Probabilistic Properties of Highly Connected Random Geometric Graphs

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In this paper we study the probabilistic properties of reliable networks of minimal total edge lengths. We study reliability in terms of k-edge-connectivity in graphs in d-dimensional space. We show this problem fits into Yukich's framework for Euclidean functionals for arbitrary k, dimension d and distant-power gradient p, with $p < d$. With this framework several theorems on the convergence of optimal solutions follow. We apply Yukich's framework for functionals so that we can use partitioning algorithms that rapidly compute near-optimal solutions on typical examples. These results are then extended to optimal k-edge-connected power assignment graphs, where we assign power to vertices and charge per vertex. The network can be modelled as a wireless network.

1 Introduction

The design of fault tolerant networks is an important issue in today's research, due to their numerous applications [1]. The goal is to find cheap and reliable networks with some specific characteristics. Reliability is often expressed in terms of the connectivity of a network. For example, we might want to have multiple paths between each pair of nodes to account for possible failures in a link.

Wireless ad hoc networks have also received significant attention in recent studies [4, 6]. Instead of direct connections between nodes, communication takes place through single-hop transmissions or by relaying through intermediate nodes. Here we assign a transmission power to each node. As transmission range is directly related to power usage and therefore to battery lifetime, the goal is to find a fault tolerant network with minimal total power usage.

Finding a cheapest k-edge-connected network is NP-hard [5], and so is finding a minimal power wireless network [3]. As we still want to have reasonably good solutions in acceptable computation time, we need to find good heuristics. We fit the problems into Yukich's framework for Euclidean functional [8] to get limit theorems and concentration results, as well as using them for analysis of the partitioning algorithm.

Partitioning algorithms have shown a lot of potential with similar problems [2]. In practice, partitioning algorithms are very fast. Partitioning algorithms divide the whole problems into smaller cells and compute optimal solutions on these. Then these solutions are joined to obtain a solution for the whole problem.

2 Definitions and Results

All graphs in this paper are undirected and simple. Let $G = (V, E)$ be a graph. We assume $V \subset \mathbb{R}^d$, where d is a constant and V is finite. The cost of an edge is its length raised to the power of the distant-power gradient $p > 0$. So adding edge (u, v) to a graphs increases the cost by $|(u, v)|^p$, where $|(u, v)|$ denotes the Euclidean distance between u and v. Here, we assume p is a constant.

A graph is k-edge-connected if the graph is still connected when at most $k - 1$ edges are removed, or if it is complete. The latter is to make sure k-edge-connected graphs on less than $k + 1$ nodes still exist, which saves us from dealing with all kind of exceptions in proofs. Alternatively, a network is k-edge-connected if there exist at least k edge-disjoint paths between every pair of vertices.

Let $d \in \mathbb{N}$ be arbitrary and let $p > 0$. Then $\mathrm{MkEE}^p(V)$ is the minimal length of a k-edge-connected graph in terms of summed edge lengths on V with pth power-weighted edges. Thus

$$\mathrm{MkEE}^p(V) = \min_{X \in \mathcal{S}(V,k)} \sum_{e \in X} |e|^p, \tag{1}$$

where $\mathcal{S}(V, k)$ is the set of k-edge-connected simple graphs on V and $|e|$ denotes the Euclidean length of an edge e. Following Yukich [8], we call MkEE^p a functional.

One of the desired properties for functionals is subadditivity. Roughly speaking, this shows that the function value of a whole set is not larger than the sum of function values of the sets in a partition of this set (with some error term).

Theorem 1. *For $p \geq 1$, MkEE^p is geometrically subadditive, i.e. for all finite sets V, all rectangles R and all partitions of R into rectangles R_1 and R_2 we have*

$$\mathrm{MkEE}^p(V \cap R) \leq \mathrm{MkEE}^p(V \cap R_1) + \mathrm{MkEE}^p(V \cap R_2) + C_1(\mathrm{diam}\, R)^p, \tag{2}$$

where $C_1 = C_1(d, p)$ is a constant.

We would also want MkEE^p to be superadditive. Roughly speaking, this would show that the function value of a whole set is not lower than the sum of function values of the sets in a partition. Combining sub- and superadditivity makes the functional nearly additive in the sense that $\mathrm{MkEE}^p(F, R) \approx \mathrm{MkEE}^p(F, R_1) + \mathrm{MkEE}^p(F, R_2)$.

We could then approximate the optimal solution value of the whole set by the sum of optimal solutions on its partitions. It is easily checked however that MkEE^p does not possess superadditivity. This is why we introduce the canonical boundary functional, an idea first articulated in Redmond's thesis [7]. In boundary functionals, the entire boundary of the rectangle is considered as one additional vertex that can be used. We also refer to Yukich [8] for more on this topic.

MkEE^p_B is the boundary functional of MkEE^p, so that $\mathrm{MkEE}^p_B(V \cap R)$ is the minimal length of a k-edge-connected boundary graph in terms of summed edge lengths on $V \cup \partial R$ in d-dimensional rectangle R with pth power-weighted edges. Here ∂R denotes the boundary of R. A vertex v is connected to ∂R by adding edge (v, v_∂) where $v_\partial = \arg\min_{w \in \partial R} |(v, w)|$.

Theorem 2. *For $p \geq 1$, MkEE^p_B is a superadditive functional, i.e. for all finite sets V, all rectangles R and all partitions of R into rectangles R_1 and R_2 we have*

$$\mathrm{MkEE}^p_B(V \cap R) \geq \mathrm{MkEE}^p_B(V \cap R_1) + \mathrm{MkEE}^p_B(V \cap R_2). \tag{3}$$

As we cannot directly show near additivity, we want to show that MkEE^p and MkEE^p_B are pointwise close. Then we would get approximately get sub- and superadditivity for both functionals.

Theorem 3. *For $1 \leq p < d$, MkEE^p is pointwise close to MkEE^p_B, i.e. for all finite sets $V \subset [0,1]^d$ we have*

$$|\,\mathrm{MkEE}^p(V) - \mathrm{MkEE}^p_B(V)| = o(|V|^{(d-p)/d}). \tag{4}$$

We have shown geometric subadditivity, superadditivity and pointwise closeness, creating a powerful set of properties. These properties are more useful for obtaining other results when the functional also is smooth. This describes how strong the variations of a functional are if vertices are added or deleted. Smooth functionals behave a lot more predictable and therefore it plays an important role in many limit theories.

Theorem 4. *For $1 \leq p < d$, MkEE^p is smooth, i.e. for all finite sets U and V we have*

$$|\,\mathrm{MkEE}^p(U \cup V) - \mathrm{MkEE}^p(U)| = O(|V|^{(d-p)/d}). \tag{5}$$

One of the concentration results we have obtained is stated below. It shows that the functional values are not far from their expected value.

Theorem 5. *For $1 \leq p < d$ and $k \in \mathbb{N}$, there exists a constant $\alpha = \alpha(d,k) \geq 0$ such that*

$$\lim_{n\to\infty} \mathrm{MkEE}^p(V,R)/n^{(d-p)/p} = \alpha \quad \text{c.c., and} \tag{6}$$

$$\lim_{n\to\infty} \mathrm{MkEE}^p_B(V,R)/n^{(d-p)/p} = \alpha \quad \text{c.c.,} \tag{7}$$

where $n = |V|$. Here c.c. denotes complete convergence.

3 Partitioning algorithm

In a partitioning algorithm, the Euclidean plane is divided into a number of cells that all contain only a few points. On each cell an optimal solution is calculated. This is generally much faster than calculating a solution on all points at once, as these problems are often NP-hard. The solutions of all cells are then joined to obtain a solution for the whole set.

We implement a partitioning scheme for MkEE^p having a polynomial running time, for which we derive approximation guarantees.

Algorithm 6 (Partitioning Scheme)**.**
Input: *set $V \subseteq [0,1]^d$ of n points and number of points per cell s*

1. *Partition $[0,1]^d$ into $\ell = \sqrt[d]{n/s}$ stripes of dimension $d-1$ such that each stripe contains exactly $n/\ell = (n^{d-1}s)^{1/d}$ points.*
2. *Keep partitioning each $i+1$-dimensional stripe into ℓ stripes of dimension i such that each stripe contains exactly $n/\ell^i = (n^{d-i}s^i)^{1/d}$ points. Stop at $i = 1$ so that each 2-dimensional stripe is partitioned into ℓ cells with $n/\ell^d = s$ points. In this way we end up with $\ell^d = n/s$ cells. Here we assume $s > k$.*
3. *Compute a graph achieving the optimal solution of MkEE^p for each cell.*
4. *Join the graphs to obtain a k-edge-connected graph on V.*

It can be easily verified that the graph we get as an output from Algorithm 6 is k-edge-connected. With this algorithm and the properties obtained in Section 2 we can now give running time and approximation guarantees. Depending on the way we compute the optimal solution on each cell, we need to vary s to get a polynomial running-time.

Theorem 7. *If the algorithm for computing an optimal solution on each cell in Algorithm 6 has a running time of $O(C^{n^2})$ for some constant C, the Partitioning Scheme has a polynomial running time if we choose $s = O(\sqrt{\log n})$. The approximation guarantee then becomes* $\mathrm{MkEE}^p(V) + O((n/s)^{(d-p)/d})$ *for k-edge-connected graphs.*

4 Extention to wireless networks

Besides our model for wired networks, we consider a different model for wireless networks. These are defined by assigning power to each vertex. A power assignment PA assigns a real, positive value to all vertices $v \in V$. The corresponding power assignment graph then contains all edges (u, v) for which $\mathrm{PA}(u), \mathrm{PA}(v) \geq |(u, v)|^p$. The costs of k-edge-connected power assignment graphs is then simply the sum of all assigned powers. We obtain results similar to Theorems 1 – 5 for the functional in wireless networks.

References

[1] Fatiha Bendali, I Diarrassouba, Ali Ridha Mahjoub, M Didi Biha, and Jean Mailfert, *A branch-and-cut algorithm for the k-edge connected subgraph problem*, Networks **55** (2010), no. 1, 13–32.

[2] Markus Bläser, Bodo Manthey, and BV Raghavendra Rao, *Smoothed analysis of partitioning algorithms for euclidean functionals*, Algorithmica **66** (2013), no. 2, 397–418.

[3] Andrea EF Clementi, Paolo Penna, and Riccardo Silvestri, *On the power assignment problem in radio networks*, Mobile Networks and Applications **9** (2004), no. 2, 125–140.

[4] Maurits de Graaf and Bodo Manthey, *Probabilistic analysis of power assignments*, International Symposium on Mathematical Foundations of Computer Science, Springer, 2014, pp. 201–212.

[5] Michael R Gary and David S Johnson, *Computers and intractability: A guide to the theory of np-completeness*, 1979.

[6] Ram Ramanathan and Regina Rosales-Hain, *Topology control of multihop wireless networks using transmit power adjustment*, INFOCOM 2000. Nineteenth Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings. IEEE, vol. 2, IEEE, 2000, pp. 404–413.

[7] Charles Redmond, *Boundary rooted graphs and euclidean matching algorithms*, Ph.D. thesis, Lehigh University, Bethlehem, PA, USA, 1993.

[8] Joseph E Yukich, *Probability theory of classical euclidean optimization problems*, Lecture Notes in Mathematics, vol. 1675, Springer-Verlag, Berlin, 1998.