi\varphi_1\varphi_2 A\psi_2\psi_1\psi; u_1u_2\tau(A)) \geq \mu(\varphi\varphi_1\varphi_2 A\psi_2\psi_1\psi; u_1u_2(\varphi B\psi)) \geq$$
$$\geq \mu(\varphi\varphi_1\varphi_2 A\psi_2\psi_1\psi; u_1(\varphi\varphi_1 S\psi_1\psi) > 0$$

for some (possibly empty) words φ_1, φ_2, ψ_1, and ψ_2 over V. The first two inequalities follow from (1), (2) and $a \star b \leq a \wedge b \leq a$ for all a, b in $\mathcal{L}$; cf. the remark after Definition 2.1. The latter strict inequality follows from the reachability of A from S. But then, since φ and ψ are nonempty, G is not a fuzzy regular K_∞-grammar. We obtain the same contradiction if $\bigcup_{A\in V} \tau(A)$ contains words of both forms (ii) and (iii). Thus if $\bigcup_{A\in V} \tau(A)$ contains a word of the form (ii), then in all words of the form (ii) the word φ is in $\Sigma^\star$; otherwise it would contain also a word of the form (i) or (iii). Hence G is a "right linear" fuzzy context-free K_∞-grammar, i.e., for each A in $V-\Sigma$ we have $\tau(A) \subseteq \Sigma^\star(V-\Sigma) \cup \Sigma^\star$. By a similar argument we conclude that if $\tau(A)$ contains a word of the form (iii), then G is "left linear", i.e., for each A in $V-\Sigma$, $\tau(A) \subseteq (V-\Sigma)\Sigma^\star \cup \Sigma^\star$. So if G is right linear, then $\tau(A) = \bigcup\{L_{AX}\{X\} \mid X \in V-\Sigma\} \cup L_A$ for each A in $V-\Sigma$, where L_A and L_{AX} are languages over Σ (The left linear case is similar). Note that L_{AX} is empty whenever X does not occur in any word of $\tau(A)$. Since K_∞ is a full AFFL (Proposition 3.4), L_A, and each L_{AX} are in K_∞ and, consequently, $R_{f,1}(K_\infty) \subseteq \hat{\mathrm{S}}\mathrm{ub}(\mathrm{REG}_f, K_\infty) = K_\infty$; cf. Proposition 3.3.1 in [13] and Proposition 3.4.

Note that any fuzzy context-free K_∞-grammar with $V-\Sigma=\{S\}$ satisfies Case 1, as $\mu(S;\lambda(S))=1$.

Case 2: There exists a nonterminal symbol α such that for all sequences u in $\tau^\star$ and for all words w over V we have: if $\mu(w;u(\alpha))>0$, then w does not contain an occurrence of S.

This latter statement is equivalent to: for all words φ and ψ in $V^\star$, $\mu(\varphi S\psi; u(A)) = 0$.

The proof of the fact that $L(G)$ belongs to K_∞ proceeds by induction on the number m of nonterminal symbols in G.

Basis: For $m=1$ we have $V-\Sigma=\{S\}$. Due the remark at the end of Case 1, the basis is vacuously satisfied.

Induction step: Assume that the assertion holds for $m=n$. Let the number of nonterminal symbols in $V-\Sigma$ be $n+1$. Consider the fuzzy context-free K_∞-grammar G_1 defined by $G_1=(V-\{S\},\Sigma,\{\tau_1\},A)$ with

$$\tau_1(\alpha) = \tau(\alpha) \cap (V-\{S\})^\star \qquad \text{for each } \alpha \text{ in } V-\{S\}$$

and the fuzzy context-free K_∞-grammar $G_2=(V,\Sigma\cup\{A\},\{\tau_2\},S)$ with

$$\tau_2(\alpha) = \tau(\alpha) \qquad \text{for each } \alpha \text{ in } V-\{A\}\ ,$$
$$\tau_2(A) = \{A\}.$$

Then both G_1 and G_2 are non-self-embedding fuzzy grammars having n nonterminal symbols. Now both languages $L(G_1)$ and $L(G_2)$ are in K_∞; either by the induction hypothesis or by Case 1. But $L(G)$ equals the result of substituting $L(G_1)$ for A in $L(G_2)$. The fact that K_∞ is closed under fuzzy substitution implies that $L(G)$ is in K_∞. This completes the induction and establishes the inclusion $R_{f,1}(K_\infty) \subseteq K_\infty$.

Finally, we deal with the case when K is a fuzzy prequasoid but not a fuzzy quasoid. Then K equals FIN_f, and hence by the monotonicity of R_f, we have $R_f(\mathrm{FIN}_f) \subseteq R_f(\mathrm{REG}_f)$. Applying Lemma 5.5 for the smallest fuzzy quasoid REG_f yields $R_f(\mathrm{REG}_f) \subseteq (\mathrm{REG}_f)_\infty \cup \mathrm{REG}_f = \mathrm{REG}_f = (\mathrm{FIN}_f)_\infty \cup \mathrm{REG}_f$. Combining these inclusions results in $R_f(\mathrm{FIN}_f) \subseteq (\mathrm{FIN}_f)_\infty \cup \mathrm{REG}_f$ which completes the proof. □

We are now ready for the main result of this paper.

Theorem 5.6. *(1) If K is a fuzzy prequasoid, then $R_f(K) = K_\infty \cup \mathrm{REG}_f$.*
(2) A family K is a full substitution-closed AFFL if and only if K is a fuzzy prequasoid and $R_f(K) = K$.

Proof. (1) follows immediately from Lemmas 5.4 and 5.5.

(2) Let K be a full substitution-closed AFFL. Obviously, K is a fuzzy prequasoid, and $K \supseteq \mathrm{REG}_f$. Hence by (1), $K = K_\infty = R_f(K)$.

Conversely, let K be a fuzzy prequasoid satisfying $R_f(K)=K$. Then $\mathrm{FIN}_f \subseteq K$, and from (1) it follows that $\mathrm{REG}_f \subseteq R_f(K)$ and that $K = R_f(K) = K_\infty$. Hence by Proposition 3.4(2), K is a full substitution-closed AFFL. □

Combining Theorem 5.3 and Theorem 5.6.(1) for $K = \mathrm{FIN}_f$ yields the following analogue of Corollary 4.4.

Corollary 5.7. *For each* $m \geq 1$, $R_{f,1}(\mathrm{FIN}_f) = R_{f,m}(\mathrm{FIN}_f) = R_f(\mathrm{FIN}_f) = \mathrm{REG}_f$. □

6 Concluding Remarks

In this paper we studied the effect of the non-self-embedding restriction on the generating power of fuzzy context-free K-grammars. This type of grammar is a generalization of both the fuzzy context-free grammars of [19] and the context-free K-grammars of [21, 22]. Clearly, our results applies to these two kinds of grammars as well: in the former case we take K equal to FIN_f (Corollary 5.7), in the latter one we require that K is a family of crisp languages (cf. [9]).

Our main result (Theorem 5.6) together with Proposition 3.4.(2) imply the following counterpart of Theorem 4.6 with respect to non-self-embedding fuzzy nested iterated substitution. Remember that $\Pi_f(k)$ denotes the smallest fuzzy prequasoid that includes K; cf. Section 4.

Theorem 6.1. *(1) If* K *is a fuzzy prequasoid, then* $R_f(K)$ *is a full substitution-closed AFFL.*
(2) For each arbitrary family K *of fuzzy languages,* $R_f\Pi_f(K)$ *is the smallest full substitution-closed AFFL that includes* K.

Proof. (1) Let K be a fuzzy prequasoid but not a fuzzy quasoid. Then $K = \mathrm{FIN}_f$, and by Corollary 5.7, $R_f(K) = R_f(\mathrm{FIN}_f) = \mathrm{REG}_f$ which is a full substitution-closed AFFL [8].

Let now K be a fuzzy quasoid. Then K contains all fuzzy regular languages, and Theorem 5.6.(1) yields $R_f(K) = K_\infty$ which is a full substitution-closed AFFL according to Proposition 3.4.(2).

(2) Let $\hat{\mathcal{R}}_f(K)$ be the smallest full substitution-closed AFFL that includes K. By the inclusion $K \subseteq \hat{\mathcal{R}}_f(K)$ and the monotonicity of both R_f and Π_f, we have $R_f\Pi_f(K) \subseteq R_f\Pi_f\hat{\mathcal{R}}_f(K)$. According to Theorem 5.6 this yields $R_f\Pi_f(K) \subseteq \hat{\mathcal{R}}_f(K)$. But following Theorem 5.6.(1), $R_f\Pi_f(K)$ is a full substitution-closed AFFL that includes K. Hence $\hat{\mathcal{R}}_f(K) = R_f\Pi_f(K)$. □

According to Theorem 5.6 a family of fuzzy languages K is a full substitution-closed AFFL if and only if $\Pi_f(K) = K$ and $R_f(K) = K$. Therefore, for $\hat{\mathcal{R}}_f(K)$ we have $\hat{\mathcal{R}}_f(K) = \bigcup\{w(K) \mid w \in \{\Pi_f, R_f\}^\star\}$ or, written slightly differently, $\hat{\mathcal{R}}_f(K) = \{\Pi_f, R_f\}^\star(K)$. Now Theorem 6.2 tells us that this infinite set of strings over $\{\Pi_f, R_f\}$ can be reduced to the single string $R_f\Pi_f$.

In a way similar to Corollary 4.7 we obtain as a consequence of Theorem 6.1 the following fact.

Corollary 6.2. REG_f *is the smallest full substitution-closed AFFL.* □

Comparing the results in Sections 4, 5 and 6 rises the question whether a uniform approach to closure properties of families of fuzzy languages, similar to the one of families of crisp languages as in [3], is possible.

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