2020 American Control Conference
Denver, CO, USA, July 1-3, 2020

Semi-global state synchronization for multi-agent systems subject to actuator saturation and unknown nonuniform input delay

Meirong Zhang[1], Ali Saberi[2], Anton A. Stoorvogel[3]

Abstract— **This paper studies semi-global state synchronization of homogeneous networks with diffusive full-state coupling subject to actuator saturation and unknown nonuniform input delay. We assume that agents are at most critical unstable, that is the agents have all its eigenvalues in the closed left-half complex plane. The communication network is associated with an undirected and weighted graph. In this paper, we derive an upper bound for the input delay tolerance, which explicitly depends on the agent dynamics. Moreover, for any unknown delay less than the upper bound, we propose a linear static protocol for MAS based on a low-gain methodology such that state synchronization is achieved among agents for any initial conditions in a priori given compact set.**

I. Introduction

Synchronization problem in a multi-agent system has received considerable attention. The main reason is that it has prospective applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The purpose of synchronization is to ensure an asymptotic agreement on a common state or output trajectory through distributed control protocols (see [1], [10], [13] and references therein).

In the applications, the network model is always imperfect. In particular, time-delay effects are pervasive in any communication scheme. As explained in [3], we can categorize two kinds of time delay: input delay and communication delay. Input delay is established from processing time to generate an input for each agent while communication delay refers to the time consumed during the transfer of information between agents. Most efforts have been put into input delay problems (see [2], [8], and [21] for example). These references, although including results on linear and non-linear agents, are mostly restricted to simple agent models such as first/second-order dynamics. Recently, in [23] and [24], the synchronization problem under unknown uniform constant input delay is solved for both discrete- and continuous-time high-order linear agents that are critically unstable. This work has been recently extended to unknown nonuniform input delay in [33]. In the case of communication delay, some results can be found. [20] and [27] consider single-integrator dynamics in the network and it is demonstrated that the communication delay does not affect the synchronizability of the network. [11] and [12] give the consensus conditions for networks with higher-order but SISO dynamics. In [8], second-order dynamics are investigated, but the communication delays are assumed known. Recently, [4] and [5] dealt with nonlinear heterogeneous MAS with unknown nonuniform constant communication delay where they solved a delayed synchronization problem.

It should be also noted that actuator saturation is pretty common and indeed is ubiquitous in engineering applications. For MAS in the presence of input saturation, usually two problems are addressed: global synchronization and semi-global synchronization. Global stabilization for MAS with full-state coupling has been studied by [9] (continuous) and [28] (discrete) for neutrally stable agents. [6] has considered the global case of partial-state coupling, using an adaptive approach. Semi-global synchronization has been studied in [16] and [17] in the case of full-state coupling. For partial state coupling, there are [15], [18] and [26]. All of these papers actually require extra communication and agents to be introspective. [29] considers non-introspective agents but still requires the extra communication. So far we only find [19] that deals with non-introspective agents and requires no extra communication. However, that paper requires the solution of a nonconvex optimization problem as part of the design of a dynamic protocol. Moreover, an underlying assumption basically requires the agents to be passifiable via input feedforward. We notice that all these papers assume that the network is either undirected or is so-called detailed balanced (a slightly weaker condition than undirected). One paper dealing with networks that are not detailed balanced is in [7], which intrinsically requires the agents to be single integrator. In [22] semi-global stabilization with full-state coupling has been studied for networks which only need to contain a directed spanning tree. Moreover, the agents are not introspective. Recently, [32] addressed the semiglobal state synchronization for both general continuous/discrete-time MASs with full-state coupling or partial-state coupling, where the network is directed and contains a spanning tree.

The objective of this paper is to extend the works in [33] to the case in the presence of saturation using the idea from [32] and [31]. The idea is to avoid the activation of input saturation by squeezing the input control signal for a priori given compact initial conditions; hence the semiglobal synchronization problem becomes a normal synchronization problem. However, there exist two big differences between this paper and [32] and [31]. One is that the squeezing of the input in this paper is on a general MAS system with nonuniform input delay, rather than on a general MAS

[1]Meirong Zhang is with School of Engineering and Applied Science, Gonzaga University, Spokane,WA, USA zhangm@gonzaga.edu

[2]Ali Saberi with School of Electrical Engineering and Computer Science, Washington State University, Pullman,WA, USA saberi@eecs.wsu.edu

[3]Anton A. Stoorvogel is with Department of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, Enschede, The Netherlands A.A.Stoorvogel@utwente.nl

system. The other is that the techniques and arguments regarding how to squeeze the input are completely different. In this paper, to solve the synchronization problem without the saturation activated is directedly referred to [33] to avoid redundancy. In this paper, we derive an upper bound for the input delay tolerance, which is only dependent on the agent dynamics. Then, for any unknown input delay satisfying the upper bound, we design a linear static protocol based on a low-gain methodology, such that state synchronization is achieved among agents for any initial conditions in a priori given compact set. In particular, the saturation can be avoided by tuning a low-gain parameter in the protocols. Moreover, the protocols are designed not only for a specific network, but for a set of networks. Only the upper bound and lower bound of associated Laplacian matrices are needed for the protocol design. The additional communication of controller states is also dispensed in this paper.

This paper has a discrete-time version, which can be found in the paper [30].

A. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm, and $\lambda_i(A)$ denotes its i'th eigenvalue when $m = n$. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. We denote by $\mathrm{diag}\{a_1,\dots,a_N\}$, a diagonal matrix with a_i $(i = 1,\dots,N)$ as the diagonal elements, and by $\mathrm{col}\{x_1,\dots,x_N\}$, a column vector with x_i $(i = 1,\dots,N)$ stacked together. $A\otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix, and $\mathbf{0}_n$ (or $\mathbf{1}_n$) denotes zero (or one) column or row vector. Sometimes we drop the subscript if the dimension is clear from the context. Given a system $G(j\omega)$, $\|G(j\omega)\|_\infty$ is the H_∞ norm of the system. Suppose two functions $f : X \to Y$ and $g : Y \to Z$. Then, $(g\circ f) : X \to Z$ is the composite function, meaning $(g\circ f)(x) = g(f(x))$. Moreover, let $\mathfrak{C}^n_{\bar\tau} := \mathbb{C}([-\bar\tau,0],\mathbb{R}^n)$ denote the Banach space of all continuous functions from $[-\bar\tau,0] \to \mathbb{R}^n$ with norm $\|x\|_C = \sup_{t\in[-\bar\tau,0]} \|x(t)\|$.

A *weighted graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{V} = \{1,\dots,N\}$ is a node set, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix, with $a_{ij} > 0$ iff $(j,i) \in \mathcal{E}$ and $a_{ii} = 0$. If $a_{ij} = a_{ji}$ for all $i,j \in \{1,\dots,N\}$, the graph is called *undirected*; otherwise *directed*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1,\dots,i_k\}$ such that $(i_j,i_{j+1}) \in \mathcal{E}$ for $j = 1,\dots,k-1$. An undirected graph is *connected* if there exists a path between every pair of nodes. For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^N a_{ik}, & i = j,\\ -a_{ij}, & i \neq j,\end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. All eigenvalues of L are located in the closed right-half complex plane with at least one eigenvalue at zero which is associated with right eigenvector $\mathbf{1}$. When $\mathcal{G}$ is undirected, L is symmetric.

II. Problem formulation

Consider a multiagent system (MAS) composed of N identical linear time-invariant agents subject to actuator saturation and unknown nonuniform input delay,

$$\begin{cases} \dot x_i(t) = Ax_i(t) + B\sigma(u_i(t-\tau_i)),\\ x_i(\varsigma) = \phi_i(\varsigma), \quad \varsigma\in[-\bar\tau,0]\end{cases} \tag{1}$$

for $i = 1,\dots,N$, where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are respectively the state and input vectors of agent i, while $\tau_i \in [0,\bar\tau]$ is an unknown constant, $\bar\tau$ is a known upper bound, and $\phi_i \in \mathfrak{C}^n_{\bar\tau}$. Moreover,

$$\sigma(u_i(t-\tau_i)) = \begin{pmatrix}\mathrm{sat}(u_{i,1}(t-\tau_i))\\ \vdots\\ \mathrm{sat}(u_{i,m}(t-\tau_i))\end{pmatrix} \text{ with } u_i = \begin{pmatrix}u_{i,1}\\ \vdots\\ u_{i,m}\end{pmatrix} \tag{2}$$

with $\mathrm{sat}(u)$ being the standard saturation function,

$$\mathrm{sat}(u) = \mathrm{sgn}(u)\min\{1,|u|\}.$$

The communication network provides each agent with a linear combination of its own states relative to that of other neighboring agents. In particular, each agent $i \in \{1,\dots,N\}$ has access to the quantity,

$$\zeta_i = \sum_{j=1}^N a_{ij}(x_i - x_j), \tag{3}$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$ indicate the communication among agents. This communication topology of the network can be described by an undirected weighted graph $\mathcal{G}$ with nodes corresponding to the agents in the network and the weight of edges given by the coefficient a_{ij}. In terms of the coefficients of the Laplacian matrix L, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^N \ell_{ij}x_j. \tag{4}$$

We refer to this network as with *full-state coupling*.

We make the following standard assumption for the agent dynamics.

Assumption 1: We assume that (A,B) is stabilizable and the agents are at most critically unstable, that is A has all its eigenvalues in the closed left-half complex plane.

Definition 1: We define the following network graph sets.

- Let $\mathbb{G}^N$ denote the set of undirected, weighted, and connected graphs with N nodes ,
- For any given $\alpha \geq \beta > 0$, let $\mathbb{G}^N_{\alpha,\beta}$ denote the set of undirected, weighted, and connected graphs with N nodes and for which the corresponding Laplacian matrix L has nonzero real eigenvalues, denoted by $\lambda_2,\dots,\lambda_N$, which satisfy $\beta < \lambda_i < \alpha$.

Definition 2: We also define $\omega_{\max}$ as

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Hurwitz},\\ \max\{\omega\in\mathbb{R} \mid \det(j\omega I - A) = 0\}, & \text{otherwise}\end{cases}$$

The state synchronization problem is formulated here.

Problem 1: Consider a MAS described by (1) and (3) with a given upper bound $\bar{\tau}$ for the input delay. Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G} \subseteq \mathbb{G}^N$. The *semi-global state synchronization* problem with a set of network graphs $\mathbf{G}$ is to find, if possible, for any a priori given bounded set of initial conditions $\mathcal{W} \subset \mathfrak{C}^n_{\bar{\tau}}$, a parameterized family of linear protocols of the form,

$$u_i = F_\delta \zeta_i, \quad (i = 1, \dots, N) \tag{5}$$

where F_δ is the feedback gain and there exists a δ^* such that for all $\delta < \delta^*$, state synchronization among agents is achieved for any graph $\mathcal{G} \in \mathbf{G}$ and for any input delay $\tau_i \in [0, \bar{\tau}]$ and any initial conditions $\phi_i \in \mathcal{W}$ for $i = 1, \dots, N$.

III. Protocol design

In this section, we design following parameterized family of protocols for continuous-time MAS subject to input saturation and unknown nonuniform input delay.

$$u_i = \gamma F_\delta \zeta_i, \tag{6}$$

where $F_\delta = -B' P_\delta$ with $P_\delta > 0$ being the unique solution of the continuous-time algebraic Riccati equation

$$A' P_\delta + P_\delta A - P_\delta B B' P_\delta + \delta I = 0, \tag{7}$$

where $\delta > 0$ is a low-gain parameter.

Before the main result, we first need the following lemmas.

Lemma 1: Suppose (A, B) is stabilizable and all the eigenvalues of A are in the closed left half plane. Let $F_\delta = -B' P_\delta$ be designed with P_δ given in (7). Then, we have the following properties:

1) The closed-loop system matrix $A + \nu\lambda B F_\delta$ is Hurwitz stable for all $\delta > 0$, $\nu = 1/\beta$, and for all $\lambda > \beta$.
2) For any $\beta > 0$, there exists a $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ there exist $r_{1,\delta} > 0$ and $\rho_\delta > 0$ with $r_{1,\delta} \to 0$ as $\delta \to 0$ such that
$$\left\| F_\delta e^{(A+\nu\lambda B F_\delta)t} \right\| \leq r_{1,\delta} e^{-\rho_\delta t}, \tag{8}$$
for all $t \geq 0$, $\nu = 1/\beta$, and for all $\lambda > \beta$.
3) We have
$$\left\| \nu\lambda F_\delta (sI - A - \nu\lambda B F_\delta)^{-1} B \right\|_\infty < 2 \tag{9}$$
for all $\delta > 0$, $\nu = 1/\beta$, for all $\lambda > \beta$.

Remark 1: Note that $F_\delta \to 0$ as $\delta \to 0$ is not sufficient for (8). We need that the input signal is small for all $t \geq 0$. In [14, Definition 4.73], we basically established this property for one fixed value of λ.

Proof: Consider $\dot{x} = (A + \nu\lambda B F_\delta)x$. It is seen that

$$\frac{d}{dt} x'(t) P_\delta x(t) \leq -\delta x'(t) x(t) \leq -2\rho_\delta x'(t) P_\delta x(t),$$

using that $\lambda \geq \beta$ where $\rho_\delta = \frac{1}{2}\delta \|P_\delta\|^{-1}$. Hence

$$\|P_\delta^{1/2} x(t)\| \leq e^{-\rho_\delta t} \|P_\delta^{1/2} x(0)\|. \tag{10}$$

Finally,

$$\begin{aligned} \|F_\delta e^{(A+\nu\lambda B F_\delta)t} x(0)\| &= \|B' P_\delta x(t)\| \\ &\leq \|B\| \|P_\delta^{1/2}\| e^{-\rho_\delta t} \|P_\delta^{1/2} x(0)\|. \end{aligned} \tag{11}$$

Since (11) is true for all $x(0) \in \mathbb{R}^n$, it follows trivially that

$$\|F_\delta e^{(A+\nu\lambda B F_\delta)t}\| \leq \|B\| \|P_\delta^{1/2}\|^2 e^{-\rho_\delta t} = \|B\| \|P_\delta\| e^{-\rho_\delta t}, \tag{12}$$

The proof of the inequality (8) is then completed by taking $r_{1,\delta} = \|B\| \|P_\delta\|$.

Using the Riccati equation, we have

$$(sI - A - \nu\lambda B F_\delta)^* P_\delta + P_\delta (sI - A - \nu\lambda B F_\delta) - P_\delta B B' P_\delta - \delta I \geq 0.$$

Postmultiplying with $\sqrt{\nu\lambda}(sI - A - \nu\lambda B F_\delta)^{-1} B$ and premultiplying with its complex conjugate we get

$$\begin{aligned} &G(s) + G(s)^* + G(s)^* G(s) \leq \\ &-\delta\nu\lambda \left[(sI - A - \nu\lambda B F_\delta)^{-1} B \right]^* \left[(sI - A - \nu\lambda B F_\delta)^{-1} B \right] \leq 0, \end{aligned}$$

where $G(s) = \nu\lambda F_\delta (sI - A - \nu\lambda B F_\delta)^{-1} B$. This yields $[G(s) + I]^* [G(s) + I] \leq I$, which leads to $\|G(s) + I\|_\infty \leq 1$, and hence $\|G(s)\|_\infty \leq 2$. ■

Lemma 2: Assume L is associated with an undirected graph. If $L = R_e J_e R_e'$ with R_e unitary and $J_e = \operatorname{diag}\{J, 0\}$ with J is diagonal, then we have

$$T_1 L T_2 = R J R^{-1}, \tag{13}$$

with $R = T_1 R_e T_2$ and $R^{-1} = T_2' R_e' T_2$, where $T_1 \in \mathbb{R}^{(N-1)\times N}$ and $T_2 \in \mathbb{R}^{N\times(N-1)}$ are given by

$$T_1 = \begin{pmatrix} I & -\mathbf{1}_{N-1} \end{pmatrix}, \quad T_2 = \begin{pmatrix} I \\ \mathbf{0}_{N-1} \end{pmatrix}.$$

Proof: Since $L\mathbf{1}_N = 0$, we have

$$L T_2 T_1 = L \begin{pmatrix} I & -\mathbf{1}_{N-1} \\ 0 & 0 \end{pmatrix} = L \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} - L \begin{pmatrix} 0 & \mathbf{1}_N \end{pmatrix} = L.$$

Therefore, $L T_2 T_1$ has $N - 1$ nonzero eigenvalues, i.e., $\lambda_2, \dots, \lambda_N$. Then $T_1 L T_2$ has the same $N - 1$ nonzero eigenvalues and hence $T_1 L T_2$ is invertible. Define $R = T_1 R_e T_2$.

We have

$$R_e = \begin{pmatrix} R_{11} & \frac{1}{\sqrt{N}}\mathbf{1} \\ R_{21} & \frac{1}{\sqrt{N}} \end{pmatrix},$$

given that $L\mathbf{1} = 0$. Then, it is found that

$$\begin{aligned} R T_2' R_e' &= T_1 R_e T_2 T_2' R_e' = T_1 \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix} \begin{pmatrix} R_{11}' & R_{21}' \end{pmatrix} \\ &= T_1 \begin{pmatrix} R_{11} & \frac{1}{\sqrt{N}}\mathbf{1}_{N-1} \\ R_{21} & \frac{1}{\sqrt{N}} \end{pmatrix} \begin{pmatrix} R_{11}' & R_{21}' \\ \frac{1}{\sqrt{N}}\mathbf{1}_{N-1}' & \frac{1}{\sqrt{N}} \end{pmatrix} \\ &= T_1 R_e R_e' = T_1. \end{aligned}$$

The third equality holds because $T_1 \begin{pmatrix} \mathbf{1}_{N-1} \\ 1 \end{pmatrix} = 0$. Therefore, $R^{-1} T_1 = T_2' R_e'$, which yields $R^{-1} = T_2' R_e' T_2$, and moreover

$$\begin{aligned} T_1 L T_2 R &= T_1 L T_2 T_1 R_e T_2 = T_1 L R_e T_2 \\ &= T_1 R_e J_e T_2 = T_1 R_e T_2 J = R J. \end{aligned}$$

■

Lemma 3: Suppose (A, B) is stabilizable and all the eigenvalues of A are in the closed left half plane. Let $F_\delta = -B'P_\delta$ be designed with P_δ given in (7). Then, we have,

$$\left\|\nu(J\otimes F_\delta)e^{(I_{N-1}\otimes A+\nu J\otimes BF_\delta)t}\right\| \leq r_{2,\delta}, \quad (14)$$

where J is the Jordan form of matrix T_1LT_2 and L is the Laplacian matrix with its associated graph in $\mathbb{G}^N_{\alpha,\beta}$, and moreover, $r_{2,\delta} \to 0$ as $\delta \to 0$.

Proof: Consider a system

$$\dot{x} = (I_{N-1}\otimes A + \nu T_1LT_2\otimes BF_\delta)x \quad (15)$$

and let $\eta := (R^{-1}\otimes I_n)x = \begin{pmatrix}\eta_2 \cdots \eta_N\end{pmatrix}'$. Then, the dynamics of η can be written as

$$\dot{\eta} = (I_{N-1}\otimes A + \nu J\otimes BF_\delta)\eta$$

with

$$\dot{\eta}_i = (A + \nu\lambda_i BF_\delta)\eta_i$$

for $i = 2, \ldots, N$. Then, following the results of the above Lemma 1, we can achieve $\|F_\delta\eta_i(t)\| \leq \tilde{r}_{i,\delta}$ with $\tilde{r}_{i,\delta} \to 0$ as $\delta \to 0$ for $i = 2, \ldots, N$. Let $r_{2,\delta} = \dfrac{\alpha}{\beta}\max\{\tilde{r}_{2,\delta}, \ldots, \tilde{r}_{N,\delta}\}$. We have

$$\left\|\nu(J\otimes F_\delta)e^{(I_{N-1}\otimes A+\nu J\otimes BF_\delta)t}\right\| \leq r_{2,\delta},$$

which completes the proof. ■

The main result for continuous-time MAS with full-state coupling is stated as follows.

Theorem 1: Consider a MAS described by(1) and (3) with an input delay upper bound $\bar{\tau}$ and input saturation. Let any $\alpha > \beta > 0$ be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.

If (A, B) is stabilizable and the agents are at most weakly unstable, then the semi-global state synchronization problem stated in Problem 1 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable if

$$\bar{\tau}\omega_{\max} < \tfrac{\pi}{2}. \quad (16)$$

Moreover, for any a priori given compact set of initial conditions $\mathcal{W} \subset \mathfrak{C}^n_{\bar{\tau}}$, there exist $\gamma > 0$ and $\delta^* > 0$ such that for this γ and any $\delta \in (0, \delta^*]$, the protocol (6) achieves state synchronization for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, for any input delay $\tau_i \in [0, \bar{\tau}]$, and for any initial condition $\phi_i \in \mathcal{W}$ for $i = 1, \ldots, N$.

Proof: Let D_i $(i = 1, \ldots, N)$ be a delay operator for agent i such that $(D_iu_i)(t) = u_i(t - \tau_i)$. In the frequency domain, $\hat{D}_i(\omega) = e^{-j\omega\tau_i}$. Define $x = \text{col}\{x_1, \ldots, x_N\}$, $u = \text{col}\{u_1, \ldots, u_N\}$ and $D = \text{diag}\{D_1, \ldots, D_N\}$ and $\hat{D}(\omega) = \text{diag}\{\hat{D}_1(\omega), \ldots, \hat{D}_N(\omega)\}$, the overall dynamics of multiagent system described by (1) and (3) can be represented by

$$\begin{cases} \dot{x}(t) = (I_N\otimes A)x(t) + (I_N\otimes B)\sigma(Du(t)), \\ u(t) = (L\otimes\gamma F_\delta)x(t), \end{cases} \quad (17)$$

If the input $u(t) = (L\otimes\gamma F_\delta)x(t)$ can be squeezed small enough, i.e. the input can avoid triggering saturation, the overall dynamics (17) becomes a system without saturation

$$\begin{cases} \dot{x}(t) = (I_N\otimes A)x(t) + (I_N\otimes B)Du(t), \\ u(t) = (L\otimes\gamma F_\delta)x(t). \end{cases} \quad (18)$$

The synchronization of (18) has been proved in [33, Theorem 1], and we will show that the synchronization of (17) by establishing that the system does not saturate provided δ is small enough.

Now define $\bar{x}_i = x_i - x_N$ and $\bar{x} = \text{col}\{\bar{x}_1, \ldots, \bar{x}_{N-1}\}$. Since $u_i = \gamma F_\delta \sum \ell_{ij}(x_j - x_N) = \gamma F_\delta \sum \ell_{ij}\bar{x}_j$, we have

$$u = (LT_2\otimes\gamma F_\delta)\bar{x} \quad (19)$$

and

$$\dot{\bar{x}} = (I_{N-1}\otimes A)\bar{x} + \gamma(T_1DLT_2)\otimes BF_\delta)\bar{x}. \quad (20)$$

The following step is to show that we can avoid the saturation if $(T_1LT_2\otimes\gamma F_\delta)\bar{x}(t)$ is sufficiently small. Applying Cauchy-Schwatz inequality, we can prove that for any $t \geq 0$,

$$\begin{aligned} &\left|\|(T_1LT_2\otimes\gamma F_\delta)\bar{x}(t)\|^2 - \|(T_1LT_2\otimes\gamma F_\delta)\bar{x}(0)\|^2\right| \\ &\qquad \leq 2\|(T_1LT_2\otimes\gamma F_\delta)\dot{\bar{x}}\|_2\|(T_1LT_2\otimes\gamma F_\delta)\bar{x}\|_2, \end{aligned}$$

which means that it is sufficient to establish that

$$\|(T_1LT_2\otimes\gamma F_\delta)\bar{x}\|_2 \leq r_{3,\delta} \quad (21)$$

and

$$\|(T_1LT_2\otimes\gamma F_\delta)\dot{\bar{x}}\|_2 \leq r_{4,\delta} \quad (22)$$

where $r_{3,\delta} \to 0$ and $r_{4,\delta} \to 0$ as $\delta \to 0$.

Now we define the linear time-invariant operator $g_\delta : v_\delta \to w_\delta$ with the state space representation:

$$\begin{cases} \dot{\xi} = (I_{N-1}\otimes A + \nu(J\otimes BF_\delta))\xi + (I_{N-1}\otimes B)v_\delta, \\ w_\delta = \nu(J\otimes F_\delta)\xi. \end{cases} \quad (23)$$

We also define another linear time-invariant operator ϑ by:

$$\begin{aligned} g(t) &= \vartheta(f)(t) \\ &= \begin{cases} \left[R^{-1}T_1(\frac{\gamma}{\nu}D(\omega) - I)R_eT_2\otimes I_m\right]f(t) & \text{if } t > \bar{\tau} \\ \left[-R^{-1}T_1R_eT_2\otimes I_m\right]f(t) & \text{otherwise} \end{cases} \end{aligned}$$

We can see that the Laplace transform of these two operators are given by

$$\begin{aligned} G_\delta(j\omega) &= \nu(J\otimes F_\delta)(j\omega I - (I_{N-1}\otimes A) \\ &\qquad - \nu(J\otimes BF_\delta))^{-1}(I_{N-1}\otimes B), \\ \Delta(j\omega) &= R^{-1}T_1(\frac{\gamma}{\nu}\hat{D}(\omega) - I)R_eT_2\otimes I_m \\ &= T_2'R_e'(\frac{\gamma}{\nu}\hat{D}(\omega) - I)R_eT_2\otimes I_m. \end{aligned}$$

Given (19), we notice that u is small if and only if $(T_1LT_2\otimes\gamma F_\delta)\bar{x}$ is small, since $\text{Im}\, L \perp \ker T_1$.

Next, define $\tilde{x} = (R^{-1}\otimes I_n)\bar{x}$. Then the dynamics of $\tilde{x}$ can be written as, for $t \geq 0$

$$\begin{aligned} \dot{\tilde{x}} &= (I_{N-1}\otimes A + \nu(J\otimes BF_\delta))\tilde{x} \\ &\quad + (I_{N-1}\otimes B)\vartheta(\nu(J\otimes F_\delta)\tilde{x}) + (I_{N-1}\otimes B)v_\delta, \end{aligned} \quad (24)$$

where

$$v_\delta(t) = \begin{cases} (-\nu J\otimes F_\delta)\begin{pmatrix}\phi_1(t-\bar{\tau}) \\ \vdots \\ \phi_N(t-\bar{\tau})\end{pmatrix} & t < \bar{\tau}, \\ 0 & t \geq \bar{\tau}. \end{cases}$$

Moreover, u is small if $(J \otimes \gamma F_\delta)\tilde{x}$ is small.

Note that v_δ vanishes for $t \geq \bar{\tau}$ and $\phi \in \mathcal{W}$. Moreover, since $F_\delta \to 0$, we have $\|v_\delta\|_\infty \to 0$ and $\|v_\delta\|_2 \to 0$ as $\delta \to 0$ for all $\phi \in \mathcal{W}$.

From (24), we obtain

$$\nu(J \otimes F_\delta)\tilde{x}(t) = \nu(J \otimes F_\delta)e^{(I_{N-1}\otimes A+\nu(J\otimes BF_\delta))t}\tilde{x}(0) + (g_\delta \circ \vartheta)(\nu(J \otimes F_\delta)\tilde{x})(t) + g_\delta(v_\delta)(t)$$

and hence

$$\nu(J \otimes F_\delta)\tilde{x}(t) = (1 - g_\delta \circ \vartheta)^{-1} \left[\nu(J \otimes F_\delta)e^{(I_{N-1}\otimes A+\nu(J\otimes BF_\delta))t}\tilde{x}(0) + g_\delta(v_\delta)(t)\right]. \quad (25)$$

From (25), we get

$$\|\nu(J \otimes F_\delta)\tilde{x}\|_2 \leq \|(I - \Delta(j\omega)G_\delta(j\omega))^{-1}\|_\infty \quad (26)$$
$$\left\|\nu(J \otimes F_\delta)e^{(I_{N-1}\otimes A+\nu(J\otimes BF_\delta))t}\tilde{x}(0)\right\|_2 + \|(I - \Delta(j\omega)G_\delta(j\omega))^{-1}\|_\infty \|w_\delta\|_2. \quad (27)$$

Taking a derivative of (25), we obtain

$$\nu(J \otimes F_\delta)\dot{\tilde{x}}(t) = (1 - g_\delta \circ \vartheta)^{-1}\left[\nu(J \otimes F_\delta) e^{(I_{N-1}\otimes A+\nu(J\otimes BF_\delta))t}(I_{N-1} \otimes A + \nu(J \otimes BF_\delta))\tilde{x}(0) + \dot{w}_\delta\right].$$

Similarly, let $\tilde{\tilde{x}}(0) = (I_{N-1} \otimes A + \nu(J \otimes BF_\delta))\tilde{x}(0)$ and then we have

$$\|\nu(J \otimes F_\delta)\dot{\tilde{x}}\|_2 \leq \|(I - \Delta(j\omega)G_\delta(j\omega))^{-1}\|_\infty \quad (28)$$
$$\left\|\nu(J \otimes F_\delta)e^{(I_{N-1}\otimes A+\nu(J\otimes BF_\delta))t}\tilde{\tilde{x}}(0)\right\|_2 + \|(I - \Delta(j\omega)G_\delta(j\omega))^{-1}\|_\infty \|\dot{w}_\delta\|_2. \quad (29)$$

In order to show that the upper bound in (26) and (28) converge to 0 as $\delta \to 0$, we first note that (14) is satisfied. Next, we analyze $\|w_\delta\|_2$ and $\|\dot{w}_\delta\|_2$. According to the definition of operator g_δ, we have $w_\delta = g_\delta(v_\delta)(t)$. Then, $\|w_\delta\|_2 \leq \|G_\delta\|_\infty \|v_\delta\|_2 \leq 2\|v_\delta\|_2$. Therefore, for any given initial condition $\phi_i \in \mathcal{W}$ $(i = 1, \ldots, N)$, $\|w_\delta\|_2 \to 0$ as $\delta \to 0$.

Next, we will prove $\|\dot{w}_\delta\|_2 \to 0$ as $\delta \to 0$. For $t \in [0, \bar{\tau}]$, the derivative of $w_\delta(t)$ is

$$\begin{aligned}\dot{w}_\delta(t) &= \nu(J \otimes F_\delta)(I_{N-1} \otimes A + \nu(J \otimes BF_\delta))\xi(t) \\ &\quad + \nu(J \otimes F_\delta)(I_{N-1} \otimes B)v_\delta(t) \\ &= \nu(J \otimes F_\delta)(I_{N-1} \otimes A + \nu(J \otimes BF_\delta)) \\ &\quad \int_0^t e^{(I_{N-1}\otimes A+\nu(J\otimes BF_\delta))(t-r)}v_\delta(r)dr \\ &\quad + \nu(J \otimes F_\delta)(I_{N-1} \otimes B)v_\delta(t).\end{aligned}$$

Since $(I_{N-1} \otimes A + \nu(J \otimes BF_\delta))$ is bounded for all $0 < \delta < 1$ and $\|v_\delta\|_\infty \to 0$ as $\delta \to 0$, we will have

$$\sup_{t\in[0,\bar{\tau}]} \|\dot{w}_\delta(t)\| \to 0 \quad \text{as } \delta \to 0, \quad (30)$$

which also implies

$$\int_0^{\bar{\tau}} \|\dot{w}_\delta(t)\|^2 dt \to 0 \quad \text{as } \delta \to 0. \quad (31)$$

Note that

$$\xi(\bar{\tau}) = \int_0^{\bar{\tau}} e^{(I_{N-1}\otimes A+\nu(J\otimes BF_\delta))(\bar{\tau}-r)}v_\delta(r)dr$$

is bounded since $\|v_\delta\|_\infty \to 0$ as $\delta \to 0$. Then, for $t > \bar{\tau}$, v_δ vanishes and

$$\dot{w}_\delta(t) = \nu(J \otimes F_\delta)(I_{N-1} \otimes A + \nu(J \otimes BF_\delta)) e^{(I_{N-1}\otimes A+\nu(J\otimes BF_\delta))t}\xi(\bar{\tau}),$$

which implies (see [25])

$$\int_{\bar{\tau}}^{\infty} \|\dot{w}_\delta(t)\|^2 dt \to 0 \quad \text{as } \delta \to 0. \quad (32)$$

Combining (31) and (32) gives $\|\dot{w}_\delta(t)\|_2 \to 0$ as $\delta \to 0$.

From (26) and (28), we can obtain (21) and (22) provided that we have $\|(I - \Delta(j\omega)G_\delta(j\omega))^{-1}\|_\infty$ is bounded independent of δ, using that $(T_1 L T_2 \otimes \gamma F_\delta)\bar{x}$ is small if and only if $(J \otimes \gamma F_\delta)\tilde{x}$ is small.

Since $\bar{\tau}\omega_{\max} < \frac{\pi}{2}$, we can choose γ such that

$$\gamma\beta\cos(\bar{\tau}\omega_{\max}) > 1. \quad (33)$$

Note that this γ is independent of low-gain parameter δ and condition (33) implies that $\gamma\beta > 1$. Let this γ be fixed during the remaining proof.

Given (33), there exists a $\varpi > 0$ such that

$$\gamma\beta\cos(\bar{\tau}(\omega_{\max} + \varpi)) > 1. \quad (34)$$

For $|\omega| < \omega_{\max} + \varpi$, we find that

$$\begin{aligned}\Delta(j\omega) + \Delta(j\omega)' &= T_2' R_e' (\frac{\gamma}{\nu}\hat{D}(\omega) - I)R_e T_2 \otimes I_m \\ &\quad + T_2' R_e' (\frac{\gamma}{\nu}\hat{D}(\omega)' - I)R_e T_2 \otimes I_m \\ &= T_2' R_e' (\frac{\gamma}{\nu}\hat{D}(\omega) + \frac{\gamma}{\nu}\hat{D}(\omega)' - 2I)R_e T_2 \otimes I_m \\ &\geq T_2' R_e' (2\gamma\beta\cos(\omega\bar{\tau}) - 2)R_e T_2 \otimes I_m \geq 0,\end{aligned}$$

because γ is chosen to satisfy $\gamma\beta\cos(\bar{\tau}(\omega_{\max} + \varpi)) > 1$ in (34). Furthermore, we obtain that

$$\begin{aligned}\|\Delta(j\omega)\|_\infty &= \left\|T_2' R_e' (\frac{\gamma}{\nu}\hat{D}(\omega) - I)R_e T_2 \otimes I_m\right\|_\infty \\ &= \left\|(\frac{\gamma}{\nu}\hat{D}(\omega) - I)R_e T_2 T_2' R_e' \otimes I_m\right\|_\infty \\ &\leq \left\|(\frac{\gamma}{\nu}\hat{D}(\omega) - I) \otimes I_m\right\|_\infty \leq 1 + \frac{\gamma}{\nu},\end{aligned}$$

since $\|\hat{D}(\omega)\| \leq 1$. Then, we have that, for $|\omega| < \omega_{\max} + \varpi$

$$\begin{aligned}\Delta(j\omega)'\Delta(j\omega) &\leq \Delta(j\omega)'\Delta(j\omega) + \Delta(j\omega) + \Delta(j\omega)' \\ &\quad + \left[I - (2 + \frac{\gamma}{\nu})^{-2}(I + \Delta(j\omega)')(I + \Delta(j\omega))\right] \\ &\leq \left[1 - (2 + \frac{\gamma}{\nu})^{-2}\right](I + \Delta(j\omega)')(I + \Delta(j\omega)),\end{aligned}$$

which leads to

$$(I+\Delta(j\omega)')^{-1}\Delta(j\omega)'\Delta(j\omega)(I+\Delta(j\omega))^{-1} \leq \left[1 - (2 + \frac{\gamma}{\nu})^{-2}\right] I.$$

Hence, there exists a $\rho > 0$ that is independent of parameter δ, such that $\|\Delta(j\omega)(I + \Delta(j\omega))^{-1}\| \leq 1 - \rho$. Moreover, from Property 3 in Lemma 1, we can immediately obtain that

$\|G_\delta(j\omega)\| < 2$, and $\|I + G_\delta(j\omega)\| < 1$. Then, it follows that, for $|\omega| < \omega_{\max} + \varpi$

$$\begin{aligned}
&\det\left[I - \Delta(j\omega)G_\delta(j\omega)\right] \\
&= \det\left[I + \Delta(j\omega) - \Delta(j\omega)(I + G_\delta(j\omega))\right] \\
&= \det(I + \Delta(j\omega)) \\
&\qquad \det\left[I - (I + \Delta(j\omega))^{-1}\Delta(j\omega)(I + G_\delta(j\omega))\right] \\
&\geq 1 - (1 - \rho) = \rho,
\end{aligned}$$

which means that, for some ϱ that is independent of parameter δ, $\underline{\sigma}(I - \Delta(j\omega)G_\delta(j\omega)) > \varrho$ for all $|\omega| < \omega_{\max} + \varpi$, for all $\tau_i \in [0, \bar{\tau}]$ and all possible L associated with a network graph in $\mathbb{G}^N_{\alpha,\beta}$. Therefore, we have that, for $|\omega| < \omega_{\max} + \varpi$

$$\left\|(I - \Delta(j\omega)G_\delta(j\omega))^{-1}\right\| \leq \frac{1}{\varrho}. \tag{35}$$

For $|\omega| \geq \omega_{\max} + \varpi$, we know that $\Delta(j\omega)G_\delta(j\omega) \to 0$ as $\delta \to 0$ uniformly in ω. Therefore, we can conclude that for a small enough δ, we have (35) for $|\omega| \geq \omega_{\max} + \varpi$. This completes the proof of Theorem 1.

■

IV. Conclusion

In this paper, we developed a static protocol to achieve semi-global state synchronization for identical agents with input saturation and delay. Compared to the existing literature, the agent dynamics are more realistic, which means a potential application in the real world. In the future, we will consider extend the research to nonidentical agents and directed communication network, even time-varying network.

References

[1] H. Bai, M. Arcak, and J. Wen. *Cooperative control design: a systematic, passivity-based approach*. Communications and Control Engineering. Springer Verlag, 2011.

[2] P. Bliman and G. Ferrari-Trecate. Average consensus problems in networks of agents with delayed communications. *Automatica*, 44(8):1985–1995, 2008.

[3] Y. Cao, W. Yu, W. Ren, and G. Chen. An overview of recent progress in the study of distributed multi-agent coordination. *IEEE Trans. on Industrial Informatics*, 9(1):427–438, 2013.

[4] N. Chopra. Output synchronization on strongly connected graphs. *IEEE Trans. Aut. Contr.*, 57(1):2896–2901, 2012.

[5] N. Chopra and M. K. Spong. Output synchronization of nonlinear systems with time delay in communication. In *Proc. 45th CDC*, pages 4986–4992, San Diego, CA, 2006.

[6] H. Chu, J. Yuan, and W. Zhang. Observer-based consensus tracking for linear multi-agent systems with input saturation. *IET Control Theory and Applications*, 9(14):2124–2131, 2015.

[7] Y. Li, J. Xiang, and W. Wei. Consensus problems for linear time-invariant multi-agent systems with saturation constraints. *IET Control Theory and Applications*, 5(6):823–829, 2011.

[8] P. Lin and Y. Jia. Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies. *Automatica*, 45(9):2154–2158, 2009.

[9] Z. Meng, Z. Zhao, and Z. Lin. On global leader-following consensus of identical linear dynamic systems subject to actuator saturation. *Syst. & Contr. Letters*, 62(2):132–142, 2013.

[10] M. Mesbahi and M. Egerstedt. *Graph theoretic methods in multiagent networks*. Princeton University Press, Princeton, 2010.

[11] U. Münz, A. Papachristodoulou, and F. Allgöwer. Delay robustness in consensus problems. *Automatica*, 46(8):1252–1265, 2010.

[12] U. Münz, A. Papachristodoulou, and F. Allgöwer. Delay robustness in non-identical multi-agent systems. *IEEE Trans. Aut. Contr.*, 57(6):1597–1603, 2012.

[13] W. Ren and Y.C. Cao. *Distributed Coordination of Multi-agent Networks*. Communications and Control Engineering. Springer-Verlag, London, 2011.

[14] A. Saberi, A.A. Stoorvogel, and P. Sannuti. *Internal and external stabilization of linear systems with constraints*. Birkhäuser, Boston, MA, 2012.

[15] H. Su and M.Z.Q. Chen. Multi-agent containment control with input saturation on switching topologies. *IET Control Theory and Applications*, 9(3):399–409, 2015.

[16] H. Su, M.Z.Q. Chen, and G. Chen. Robust semi-global coordinated tracking of linear multi-agent systems with input saturation. *Int. J. Robust & Nonlinear Control*, 25(14):2375–2390, 2015.

[17] H. Su, M.Z.Q. Chen, J. Lam, and Z. Lin. Semi-global leader-following consensus of linear multi-agent systems with input saturation via low gain feedback. *IEEE Trans. Circ. & Syst.-I Regular papers*, 60(7):1881–1889, 2013.

[18] H. Su, G. Jia, and M.Z.Q. Chen. Semi-global containment control of multi-agent systems with intermittent input saturation. *Journal of the Franklin Institute*, 352(9):3504–3525, 2015.

[19] K. Takaba. A dynamic protocol for local synchronization of linear multi-agent systems subject to input saturation. In *Proc. 54th CDC*, pages 4923–4927, Osaka, Japan, 2015.

[20] Y.-P. Tian and C.-L. Liu. Consensus of multi-agent systems with diverse input and communication delays. *IEEE Trans. Aut. Contr.*, 53(9):2122–2128, 2008.

[21] Y.-P. Tian and C.-L. Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations. *Automatica*, 45(5):1347–1353, 2009.

[22] Q. Wang, C. Yu, and H. Gao. Semiglobal synchronization of multiple generic linear agents with input saturation. *Int. J. Robust & Nonlinear Control*, 24(18):3239–3254, 2014.

[23] X. Wang, A. Saberi, A.A. Stoorvogel, H.F. Grip, and T. Yang. Consensus in the network with uniform constant communication delay. *Automatica*, 49(8):2461–2467, 2013.

[24] X. Wang, A. Saberi, A.A. Stoorvogel, H.F. Grip, and T. Yang. Synchronization in a network of identical discrete-time agents with uniform constant communication delay. *Int. J. Robust & Nonlinear Control*, 24(18):3076–3091, 2014.

[25] X. Wang, A.A. Stoorvogel, A. Saberi, H.F. Grip, and P. Sannuti. H_2 and H_∞ low-gain theory. In *American Control Conference*, pages 4463–4469, San Francisco, CA, 2011.

[26] X. Wang, H. Su, X. Wang, and G. Chen. An overview of coordinated control for multi-agent systems subject to input saturation. *Perspectives in Science*, 7:133–139, 2016.

[27] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Trans. Aut. Contr.*, 53(8):1804–1816, 2008.

[28] T. Yang, Z. Meng, D.V. Dimarogonas, and K.H. Johansson. Global consensus for discrete-time multi-agent systems with input saturation constraints. *Automatica*, 50(2):499–506, 2014.

[29] L. Zhang, M.Z.Q. Chen, and H. Su. Observer-based semi-global consensus of discrete-time multi-agent systems with input saturation. *Transactions of the Institute of Measurement and Control*, 38(6):665–674, 2016.

[30] M. Zhang, A. Saberi, and A. Stoorvogel. Semi-global state synchronization for discrete-time multi-agent systems subject to actuator saturation and unknown nonuniform input delay. *European Journal of Control*. available at: https://doi.org/10.1016/j.ejcon.2019.12.006.

[31] M. Zhang, A. Saberi, and A.A. Stoorvogel. Semi-global state synchronization for continuous or discrete-time multiagent systems subject to actuator saturation. In *American Control Conference*, pages 3380–3385, Milwaukee, WI, 2018.

[32] M. Zhang, A. Saberi, and A.A. Stoorvogel. Semiglobal state synchronization for continuous- or discrete-time multiagent systems subject to actuator saturation. *Int. J. Robust & Nonlinear Control*, 28(16):4966–4980, 2018.

[33] M. Zhang, A. Saberi, and A.A. Stoorvogel. Synchronization in a network of identical continuous- or discrete-time agents with unknown nonuniform constant input delay. *Int. J. Robust & Nonlinear Control*, 28(13):3959–3973, 2018.