



Editing to a planar graph of given degrees [☆]



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ABSTRACT

We consider the following graph modification problem. Let the input consist of a graph $G = (V, E)$, a weight function $w: V \cup E \rightarrow \mathbb{N}$, a cost function $c: V \cup E \rightarrow \mathbb{N}_0$ and a degree function $\delta: V \rightarrow \mathbb{N}_0$, together with three integers k_v , k_e and C . The question is whether we can delete a set of vertices of total weight at most k_v and a set of edges of total weight at most k_e so that the total cost of the deleted elements is at most C and every non-deleted vertex v has degree $\delta(v)$ in the resulting graph G' . We also consider the variant in which G' must be connected. Both problems are known to be NP-complete and W[1]-hard when parameterized by $k_v + k_e$. We prove that, when restricted to planar graphs, they stay NP-complete but have polynomial kernels when parameterized by $k_v + k_e$.

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1. Introduction

Graph modification problems capture a variety of fundamental graph-theoretic problems, and as such they are very well studied in algorithmic graph theory. The aim is to modify some given graph G into some other graph H , that satisfies a *certain property*, by applying at most some given number operations from a set S of *prespecified graph operations*. Well-known graph operations are the edge addition, edge deletion and vertex deletion, denoted by ea , ed and vd , respectively. For example, if $S = \{vd\}$ and H must be a clique or independent set, then we obtain two basic graph problems, namely CLIQUE and INDEPENDENT SET, respectively. To give a few more examples, if H must be a forest and either $S = \{ed\}$ or $S = \{vd\}$, then we obtain the problems FEEDBACK EDGE SET and FEEDBACK VERTEX SET, respectively. As we discuss in detail later, it is also common to consider sets S consisting of more than one graph operation.

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A property is *hereditary* if it is closed under taking induced subgraphs. A property is *non-trivial* if it is both true for infinitely many graphs and false for infinitely many graphs. A classic result of Lewis and Yannakakis [24] is that the vertex deletion problem is NP-hard for any property that is both hereditary and non-trivial. In an earlier paper, Yannakakis [33] also showed NP-hardness results for the edge deletion problem for several properties, such as being planar or outer-planar. Natanzon, Shamir and Sharan [29] and Burzyn, Bonomo and Durán [5] proved that the graph modification problem is NP-complete for several target hereditary graph properties when $S = \{ea, ed\}$.

As we can see from the above results, graph modification problems are often intractable even for elementary cases when $S \subseteq \{ea, ed\}$. As such, many papers in this area study the complexity of graph modification problems when parameterized by the total number of permitted operations k .

Cai [6] proved that the graph modification problem is FPT when parameterized by k , if $S = \{ea, ed, vd\}$ and the desired property is that of belonging to any fixed graph class characterized by a finite set of forbidden induced subgraphs. Khot and Raman [21] determined all non-trivial hereditary properties for which the vertex deletion problem is FPT on n -vertex graphs with parameter $n - k$ and proved that the problem is W[1]-hard with respect to this parameter for all other such properties.

From the aforementioned results we see that the graph modification problem has been thoroughly studied for hereditary properties. Several other natural types of properties have also been considered. For instance, Dabrowski et al. [9] combined a number of previous results [4,7,8] with new results to give a complete classification of the (parameterized) complexity of the problem of modifying an input graph into a connected graph where each vertex has some prescribed degree parity for every set $S \subseteq \{ea, ed, vd\}$.

1.1. Our focus

In this paper we consider the case when the vertices of the resulting graph must satisfy some prespecified degree constraints (note that such properties are non-hereditary, so the result of Lewis and Yannakakis does not apply to this case). This is a natural direction to consider given the classical structural results [25,32] on so-called f -factors in graphs, which are spanning subgraphs in which each vertex u must have degree $f(u)$ for some specified function f (these results immediately imply that an f -factor in a graph can be found in polynomial time if one exists, while finding connected f -factors, e.g. Hamilton cycles, is NP-complete).

Before presenting our results, we briefly discuss the known results and the general framework they fall under.

General framework. Moser and Thilikos in [28] and Mathieson and Szeider [27] initiated an investigation into the parameterized complexity of graph modification problems with respect to degree constraints. This leads to the following general problem.

DEGREE CONSTRAINT EDITING(S)

Instance: A graph G , integers d, k and a function $\delta: V(G) \rightarrow \{1, \dots, d\}$.

Question: Can G be modified into a graph G' such that $d_{G'}(v) = \delta(v)$ for each $v \in V(G')$ using at most k operations from the set S ?

Mathieson and Szeider [27] classified the parameterized complexity of this problem for $S \subseteq \{ea, ed, vd\}$. In particular they showed the following results. If $S \subseteq \{ea, ed\}$ then the problem is polynomial-time solvable. If $vd \in S$ then the problem is NP-complete, W[1]-hard with parameter k and FPT with parameter $d + k$. Moreover, they proved that the latter result holds even for a more general version, in which the vertices and edges have costs and the desired degree for each vertex should be in some given subset of $\{1, \dots, d\}$. If $\{v\} \subseteq S \subseteq \{ed, vd\}$, they proved that the problem has a polynomial kernel when parameterized by $d + k$ even if vertices and edges have costs. Recently, Mathieson [26] considered graph editing problems for a number of alternative forms of degree constraints. Golovach [19] considered the cases $S = \{ea, vd\}$ and $S = \{ea, ed, vd\}$ and proved (amongst other results) that for these cases the problem has no polynomial kernel when parameterized by $d + k$ unless $NP \subseteq coNP/poly$. Froese, Nichterlein and Niedermeier [14] gave more kernelization results for DEGREE CONSTRAINT EDITING(S).

Golovach [18] introduced a variant of DEGREE CONSTRAINT EDITING(S) with the extra condition that the resulting graph must be *connected*. He proved that, for $S = \{ea\}$, this variant is NP-complete, FPT when parameterized by k , and has a polynomial kernel when parameterized by $d + k$. The connected variant is readily seen to be W[1]-hard when $vd \in S$ by a straightforward modification of the proof of the W[1]-hardness result for DEGREE CONSTRAINT EDITING(S), when $vd \in S$, as given by Mathieson and Szeider [27].

Our results. In the light of the above NP-completeness and W[1]-hardness results when $vd \in S$ it is natural to restrict the input graph G to a special graph class. Hence, inspired by the above results, we consider the set $S = \{ed, vd\}$ and study both variants of these problems (where we insist that the resulting graph G' is connected and where we do not) for *planar* input graphs. The problem variant not demanding connectivity is defined as follows. (In fact the problems we study are slightly more general.)

PLANAR DEGREE CONSTRAINT DELETION

Instance: A planar graph $G = (V, E)$, integers k_v, k_e and a function $\delta: V \rightarrow \mathbb{N}_0$.

Question: Can G be modified into a graph G' such that $d_{G'}(v) = \delta(v)$ for each $v \in V(G')$ using at most k_v vertex deletions and at most k_e edge deletions?

We note that PLANAR DEGREE CONSTRAINT DELETION is NP-complete even if $\delta \equiv 3$ and that its connected variant is NP-complete even if $\delta \equiv 2$. These observations follow directly from the respective facts that both testing whether a planar graph of maximum degree at most 7 has a non-trivial cubic subgraph is NP-complete [31] and testing whether a cubic planar graph has a Hamiltonian cycle is NP-complete [15].

In contrast to the aforementioned W[1]-hardness results for general graphs, our two main results are that the weighted version of PLANAR DEGREE CONSTRAINT DELETION and its connected variant both have polynomial kernels when parameterized by $k_v + k_e$ (see Section 2.2 for the exact definition of these weighted versions). Note that by setting $k_v = 0$ or $k_e = 0$ we obtain the same results for DEGREE CONSTRAINED EDITING(S) when $S = \{\text{ed}\}$ and $S = \{\text{vd}\}$, respectively (though the $S = \{\text{ed}\}$ case is not surprising, since this problem is solvable in polynomial time on general graphs [27]).

In order to obtain our results we first show that both problems are polynomial-time solvable for any graph class of bounded treewidth. We then use a variant of the *protrusion decomposition/replacement* techniques introduced by Bodlaender et al. [3]. These techniques were successfully used for various problems on sparse graphs [13,16,17,22]. We stress that our problems do not fit into the meta-algorithmic framework of Bodlaender et al. [3] on kernelization. Our kernels require protrusion replacement machinery that is different from the general one in [3]. Hence our approach is, unavoidably, problem-specific.

2. Preliminaries

In this section we state terminology and notation used throughout the paper.

2.1. Basic terminology and notation

All graphs in this paper are finite, undirected and without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and the edge set is denoted by $E(G)$. For a set $X \subseteq V(G)$, we let $G[X]$ denote the subgraph of G induced by X . We let $G - X = G[V(G) \setminus X]$; note that we allow the case where $X \not\subseteq V(G)$. If $X = \{x\}$, we may write $G - x$ instead. For a set $L \subseteq E(G)$, we let $G - L$ be the graph obtained from G by deleting all edges of L . If $L = \{e\}$ then we may write $G - e$ instead. For $v \in V(G)$, let $E_G(v) = \{e \in E(G) \mid e \text{ is incident to } v\}$. For $X \subseteq V(G)$, let $E_G(X) = \bigcup_{v \in X} E_G(v)$. For $e \in E(G)$ with $e = uv$, let $V(e) = \{u, v\}$. For a set $L \subseteq E(G)$ let $V(L) = \bigcup_{e \in L} V(e)$.

Let G be a graph. For a vertex v , we let $N_G(v)$ denote its (*open*) *neighbourhood*, that is, the set of vertices adjacent to v . The *degree* of a vertex v is denoted by $d_G(v) = |N_G(v)|$. For a set $X \subseteq V(G)$, we write $N_G(X) = (\bigcup_{v \in X} N_G(v)) \setminus X$. The *closed neighbourhood* $N_G[v] = N_G(v) \cup \{v\}$, and for a non-negative integer r , $N_G^r[v]$ is the set of vertices at distance at most r from v ; note that $N_G^0[v] = \{v\}$ and that $N_G^1[v] = N_G[v]$. Let $X \subseteq V(G)$, and let r be a positive integer. We let $N_G^r[X] = \bigcup_{v \in X} N_G^r[v]$. The set X is an *r -dominating set* of G if $V(G) \subseteq N_G^r[X]$. We let $\partial_G(X) = X \cap N_G(V(G) \setminus X)$ be the *boundary* of X in G , i.e. the set of vertices in X that have neighbours G outside of X .

A *tree decomposition* of a graph G is a pair (\mathcal{X}, T) where T is a tree and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a collection of subsets (called *bags*) of $V(G)$ such that

- (i) $\bigcup_{i \in V(T)} X_i = V(G)$,
- (ii) for each edge $xy \in E(G)$, there is an $i \in V(T)$ such that $x, y \in X_i$, and
- (iii) for each $x \in V(G)$, the set $\{i \mid x \in X_i\}$ induces a connected subtree of T .

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ is $\max_{i \in V(T)} \{|X_i| - 1\}$. The *treewidth* of a graph G (denoted $\mathbf{tw}(G)$) is the minimum width over all tree decompositions of G . A tree decomposition (\mathcal{X}, T) of a graph G is *nice*, if T is a rooted binary tree such that the nodes of T are of four types:

- (i) a *leaf node* i is a leaf of T with $X_i = \emptyset$;
- (ii) an *introduce node* i has one child i' with $X_i = X_{i'} \cup \{v\}$ for some vertex $v \in V(G)$;
- (iii) a *forget node* i has one child i' with $X_i = X_{i'} \setminus \{v\}$ for some vertex $v \in V_G$; and
- (iv) a *join node* i has two children i' and i'' with $X_i = X_{i'} = X_{i''}$,

and, moreover, the root r is a forget node with $X_r = \emptyset$. Kloks [23] proved that every tree decomposition of a graph can be converted in linear time to a nice tree decomposition of the same width such that the size of the obtained tree is linear in the size of the original tree.

We need the following known observation. We include a simple proof.

Lemma 1. Let V_1 and V_2 be the bipartition classes of a planar bipartite graph G such that $d_G(v) \geq 3$ for every $v \in V_2$ and V_2 is non-empty. Then $|V_2| \leq 2|V_1| - 4$.

Proof. Let G be such a graph. Let $C(G)$ and $F(G)$ be the set of components and faces in G , respectively. Since G is bipartite, the border of every internal face of G must contain at least four edges. This also applies to the infinite outer face, since G contains a vertex of degree at least 2 (edges contained in the border of a face that are not part of a cycle are counted twice). Every edge is part of at most two faces. It follows that $4|F(G)| \leq 2|E(G)|$. Euler's Formula for planar graphs states that $|V(G)| - |E(G)| + |F(G)| - |C(G)| = 1$. Combining this with the above inequality, we find that $|E(G)| \leq 2|V(G)| - 2|C(G)| - 2 \leq 2|V(G)| - 4$.

Now $3|V_2| \leq \sum_{v \in V_2} d_G(v)$, since every vertex in V_2 has degree at least 3. We know that $\sum_{v \in V_2} d_G(v) = |E(G)|$ since G is bipartite. Combining these observations with the inequality found above implies that $3|V_2| \leq 2(|V_1| + |V_2|) - 4$. Therefore $|V_2| \leq 2|V_1| - 4$. \square

2.2. Full problem description

As mentioned in Section 1 the problems we solve are more general than PLANAR DEGREE CONSTRAINT DELETION and its connected variant. The generalizations we study are analogous to those used for other editing problems in the literature (see e.g. [27]). The unconnected variant that we solve is defined as follows:

DELETION TO A PLANAR GRAPH OF GIVEN DEGREES (DPGGD)

Instance: A planar graph $G = (V, E)$, integers k_v, k_e, C and functions $\delta: V \rightarrow \mathbb{N}_0, w: V \cup E \rightarrow \mathbb{N}, c: V \cup E \rightarrow \mathbb{N}_0$.

Question: Can G be modified into a graph G' by deleting a set $U \subseteq V$ with $w(U) \leq k_v$ and a set $D \subseteq E$ with $w(D) \leq k_e$ such that $c(U \cup D) \leq C$ and $d_{G'}(v) = \delta(v)$ for $v \in V(G')$?

In this problem, w is the *weight* and c is the *cost* function. The question is whether it is possible to delete vertices and edges of total weight at most k_v and k_e , respectively, so that the total cost of the deleted elements is at most C and the obtained graph satisfies the degree restrictions prescribed by the given function δ . Note that if we delete a vertex, the edges incident to that vertex can no longer be present, so they are automatically deleted; these edges do not contribute to the weight or cost of the solution. We include costs to make our results as general as possible. In particular note that the integer C is neither a constant nor a parameter, but part of the input. Adding costs in this way does not fundamentally complicate our proof. As the goal is to minimize the total costs, the costs for edges and vertices are combined. Besides costs, we also include weights, mainly for technical reasons.

We call the variant of DPGGD, in which the desired graph G' must be connected, the DELETION TO A CONNECTED PLANAR GRAPH OF GIVEN DEGREES problem (DCPGGD).

2.3. Protrusion decompositions

For a graph G and a positive integer r , a set $X \subseteq V(G)$ is an r -*protrusion* of G if $|\partial_G(X)| \leq r$ and $\mathbf{tw}(G[X]) \leq r$. For positive integers s and s' , an (s, s') -*protrusion decomposition* of a graph G (see also Fig. 1) is a partition $\Pi = \{R_0, \dots, R_p\}$ of $V(G)$ such that

- (i) $\max\{p, |R_0|\} \leq s$,
- (ii) for each $i \in \{1, \dots, p\}$, $R_i^+ = N_G[R_i]$ is an s' -protrusion of G , and
- (iii) for each $i \in \{1, \dots, p\}$, $N_G(R_i) \subseteq R_0 \cap \partial_G[R_i^+]$.

The sets R_1^+, \dots, R_p^+ are called the *protrusions* of Π . Originally, condition (iii) only demanded that $N_G(R_i) \subseteq R_0$ holds for each $i \in \{1, \dots, p\}$. However, we can move every vertex in $N_G(R_i) \setminus \partial_G[R_i^+]$ to R_i without affecting any of the other properties. Hence we assume without loss of generality that such vertices do not exist and may indeed state condition (iii) as above (which is convenient for our purposes). Note that if a vertex $v \in R_i^+$ has a neighbour outside of R_i^+ then $v \in N_G(R_i)$ by the definition of R_i^+ . It follows that every vertex of $\partial_G[R_i^+]$ also lies in $N_G(R_i)$ and therefore $N_G(R_i) = \partial_G[R_i^+]$.

The following statement is implicit in [3] (see Lemmas 6.1 and 6.2).

Lemma 2 ([3]). Let r and k be positive integers and let G be a planar graph that has an r -dominating set of size at most k . Then G has an $(O(kr), O(r))$ -protrusion decomposition, which can be constructed in polynomial time.

2.4. Parameterized complexity

Parameterized complexity is a two dimensional framework for studying the computational complexity of a problem. One dimension is the input size n and the other is a parameter k . A problem is said to be *fixed parameter tractable* (or FPT)

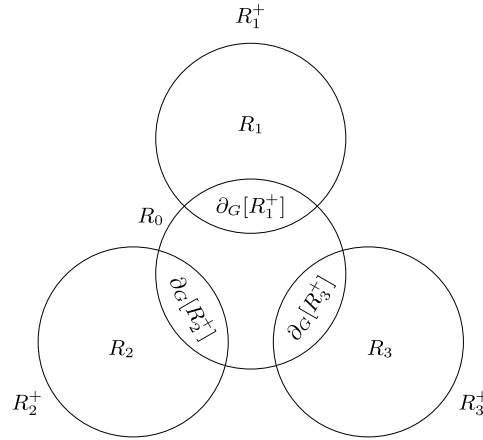


Fig. 1. A protrusion decomposition with $p = 3$. Recall that $R_i^+ = N_G[R_i]$ and note that the sets $\partial_G(R_i^+) = N_G(R_i)$ are not necessarily pairwise disjoint.

if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f . A *kernelization* for a parameterized problem is a polynomial algorithm that maps each instance (x, k) with input x and parameter k to an instance (x', k') such that

- (i) (x, k) is a yes-instance if and only if (x', k') is a yes-instance, and
- (ii) the size of x' and k' is bounded by $f(k)$ for a computable function f .

The output (x', k') is called a *kernel*. The function f is said to be the *size* of the kernel. A kernel is *polynomial* if f is polynomial. We refer to the books of Downey and Fellows [11], Flum and Grohe [12], and Niedermeier [30] for detailed introductions to parameterized complexity.

3. The polynomial kernels

In this section we construct polynomial kernels for DPGGD and DCPGGD. We say that a pair (U, D) with $U \subseteq V(G)$ and $D \subseteq E(G)$ is a *solution* for an instance $(G, k_v, k_e, C, \delta, w, c)$ of DPGGD if $w(U) \leq k_v$, $w(D) \leq k_e$, $c(U \cup D) \leq C$ and $G' = G - U - D$ satisfies $d_{G'}(v) = \delta(v)$ for all $v \in V(G')$. If $(G, k_v, k_e, C, \delta, w, c)$ is an instance of DCPGGD then (U, D) is a solution if in addition G' is connected. Notice that it can happen that $U = V(G)$ for a solution (U, D) .

In order to prove our main results, we first need to introduce some additional terminology and prove some structural results. We say that a solution (U, D) is of *minimum cost* if $c(\hat{U}, \hat{D}) \geq c(U, D)$ for every solution (\hat{U}, \hat{D}) . We say that a solution (U, D) for an instance of DPGGD or DCPGGD is *efficient* if D has no edges incident to the vertices of U . Since deleting a vertex automatically removes all incident edges with no weight or cost penalty, we can make the following observation.

Observation 1. Any yes-instance of DPGGD or DCPGGD has an efficient solution of minimum cost.

We will also make use of the following simple observation.

Observation 2. Let $(G, k_v, k_e, C, \delta, w, c)$ be instance of DPGGD or DCPGGD that has an efficient solution (U, D) . If $d_G(v) = \delta(v)$ for some $v \in V(G)$ then v is not incident to an edge of D .

We say that an instance $(G, k_v, k_e, C, \delta, w, c)$ of DPGGD (DCPGGD respectively) is *normalized* if

- (i) for every $v \in V(G)$, $\delta(v) \leq d_G(v) \leq \delta(v) + k_v + k_e$, and
- (ii) every vertex v in the set $S = \{u \in V(G) \mid d_G(u) = \delta(u)\}$ is adjacent to a vertex in $\bar{S} = V(G) \setminus S$.

Lemma 3. There is a polynomial-time algorithm that for each instance of DPGGD or DCPGGD either solves the problem or returns an equivalent normalized instance.

Proof. Let $(G, k_v, k_e, C, \delta, w, c)$ be an instance of DPGGD. To simplify notation, we keep the same notation for the functions δ, w, c if we delete vertices or edges and do not modify the values of the functions for the remaining elements if this does not create confusion.

We say that a reduction rule is *safe* if by applying the rule we either solve the problem or obtain an equivalent instance. It is straightforward to see that the following reduction rules are safe.

Yes-instance rule. If $S = V(G)$ then (\emptyset, \emptyset) is a solution, return a yes-answer and stop.

Vertex deletion rule. If G has a vertex v with $d_G(v) < \delta(v)$ or $d_G(v) > \delta(v) + k_v + k_e$, then delete v and set $k_v = k_v - w(v)$, $C = C - c(v)$. If $k_v < 0$ or $C < 0$, then stop and return a no-answer.

Observe that by the exhaustive application of the **vertex deletion rule** and applying the **yes-instance rule** whenever possible, we either solve the problem or we obtain an instance which satisfies (i) of the definition of normalized instances, but where $S \neq V(G)$. Notice that, in particular, the **yes-instance rule** is applied if the set of vertices becomes empty. To ensure (ii), we apply the following two rules.

Contraction rule. If G has two adjacent vertices $u, v \in S = \{x \in V(G) \mid d_G(x) = \delta(x)\}$ such that $N_G(v) \subseteq S$, then we construct the instance $(G', k_v, k_e, C, \delta', w', c')$ as follows.

- Contract uv . Denote the obtained graph $G' = G/uv$ and let z be the vertex obtained from u and v .
- Set $\delta'(z) = d_{G'}(z)$ and set $\delta'(x) = d_{G'}(x)$ for any $x \in S \setminus \{u, v\}$. For each $x \in \bar{S}$, set $\delta'(x) = \delta(x)$.
- Set $w'(z) = w(u) + w(v)$ and $c'(z) = c(u) + c(v)$. For $x \in V(G) \setminus \{u, v\}$, set $w'(x) = w(x)$ and $c'(x) = c(x)$.
- For each $xz \in E(G')$, set $w'(xz) = k_e + 1$ and $c'(xz) = 0$. For all other edges $xy \in E(G')$, set $w'(xy) = w(xy)$ and $c'(xy) = c(xy)$.

Let (U, D) be an efficient solution for $(G, k_v, k_e, C, \delta, w, c)$. By **Observation 2**, D has no edges incident to u or v . Also either $u, v \in U$ or $u, v \notin U$, because u and v are adjacent and $d_G(u) = \delta(u)$ and $d_G(v) = \delta(v)$. Let $U' = (U \setminus \{u, v\}) \cup \{z\}$ if $u, v \in U$ and $U' = U$ otherwise. We have that (U', D) is a solution for $(G', k_v, k_e, C, \delta', w', c')$. If (U', D') is an efficient solution for $(G', k_v, k_e, C, \delta', w', c')$, then D' has no edges incident to z by **Observation 2**. If $z \in U'$, let $U = (U' \setminus \{z\}) \cup \{u, v\}$ and $U = U'$ otherwise. We obtain that (U, D) is a solution for the original instance.

We exhaustively apply the above rule. Assume that it cannot be applied for $(G, k_v, k_e, C, \delta, w, c)$. Then we have that this instance satisfies (i) and the following holds: for any $v \in S \neq V(G)$, either v is adjacent to a vertex in \bar{S} or v is an isolated vertex. It remains to deal with isolated vertices.

Isolates removal rule. If G has an isolated vertex v , then delete v .

To see that above rule is safe, notice that, because the considered instance satisfies (i), it follows that $\delta(v) \leq d_G(v) = 0$, so $v \in S$. Clearly, by the exhaustive application of the **isolates removal rule**, we either solve the problem or obtain an instance that satisfies (i) and (ii).

Now consider an instance $(G, k_v, k_e, C, \delta, w, c)$ of DCPGGD.

We replace the **yes-instance rule** by the following variant.

Yes-instance rule (connected). If $S = V(G)$ and G is connected, then (\emptyset, \emptyset) is a solution, return a yes-answer and stop.

It is straightforward to verify that the **vertex deletion rule** and the **contraction rule** are safe for this problem. By applying these rules and by the application of the connected variant of the **yes-instance rule** whenever possible, we either solve the problem or obtain an equivalent instance that satisfies (i) and has the property that for any $v \in S$, either v is adjacent to a vertex in \bar{S} or v is an isolated vertex. Suppose that $(G, k_v, k_e, C, \delta, w, c)$ satisfies these properties. Observe that if H is a component of G , then for any solution (U, D) , either $V(H) \subseteq U$ or $V(G) \setminus V(H) \subseteq U$. Therefore, it is safe to apply the following variant of the **isolates removal rule**.

Isolates removal rule (connected). If G has an isolated vertex v , then if $w(V(G) \setminus \{v\}) \leq k_v$ and $c(V(G) \setminus \{v\}) \leq C$, then $(V(G) \setminus \{v\}, \emptyset)$ is a solution, return a yes-answer and stop. Otherwise, if $w(V(G) \setminus \{v\}) > k_v$ or $c(V(G) \setminus \{v\}) > C$, delete v and set $k_v = k_v - w(v)$ and $C = C - c(v)$; if $k_v < 0$ or $C < 0$, then stop and return a no-answer.

It is easy to see that if the input graph was planar then the graph formed after applying the rules above will also be planar. \square

Lemma 4. *If $(G, k_v, k_e, C, \delta, w, c)$ is a normalized yes-instance of DPGGD (DCPGGD respectively) then G has a 2-dominating set of size at most $k_v + 2k_e$.*

Proof. We prove the lemma for DPGGD; the proof for DCPGGD is the same. Let $(G, k_v, k_e, C, \delta, w, c)$ be a normalized yes-instance of the problem. Let (U, D) be a solution and $W = U \cup V(D)$. Clearly, $|W| \leq k_v + 2k_e$, because the weights are positive integers. We show that W is a 2-dominating set of G .

Let $S = \{v \in V(G) \mid d_G(v) = \delta(v)\}$ and $\bar{S} = V(G) \setminus S$. For any vertex $v \in \bar{S}$, either $v \in U$ or v is adjacent to a vertex of U or v is incident to an edge of D . Hence, $\bar{S} \subseteq N_G[W]$. Let $v \in S$. Because the considered instance is normalized, v is adjacent to a vertex $u \in \bar{S}$. It implies, that $S \subseteq N_G^2[W]$. \square

The following is a direct consequence of [Lemmas 2 and 4](#).

Lemma 5. *There is a fixed constant α such that, if $(G, k_v, k_e, C, \delta, w, c)$ is a normalized yes-instance of DPGGD (DCPGGD respectively), then G has an $(\alpha(k_v + 2k_e), \alpha)$ -protrusion decomposition. Moreover, if there is such a decomposition, one can be constructed in polynomial time.*

The next lemma states that, for both DPGGD and DCPGGD, an optimal solution can be found in polynomial time on graphs of bounded treewidth. The proof is based on the standard techniques for dynamic programming over tree decompositions.

Lemma 6. *DPGGD (DCPGGD respectively) can be solved, and an efficient solution (U, D) of minimum cost can be obtained in $(k_v + k_e)^{O(q)} \cdot \text{poly}(n)$ time (in $(q(k_v + k_e))^{O(q)} \cdot \text{poly}(n)$ time respectively) for instances $(G, k_v, k_e, C, \delta, w, c)$ where G is an n -vertex graph of treewidth at most q and $\delta(v) \leq d_G(v) \leq \delta(v) + k_v + k_e$ for $v \in V(G)$.*

Proof. We use a more or less standard approach for construction of dynamic programming algorithms for graphs of bounded treewidth.

First, we consider DPGGD. Let $(G, k_v, k_e, C, \delta, w, c)$ be an instance of the problem where $\text{tw}(G) \leq q$ and $\delta(v) \leq d_G(v) \leq \delta(v) + k_v + k_e$ for all $v \in V(G)$. We first of all assume that a nice tree decomposition (\mathcal{X}, T) of G with width $t = O(q)$ is given. To simplify later arguments, we may assume $t \geq 2$. For this, we may use the algorithm of [\[2\]](#) to obtain a decomposition whose width is at most five times the optimal in $2^{O(q)} \cdot n$ steps and then convert it to a nice tree decomposition using the aforementioned results of Kloks [\[23\]](#).

Let r denote the root of T . For any node $i \in V(T)$, let T_i denote the subtree of T induced by i and its descendants and let $G_i = G[\bigcup_{j \in V(T_i)} X_j]$. We apply a dynamic programming algorithm over (\mathcal{X}, T) .

First, we describe the tables that are constructed for the nodes of T . Let $i \in V(T)$. We define **table_i** as a partial function whose inputs are quintuples (X, Y, γ, h_v, h_e) where

- $X \subseteq X_i$,
- $Y \subseteq E(G[X_i])$,
- $\gamma : X_i \setminus X \rightarrow \{0, \dots, k_v + k_e\}$,
- $h_v \leq k_v$ and
- $h_e \leq k_e$.

The value of **table_i** is a minimum cost pair $(U, D) \in 2^{V(G_i)} \times 2^{E(G_i)}$ with the following properties:

- (i) for any $v \in U$ and any $e \in D$, v and e are not incident,
- (ii) $w(U) \leq h_v$ and $w(D) \leq h_e$,
- (iii) $U \cap X_i = X$ and $D \cap E(G[X_i]) = Y$,
- (iv) for every $v \in X_i \setminus X$, the number of neighbours of v in G_i that belong in $U \setminus X_i$ plus the number of edges of $D \setminus E(G[X_i])$ that are incident to v is exactly $\gamma(v)$,
- (v) for each $v \in V(G_i) \setminus X_i$, $d_{G'_i}(v) = \delta(v)$ where $G'_i = G_i - U - D$,

and, if no such pair (U, D) exists, then **table_i** (X, Y, d, h_v, h_e) is void.

Recall that $X_r = \emptyset$. Observe that $(G, k_v, k_e, C, \delta, w, c)$ is a yes-instance if and only if **table_r** $(\emptyset, \emptyset, \emptyset, k_v, k_e)$ is non-void (where $\emptyset : \emptyset \rightarrow \{0, \dots, k_v + k_e\}$). Moreover, in such a case, the value of **table_r** $(\emptyset, \emptyset, \emptyset, k_v, k_e)$ is a minimum-cost solution for this instance.

Now we explain how we construct **table_i** for each $i \in V(T)$. If i is a leaf node, **table_i** is constructed in a straightforward way because $X_i = \emptyset$. Indeed, for $0 \leq h_v \leq k_v$ and $0 \leq h_e \leq k_e$ we set **table_i** $(\emptyset, \emptyset, \emptyset, h_v, h_e) = (\emptyset, \emptyset)$ and have **table_i** void in all other cases. Hence, it remains to give the construction for *introduce*, *forget*, and *join* nodes. Let $i \in V(T)$ be a node of one of these types. Assume inductively that the function **table_{i'}** for every child i' of i has already been constructed.

In what follows we write **table_i** $(X, Y, \gamma, h_v, h_e) \leftarrow (U, D)$ to refer to the following procedure: If **table_i** (X, Y, γ, h_v, h_e) is undefined, set it to be equal to (U, D) . If **table_i** $(X, Y, \gamma, h_v, h_e) = (\hat{U}, \hat{D})$ and $c(\hat{U} \cup \hat{D}) > c(U \cup D)$, change **table_i** (X, Y, γ, h_v, h_e) to be equal to (U, D) . Otherwise, do not change **table_i** (X, Y, γ, h_v, h_e) .

Construction for an introduce node. Let i' be the child of i and $X_i = X_{i'} \cup \{v\}$. Notice that $N_{G_i}(v) \subseteq X_{i'}$. We start with **table_i** empty. Then, for each pair h_v, h_e where $h_v \leq k_v$ and $h_e \leq k_e$ and each pair $((X', Y', \gamma', h'_v, h'_e), (U', D')) \in \text{table}_{i'}$ where $h'_v \leq h_v$ and $h'_e \leq h_e$, we do the following:

- Let $X \leftarrow X' \cup \{v\}$, $Y \leftarrow Y'$, $\gamma \leftarrow \gamma'$, $U \leftarrow U' \cup \{v\}$, and $D \leftarrow D'$.
If $h_v \geq h'_v + w(v)$, then $\mathbf{table}_i(X, Y, \gamma, h_v, h_e) \leftarrow (U, D)$.
- Let $X \leftarrow X'$, $U \leftarrow U'$, $\gamma \leftarrow \gamma' \cup \{(v, 0)\}$.
For every $L \subseteq \{vu \mid vu \in E(G), u \in X_i \setminus X'\}$, let $Y \leftarrow Y' \cup L$, $D \leftarrow D' \cup L$, and if $h_e \geq h'_e + w(L)$, then $\mathbf{table}_i(X, Y, \gamma, h_v, h_e) \leftarrow (U, D)$.

Construction for a forget node. Let i' be the child of i and $X_i = X_{i'} \setminus \{v\}$. We start with \mathbf{table}_i empty. For each pair $((X', Y', \gamma', h_v, h_e), (U, D)) \in \mathbf{table}_{i'}$, we do the following.

- If $v \in X'$ then let $X \leftarrow X' \setminus \{v\}$, $Y \leftarrow Y'$, and define γ by replacing in γ' each pair $(u, \gamma'(u))$ where $uv \in E(G)$ and $u \in X_i \setminus X$ by the pair $(u, \gamma'(u) + 1)$.
If $\max_{u \in X_i \setminus X} \gamma(u) \leq k_v + k_e$, then $\mathbf{table}_i(X, Y, \gamma, h_v, h_e) \leftarrow (U, D)$.
- If $v \notin X'$, then let $X \leftarrow X'$, $L \leftarrow \{vu \in E(G) \mid u \in X_i\} \cap Y'$, $Y \leftarrow Y' \setminus L$, and define γ by replacing in $\gamma^- = \gamma' \setminus \{(v, \gamma'(v))\}$ each pair $(u, \gamma'(u))$ where $uv \in L$ by the pair $(u, \gamma'(u) + 1)$.
If $\delta(v) = d_G(v) - |L| - \gamma'(v)$ and $\max_{u \in X_i \setminus X} \gamma(u) \leq k_v + k_e$, then $\mathbf{table}_i(X, Y, \gamma, h_v, h_e) \leftarrow (U, D)$.

Construction for a join node. Let i' and i'' be the children of i . We start with \mathbf{table}_i empty. For each pair $((X, Y, \gamma', h'_v, h'_e), (U', D')) \in \mathbf{table}_{i'}$ and each pair $((X, Y, \gamma'', h''_v, h''_e), (U'', D'')) \in \mathbf{table}_{i''}$ we do the following.

- Let $\gamma \leftarrow \gamma' + \gamma''$, $U \leftarrow U' \cup U''$ and $D \leftarrow D' \cup D''$.
If $\max_{u \in X_i \setminus X} \gamma(u) \leq k_v + k_e$, then for any two integers h_v, h_e such that $h'_v + h''_v - w(X) \leq h_v \leq k_v$ and $h'_e + h''_e - w(Y) \leq h_e \leq k_e$, $\mathbf{table}_i(X, Y, \gamma, h_v, h_e) \leftarrow (U, D)$.

Using standard arguments, it is straightforward to verify the correctness of the algorithm. To evaluate the running time, recall that \mathbf{table}_i receives a quintuple (X, Y, γ, h_v, h_e) as input. There are at most 2^{t+1} possible choices for X , $2^{3(t+1)-6} = 2^{3t-3}$ choices of Y (because of the planarity of G), $(k_v + k_e + 1)^{t+1}$ choices of γ , $k_v + 1$ possible values of h_v and $k_e + 1$ possible values for h_e . We therefore have that each \mathbf{table}_i has $(k_v + k_e)^{O(t)}$ entries. This implies that the running time of the dynamic programming algorithm is $(k_v + k_e)^{O(t)} \cdot n$.

Now we consider DCPGGD. The difference is that we have to keep track of components of a partial solution as is standard for dynamic programming algorithms for graphs of bounded treewidth with a connectivity condition such as, e.g. the STEINER TREE problem. Let $(G, k_v, k_e, C, \delta, w, c)$ be an instance of DCPGGD where $\mathbf{tw}(G) \leq t$ and $\delta(v) \leq d_G(v) \leq \delta(v) + k_v + k_e$ for $v \in V(G)$. Without loss of generality we assume that a nice tree decomposition (\mathcal{X}, T) of G with treewidth at most t is given and apply a dynamic programming algorithm over (\mathcal{X}, T) . Let $i \in V(T)$.

We define \mathbf{table}_i^c as a partial function whose inputs are quintuples $(\mathcal{P}, Y, \gamma, h_v, h_e)$ where

- $\mathcal{P} = \{P_0, \dots, P_s\}$ is a partition of X_i ,
- $Y \subseteq E(G[X_i])$,
- $\gamma : X_i \setminus X \rightarrow \{0, \dots, k_v + k_e\}$,
- $h_v \leq k_v$ and
- $h_e \leq k_e$.

The value of \mathbf{table}_i^c is a minimum cost pair $(U, D) \in 2^{V(G_i)} \times 2^{E(G_i)}$ with the following properties:

- (i) for any $v \in U$ and any $e \in D$, v and e are not incident,
- (ii) $w(U) \leq h_v$ and $w(D) \leq h_e$,
- (iii) $U \cap X_i = P_0$ and $D \cap E(G[X_i]) = Y$,
- (iv) for every $v \in X_i \setminus X$, the number of neighbours of v in G_i that belong in $U \setminus X_i$, plus the number of edges of $D \setminus E(G[X_i])$ that are incident to v is exactly $\gamma(v)$,
- (v) for each $v \in V(G_i) \setminus X_i$, $d_{G'_i}(v) = \delta(v)$ where $G'_i = G_i - U - D$,
- (vi) if $s = 0$, then $G'_i = G_i - U - D$ is connected and if $s \geq 1$, then G'_i has s components H_1, \dots, H_s such that $V(H_h) \cap X_h = P_h$ for $h \in \{1, \dots, s\}$,

and, if no such pair (U, D) exists, then $\mathbf{table}_i^c(\mathcal{P}, Y, d, h_v, h_e)$ is void.

As in the non-connected case, $(G, k_v, k_e, C, \delta, w, c)$ is a yes-instance if and only if $\mathbf{table}_r^c(\emptyset, \emptyset, \emptyset, k_v, k_e)$ is non-void and the value of $\mathbf{table}_r^c(\emptyset, \emptyset, \emptyset, k_v, k_e)$, if exists, is a minimum-cost solution for this instance.

The partial function \mathbf{table}_i^c is constructed for every $i \in V(T)$ similarly to the construction of \mathbf{table}_i for DPGGD. Because there are at most $(t+1)^{t+1}$ partitions \mathcal{P} of each X_i , we have that each table contains $(t(k_v + k_e))^{O(t)}$ entries. Therefore, the running time of the dynamic programming algorithm is $(t(k_v + k_e))^{O(t)}n$. \square

We are now ready to present our two main results, starting with the one for DPGGD.

Theorem 1. DPGGD has a polynomial kernel when parameterized by $k_v + k_e$.

Proof. Let $(G, k_v, k_e, C, \delta, w, c)$ be an instance of DPGGD. By Lemma 3, we may assume that this instance is normalized. By Lemma 4, if $(G, k_v, k_e, C, \delta, w, c)$ is a yes-instance, then G has a 2-dominating set of size at most $k_v + 2k_e$. By Lemma 5, there is a fixed constant α such that G has an $(\alpha(k_v + 2k_e), \alpha)$ -protrusion decomposition, and such a decomposition, if it exists, can be constructed in polynomial time. To simplify later arguments, we may assume $\alpha \geq 3$. Clearly, if we fail to obtain such a decomposition, we return a no-answer and stop. Hence, from now on we assume that an $(\alpha(k_v + 2k_e), \alpha)$ -protrusion decomposition $\Pi = \{R_0, \dots, R_p\}$ of G is given. As before, we keep the same notation δ, w, c for the restrictions of these functions. Again, we will introduce new reduction rules. We will keep the notation for G and for the parameters unchanged where this is well-defined. We also assume that if we consider sets of vertices or edges associated with the considered instance and delete vertices or edges from the graph, then we also delete these elements from the associated sets.

For each $i \in \{1, \dots, p\}$, we construct $W_i \subseteq R_i$ and $L_i \subseteq E_G(R_i)$. To do this, we consider the set \mathcal{Q} of all possible quintuples $\mathbf{q} = (h_v, h_e, X, Y, \delta')$ such that

- $0 \leq h_v \leq k_v$ and $0 \leq h_e \leq k_e$,
- $X \subseteq N_G(R_i)$ and $Y \subseteq E(G[N_G(R_i) \setminus X])$, and
- we define $F = G[R_i^+] - X - Y$ and require that $\delta': V(F) \rightarrow \mathbb{N}_0$ is a function such that $\delta'(v) \leq d_F(v) \leq \delta'(v) + k_v + k_e$ for $v \in N_G(R_i) \setminus X$ and $\delta'(v) = \delta(v)$ for $v \in R_i$.

Observe that there are at most 2^α sets X , at most $2^{3\alpha-6}$ sets Y , at most $(k_v + 1)(k_e + 1)$ pairs h_v, h_e , and for each X , there are at most $(k_v + k_e + 1)^\alpha$ possibilities for δ' . Therefore $|\mathcal{Q}| \leq 2^\alpha 2^{3\alpha-6} (k_v + 1)(k_e + 1)(k_v + k_e + 1)^\alpha = (k_v + k_e)^{O(\alpha)}$.

For each $\mathbf{q} = (h_v, h_e, X, Y, \delta') \in \mathcal{Q}$, we construct an instance $I_{\mathbf{q}} = (F, h_v, h_e, C, \delta', w', c)$ of DPGGD such that

- $w'(v) = k_v + 1$ for $v \in N_G(R_i) \setminus X$ and $w'(v) = w(v)$ for $v \in R_i$ and
- $w'(e) = k_e + 1$ for $e \in E(G[N_G(R_i) \setminus X]) \setminus Y$ and $w'(e) = w(e)$ for all other edges of F .

By Lemma 6, we can solve the problem for this instance in $(k_v + k_e)^{O(\alpha)}$ time. Let $(U_{\mathbf{q}}, D_{\mathbf{q}})$ denote the obtained solution of minimum cost and set $U_{\mathbf{q}} = D_{\mathbf{q}} = \emptyset$ if no solution exists for $I_{\mathbf{q}}$. Let

$$W_i = \bigcup_{\mathbf{q} \in \mathcal{Q}} U_{\mathbf{q}} \text{ and } L_i = \bigcup_{\mathbf{q} \in \mathcal{Q}} D_{\mathbf{q}}.$$

Because each $U_{\mathbf{q}}$ has at most k_v vertices and each $D_{\mathbf{q}}$ has at most k_e edges, we obtain that $|W_i| \leq |\mathcal{Q}|k_v \leq (k_v + 1)(k_e + 1) \cdot 2^\alpha \cdot 2^{3\alpha-6} \cdot (k_v + k_e + 1)^\alpha \cdot k_v$ and $|L_i| \leq |\mathcal{Q}|k_e \leq (k_v + 1)(k_e + 1) \cdot 2^\alpha \cdot 2^{3\alpha-6} \cdot (k_v + k_e + 1)^\alpha \cdot k_e$. Hence, the size of W_i and L_i is $(k_v + k_e)^{O(\alpha)}$.

Let $W = R_0 \cup \bigcup_{i \in \{1, \dots, p\}} W_i$ and $L = E(G[R_0]) \cup \bigcup_{i \in \{1, \dots, p\}} L_i$. Because $\max\{p, |R_0|\} \leq \alpha(k_v + 2k_e)$, we have that $|W| = (k_v + k_e)^{O(\alpha)}$ and $|L| = (k_v + k_e)^{O(\alpha)}$. We prove the following claim.

Claim A. If $(G, k_v, k_e, C, \delta, w, c)$ is a yes-instance of DPGGD, then it has an efficient solution (U, D) of minimum cost such that $U \subseteq W$ and $D \subseteq L$.

We prove Claim A as follows. Let (U, D) be an efficient solution for $(G, k_v, k_e, C, \delta, w, c)$ of minimum cost such that $s = |U \setminus W| + |D \setminus L|$ is minimum. If $s = 0$, then the claim is fulfilled. Suppose, for contradiction, that $s > 0$. This means that there is an $i \in \{1, \dots, p\}$ such that $(U \cap R_i) \setminus W_i \neq \emptyset$ or $(D \cap E_G(R_i)) \setminus L_i \neq \emptyset$. Let $X = U \cap N_G(R_i)$, $Y = D \cap E(N_G(R_i))$ and $F = G[R_i^+] - X - Y$. Let $h_v = |U \cap V(F)|$ and $h_e = |D \cap E(F)|$. For each vertex $v \in N_G(R_i) \setminus X$, let d_v be the total number of vertices in $U \cap V(F)$ adjacent to v plus the number of edges in $D \cap E(F)$ incident to v . Let $\delta'(v) = d_F(v) - (d_G(v) - \delta(v) - d_v)$ for $v \in N_G(R_i) \setminus X$ and $\delta'(v) = \delta(v)$ for all other vertices of F .

Clearly, $(F, h_v, h_e, C, \delta', w', c) = I_{\mathbf{q}}$ is the instance of DPGGD when $\mathbf{q} = (h_v, h_e, X, Y, \delta')$ if we set w' as before. Let $U' = U \cap V(F)$ and $D' = D \cap E(F)$. Then (U', D') is a solution for the instance $I_{\mathbf{q}}$ and, therefore $I_{\mathbf{q}}$ is a yes-instance. In particular, this means that there is a solution (U'', D'') for $I_{\mathbf{q}} = (F, h_v, h_e, C, \delta', w', c)$ that was constructed by the aforementioned procedure for the construction of W_i and L_i . Clearly, $U'' \subseteq W_i \subseteq W$ and $D'' \subseteq L_i \subseteq L$. Because our algorithm for graphs of bounded treewidth finds a solution of minimum cost, it follows that $c(U'' \cup D'') \leq c(U' \cup D')$. It remains to observe that (\hat{U}, \hat{D}) , where $\hat{U} = (U \setminus U') \cup U''$ and $\hat{D} = (D \setminus D') \cup D''$, is a solution for $(G, k_v, k_e, C, \delta, w, c)$ with $c(\hat{U} \cup \hat{D}) \leq c(U \cup D)$, but this contradicts the choice of (U, D) because $|\hat{U} \setminus W| + |\hat{D} \setminus L| < s$. This completes the proof of Claim A.

Let $S = \{v \in V(G) \mid d_G(v) = \delta(v)\} \setminus W$ and $T = \{v \in V(G) \mid d_G(v) > \delta(v)\} \setminus W$; because the instance we consider is normalized, these sets form a partition of $V(G) \setminus W$ (note that these sets may be empty). If $v \in S$, then for any efficient solution (U, D) such that $U \subseteq W$ and $D \subseteq L$, v is not adjacent to any vertex of U and not incident to any edge of L . This implies that it is safe to exhaustively apply the following rule without destroying the statement of Claim A.

Set adjustment rule. If there is a vertex $v \in S$ that is adjacent to a vertex $u \in W$, then set $W = W \setminus \{u\}$ and set $S = S \cup \{u\}$ if $d_G(u) = \delta(u)$ and set $T = T \cup \{u\}$ if $d_G(u) > \delta(u)$. If $v \in S$, remove any edge incident to v from L .

By [Claim A](#), it is safe to modify the weights as follows.

Weight adjustment rule. Set $w(v) = k_v + 1$ for $v \in V(G) \setminus W$ and set $w(e) = k_e + 1$ for $e \in E(G) \setminus L$.

After the exhaustive application of the **set adjustment rule**, we have that $N_G(S) \subseteq T$. Now it is safe to remove S .

S-reduction rule. If $v \in S$, then remove v and set $\delta(u) = \delta(u) - 1$ for $u \in N_G(v)$. If $\delta(u) < 0$ for some $u \in N_G(v)$, then return a no-answer and stop.

To show that the above rule is safe, let $G' = G - S$ and let δ' be the function obtained from δ by the application of the rule. Suppose that $(G, k_v, k_e, C, \delta, w, c)$ is a yes-instance. Then, by [Claim A](#), we have a solution (U, D) such that $U \subseteq W$ and $D \subseteq L$. Because $N_G(S) \subseteq T$, $T \cap W = \emptyset$ and the vertices of S are not incident to edges of L , it follows that we do not stop and (U, D) is a solution for $(G', k_v, k_e, C, \delta', w, c)$. Now let (U, D) be a solution for $(G', k_v, k_e, C, \delta', w, c)$. Because of the application of the **weight adjustment rule**, $U \subseteq W$ and $D \subseteq L$. Because $N_G(S) \subseteq T$, $T \cap W = \emptyset$ and the vertices of S are not incident to edges of L , we have that (U, D) is a solution for $(G, k_v, k_e, C, \delta, w, c)$. This completes the proof that the **S-reduction rule** is safe.

Let $W' = W \cup V(L)$ and $T' = T \setminus V(L)$. Clearly, $|W'| \leq |W| + 2|L| = (k_v + k_e)^{O(\alpha)}$.

Using similar arguments to those for the **S-reduction rule**, the following rule is also safe.

T'-reduction rule. If $uv \in E(G[T'])$, then remove uv and set $\delta(u) = \delta(u) - 1$ and $\delta(v) = \delta(v) - 1$. If $\delta(u) < 0$ or $\delta(v) < 0$, then return a no-answer and stop.

After the exhaustive application of the above rule, T' is an independent set in the obtained graph G . Some of the vertices of this independent set may have the same neighbourhoods. We deal with them using the next rule.

Twin reduction rule. Suppose there are $u, v \in T'$ with $N_G(u) = N_G(v)$. If $\delta(u) = \delta(v)$, then remove v and set $\delta(x) = \max\{0, \delta(x) - 1\}$ for $x \in N_G(u)$. If $\delta(u) \neq \delta(v)$ then return a no-answer and stop.

To prove that the above rule is safe, consider a pair of vertices $u, v \in T'$ with $N_G(u) = N_G(v)$ and $\delta(u) = \delta(v)$. Let $G' = G - v$ and let δ' denote the function obtained from δ by the rule. Suppose that $(G, k_v, k_e, C, \delta, w, c)$ is a yes-instance. Then we have a solution (U, D) such that $U \subseteq W$ and $D \subseteq L$. Notice that $T' \cap U = \emptyset$ and the vertices of T' are not incident to the edges of L . Note that $u, v \notin U$ and if $x \in N_G(u)$ then $ux, vx \notin D$. We have that U contains exactly $d_G(u) - \delta(u)$ vertices that are adjacent to u . Therefore, (U, D) is a solution for $(G', k_v, k_e, C, \delta', w, c)$. Now assume that (U, D) is a solution for $(G', k_v, k_e, C, \delta', w, c)$. By the same arguments, U contains exactly $d_{G'}(u) - \delta'(u)$ vertices that are adjacent to u . Also if $x \in N_G(u)$ and $\delta'(x) = 0$, then $x \in U$, because $u \notin U$ and $ux \notin D$. Because $N_G(u) = N_G(v)$, $\delta(u) = \delta(v)$ and T' is an independent set, U contains $d_G(u) - \delta(u)$ vertices that are adjacent to u and $d_G(v) - \delta(v)$ vertices that are adjacent to v . It follows that (U, D) is a solution for $(G, k_v, k_e, C, \delta, w, c)$. Now consider the case when $N_G(u) = N_G(v)$ and $\delta(u) \neq \delta(v)$. Suppose, for contradiction that there is a solution (U, D) . By the above arguments, U contains exactly $d_G(u) - \delta(u)$ vertices that are adjacent to u and $d_G(v) - \delta(v)$ vertices that are adjacent to v . Since $N_G(u) = N_G(v)$ and $\delta(u) \neq \delta(v)$, this is a contradiction, so there cannot be such a solution.

After the exhaustive application of the above rule for any two vertices $u, v \in T'$, we have that $N_G(u) \neq N_G(v)$. Let $T'_0, T'_1, T'_2, T'_{\geq 3}$ denote the sets of vertices in T' that are of degree 0, 1, 2 and at least 3 respectively. Observe that $d_G(v) > \delta(v) \geq 0$ for $v \in T'$. Therefore, $T'_0 = \emptyset$ and $T'_1, T'_2, T'_{\geq 3}$ form a partition of T' (note that these sets may be empty). By the **twin reduction rule** $|T'_1| = |N_G(T'_1)| \leq |W'|$ and $|T'_2| \leq \binom{|N_G(T'_2)|}{2} \leq \frac{1}{2}|W'|(|W'| - 1)$. By [Lemma 1](#), $|T'_{\geq 3}| \leq 2|N_G(T')| - 4 \leq 2|W'| - 4$ (or $|T'_{\geq 3}| = 0$). We have that $|V(G)| = |W'| + |T'| = |W'| + |T'_1| + |T'_2| + |T'_{\geq 3}| \leq \frac{1}{2}|W'|^2 + \frac{7}{2}|W'|$. Since W' has $(k_v + k_e)^{O(\alpha)}$ vertices, we obtain that the obtained graph G has size $k^{O(1)}$ where $k = k_v + k_e$, i.e. we have a polynomial kernel for DPGGD.

To complete the proof, it remains to observe that the construction of the normalized instance can be done in polynomial time by [Lemma 3](#), the construction of W and L can be done in polynomial time by [Lemma 6](#), and all the subsequent reduction rules can be applied in polynomial time. \square

The proof of our second main result is based on the same approach as the proof of [Theorem 1](#), but it is more technically involved because we have to ensure connectivity of the graph obtained by the editing.

Theorem 2. DCPGGD has a polynomial kernel when parameterized by $k_v + k_e$.

Proof. Let $(G, k_v, k_e, C, \delta, w, c)$ be an instance of DCPGGD. By [Lemma 3](#), we may assume that this instance is normalized. By [Lemma 4](#), if $(G, k_v, k_e, C, \delta, w, c)$ is a yes-instance, then G has a 2-dominating set of size at most $k_v + 2k_e$. By [Lemma 5](#), there is a fixed constant α such that G has an $(\alpha(k_v + 2k_e), \alpha)$ -protrusion decomposition, and such a decomposition, if it

exists, can be constructed in polynomial time. To simplify later arguments, we may assume $\alpha \geq 3$. Clearly, if we fail to obtain such a decomposition, we return a no-answer and stop. Hence, from now on we assume that an $(\alpha(k_v + 2k_e), \alpha)$ -protrusion decomposition $\Pi = \{R_0, \dots, R_p\}$ of G is given. As before, we keep the same notation δ, w, c for the restrictions of these functions. Again, we will introduce new reduction rules. We will keep the notation for G and for the parameters unchanged where this is well-defined. We also assume that if we consider sets of vertices or edges associated with the considered instance and delete vertices or edges from the graph, then we also delete these elements from the associated sets.

For each $i \in \{1, \dots, p\}$, we construct $W_i \subseteq R_i$ and $L_i \subseteq E_G(R_i)$. To do this, we consider the set \mathcal{Q} of all possible sextuples $\mathbf{q} = (h_v, h_e, X, Y, \mathcal{P}, \delta')$ such that

- $0 \leq h_v \leq k_v$ and $0 \leq h_e \leq k_e$,
- $X \subseteq N_G(R_i)$ and $Y \subseteq E(G[N_G(R_i) \setminus X])$,
- $\mathcal{P} = \{P_1, \dots, P_s\}$ is a set covering of $N_G(R_i) \setminus X$, with $s \leq |N_G(R_i) \setminus X|$,
- we define $F = G[R_i^+] - X - Y$ and require that $\delta': V(F) \rightarrow \mathbb{N}_0$ is a function such that $\delta'(v) \leq d_F(v) \leq \delta'(v) + k_v + k_e$ for $v \in N_G(R_i) \setminus X$ and $\delta'(v) = \delta(v)$ for $v \in R_i$.

Observe that there are at most 2^α sets X , at most $2^{3\alpha-6}$ sets Y , at most $(k_v + 1)(k_e + 1)$ pairs h_v, h_e , and for each X , there are at most 2^{α^2} possible set covers \mathcal{P} and at most $(k_v + k_e + 1)^\alpha$ possibilities for δ' . Therefore $|\mathcal{Q}| \leq 2^\alpha 2^{3\alpha-6} (k_v + 1)(k_e + 1) 2^{\alpha^2} (k_v + k_e + 1)^\alpha = (k_v + k_e)^{O(\alpha^2)}$.

For each $\mathbf{q} = (h_v, h_e, X, Y, \mathcal{P}, \delta') \in \mathcal{Q}$, we construct an instance $I_{\mathbf{q}} = (F_{\mathcal{P}}, h_v, h_e, C, \delta'', w', c')$ of DCPGGD such that

- $F_{\mathcal{P}}$ is the graph obtained from F by adding a set of s new vertices $Z = \{z_1, \dots, z_s\}$ and making z_i adjacent to all the vertices of P_i . If $\mathcal{P} = \emptyset$, which means that $N_G(R_i^+) = X$, then we simply have that $Z = \emptyset$ and $F_{\mathcal{P}} = F$.
- $\delta''(v) = d_{F_{\mathcal{P}}}(v)$ for $v \in Z$ and $\delta''(v) = \delta'(v)$ for $v \in V(F_{\mathcal{P}}) \setminus Z$.
- $w'(v) = k_v + 1$ for $v \in (N_G(R_i) \setminus X) \cup Z$ and $w'(v) = w(v)$ for $v \in R_i$.
- $w'(e) = k_e + 1$ for $e \in (E(G[N_G(R_i) \setminus X]) \setminus Y) \cup E_{F_{\mathcal{P}}}(Z)$, and $w'(e) = w(e)$ for all other edges of $F_{\mathcal{P}}$.
- $c'(v) = 0$ for $v \in Z$ and $c'(v) = c(v)$ for $v \in V(F_{\mathcal{P}}) \setminus Z$; $c'(e) = 0$ for $e \in E_{F_{\mathcal{P}}}(Z)$ and $c'(e) = c(e)$ for all other edges in $F_{\mathcal{P}}$.

Since $|Z| \leq |N_G(R_i)| \leq \alpha$, it follows that $|Z| \leq \alpha$ and therefore $\mathbf{tw}(F_{\mathcal{P}}) \leq \mathbf{tw}(F) + \alpha \leq 2\alpha$. We can check in linear time whether $F_{\mathcal{P}}$ is planar [20]. If it is not, then $I_{\mathbf{q}}$ is not a valid instance of DCPGGD and we set $(U_{\mathbf{q}}, D_{\mathbf{q}}) = (\emptyset, \emptyset)$. Otherwise, by Lemma 6, we can solve DCPGGD for $I_{\mathbf{q}}$ in $(\alpha(k_v + k_e))^{O(\alpha)} \cdot \text{poly}(n)$ time and find a solution of minimum cost. Let $(U_{\mathbf{q}}, D_{\mathbf{q}})$ be the obtained solution of minimum cost and let $U_{\mathbf{q}} = D_{\mathbf{q}} = \emptyset$ if no solution exists. Notice that $Z \cap U_{\mathbf{q}} = \emptyset$, because the vertices of Z have weight $k_v + 1$, and $D_{\mathbf{q}}$ has no edges incident to the vertices of Z , because these edges have weight $k_e + 1$. Let

$$W_i = \bigcup_{\mathbf{q} \in \mathcal{Q}} U_{\mathbf{q}} \text{ and } L_i = \bigcup_{\mathbf{q} \in \mathcal{Q}} D_{\mathbf{q}}.$$

Because each $U_{\mathbf{q}}$ has at most k_v vertices and each $D_{\mathbf{q}}$ has at most k_e edges, we obtain that $|W_i| \leq |\mathcal{Q}|k_v \leq (k_v + 1)(k_e + 1) \cdot 2^\alpha \cdot 2^{3\alpha-6} \cdot 2^{\alpha^2} \cdot (k_v + k_e + 1)^\alpha \cdot k_v$ and $|L_i| \leq |\mathcal{Q}|k_e \leq (k_v + 1)(k_e + 1) \cdot 2^\alpha \cdot 2^{3\alpha-6} \cdot 2^{\alpha^2} \cdot (k_v + k_e + 1)^\alpha \cdot k_e$. Hence, the size of W_i and L_i is $(k_v + k_e)^{O(\alpha^2)}$.

Let $W = R_0 \cup \bigcup_{i \in \{1, \dots, p\}} W_i$ and $L = E(G[R_0]) \cup \bigcup_{i \in \{1, \dots, p\}} L_i$. Because $\max\{p, |R_0|\} \leq \alpha(k_v + 2k_e)$, we have that $|W| = (k_v + k_e)^{O(\alpha^2)}$ and $|L| = (k_v + k_e)^{O(\alpha^2)}$. We prove the following claim.

Claim A. *If $(G, k_v, k_e, C, \delta, w, c)$ is a yes-instance of DCPGGD, then it has an efficient solution (U, D) of minimum cost such that $U \subseteq W$ and $D \subseteq L$.*

We prove Claim A as follows. Let (U, D) be an efficient solution for $(G, k_v, k_e, C, \delta, w, c)$ of minimum cost such that $s = |U \setminus W| + |D \setminus L|$ is minimum. If $s = 0$, then the claim is fulfilled. Suppose, for contradiction, that $s > 0$. This means that there is an $i \in \{1, \dots, p\}$ such that $(U \cap R_i) \setminus W_i \neq \emptyset$ or $(D \cap E_G(R_i)) \setminus L_i \neq \emptyset$.

Let $X = U \cap N_G(R_i)$, $Y = D \cap E(N_G(R_i))$ and $F = G[R_i^+] - X - Y$. Let $h_v = |U \cap V(F)|$ and $h_e = |D \cap E(F)|$. If $X \neq N_G(R_i)$, then consider the graph $H = G - U - D - R_i^+$ and let H_1, \dots, H_s denote the components of H . Next, starting with the graph $H' = G - U - D - R_i$, contract each H_j to a single vertex z_j and call the resulting graph H'' . Note that $Z = \{z_1, \dots, z_s\}$ is an independent set in H'' . By the definition of protrusion decomposition, every vertex of $N_G(R_i) \setminus X$ is adjacent to at least one vertex in Z . Likewise, since $G - U - D$ is connected, every vertex z_j must have a neighbour in $N_G(R_i) \setminus X$. If there is a vertex $z_j \in Z$ such that removing it from H'' does not increase the number of components in H'' and every vertex in $N_G(R_i) \setminus X$ has a neighbour in $Z \setminus \{z_j\}$ then we remove z_j from H'' and from Z . Doing this exhaustively, we obtain a graph with $|Z| \leq |N_G(R_i) \setminus X| \leq \alpha$. Call this graph $F_{\mathcal{P}}$. Without loss of generality assume $Z = \{z_1, \dots, z_t\}$. Let $P_j = N_{H''}(z_j)$ for $j \in \{1, \dots, t\}$. Then $\mathcal{P} = \{P_1, \dots, P_t\}$ is a set cover of $N_G(R_i) \setminus X$ containing at most α sets. If $X = N_G(R_i)$, then set $\mathcal{P} = \emptyset$ and $F_{\mathcal{P}} = F$. Now $F_{\mathcal{P}}$ is precisely the graph constructed from F and \mathcal{P} earlier. Note that $F_{\mathcal{P}}$ is planar since it is obtained from G by contractions, vertex deletions and edge deletions.

For each vertex $v \in N_G(R_i) \setminus X$, let d_v be the total number of vertices in $U \setminus V(F)$ adjacent to v plus the number of edges in $D \setminus E(F)$ incident to v .

Let $\delta'(v) = d_{F_{\mathcal{P}}}(v) - (d_G(v) - \delta(v) - d_v)$ for $v \in N_G(R_i) \setminus X$ and $\delta'(v) = \delta(v)$ for other vertices of $F_{\mathcal{P}}$. Set w' , c' and δ'' as before.

Clearly, $I_{\mathbf{q}} = (F_{\mathcal{P}}, h_v, h_e, C, \delta', w', c')$ is an instance of DCPGGD when $\mathbf{q} = (h_v, h_e, X, Y, \mathcal{P}, \delta')$. Let $U' = U \cap V(F)$ and $D' = D \cap E(F)$. Then (U', D') is a solution for the instance $I_{\mathbf{q}}$ and, therefore $I_{\mathbf{q}}$ is a yes-instance.

In particular, this means that there is a solution (U'', D'') for $I_{\mathbf{q}} = (F_{\mathcal{P}}, h_v, h_e, C, \delta'', w', c')$ that was constructed by the aforementioned procedure for the construction of W_i and L_i . Clearly, $U'' \subseteq W_i \subseteq W$ and $D'' \subseteq L_i \subseteq L$. Because our algorithm for graphs of bounded treewidth finds a solution of minimum cost, it follows that $c(U'' \cup D'') \leq c(U' \cup D')$. It remains to observe that (\hat{U}, \hat{D}) , where $\hat{U} = (U \setminus U') \cup U''$ and $\hat{D} = (D \setminus D') \cup D''$, is a solution for $(G, k_v, k_e, C, \delta, w, c)$ with $c(\hat{U} \cup \hat{D}) \leq c(U \cup D)$, but this contradicts the choice of (U, D) because $|\hat{U} \setminus W| + |\hat{D} \setminus L| < s$. This completes the proof of [Claim A](#).

If $v \in \overline{W} = V(G) \setminus W$ and $d_G(v) = \delta(v)$, then for any efficient solution (U, D) such that $U \subseteq W$ and $D \subseteq L$, v is not adjacent to a vertex of U . Moreover, $E_G(v) \cap D = \emptyset$, by [Observation 2](#). This implies that it is safe to apply the following rule without destroying the statement of [Claim A](#).

Set adjustment rule. If there is a vertex $v \in \overline{W}$ with $d_G(v) = \delta(v)$, then set $W = W \setminus N_G(v)$ and set $L = L \setminus E_G(v)$.

The sets W and L give us the following possibility to remove some vertices when there is the unique possibility to satisfy degree restrictions.

Vertex deletion rule. If there is a vertex $v \in \overline{W}$ with $d_G(v) > \delta(v)$ such that $E_G(v) \cap L = \emptyset$ then

- if $|N_G(v) \cap W| < d_G(v) - \delta(v)$, then return a no-answer and stop;
- if $|N_G(v) \cap W| = d_G(v) - \delta(v)$, then delete the vertices of $N_G(v) \cap W$ and set $k_v = k_v - w(N_G(v) \cap W)$ and $C = C - c(N_G(v) \cap W)$; if $k_v < 0$ or $C < 0$, then return a no-answer and stop.

We exhaustively apply the above two rules until they can no longer be further applied. Let $S = \{v \in V(G) \mid d_G(v) = \delta(v)\} \setminus W$. Notice that $N_G(S) \subseteq \overline{W}$ by the **set adjustment rule**. It is easy to see that the following rule is safe.

S-neighbour rule. If v has k neighbours in S , and $\delta(v) < k$ then return a no-answer and stop.

We apply the **S-neighbour rule** exhaustively. Next, we contract the edges of $G[S]$.

S-contraction rule 1. If G has two adjacent vertices $u, v \in S$, then we do as follows.

- For any vertex $x \in V(G) \setminus \{u, v\}$ such that $xu, xv \in E(G)$, set $\delta(x) = \delta(x) - 1$.
- Contract uv ; let z denote the vertex obtained from u and v .
- Set $w(z) = k_v + 1$ and $c(z) = 0$.
- For $e \in E_G(z)$, set $w(e) = k_e + 1$, $c(e) = 0$.

We now show that the **S-contraction rule 1** is safe. To do this, let $(G', k_v, k_e, C, \delta', w', c')$ denote the instance obtained by an application of the rule. Let (U, D) be an efficient solution for $(G, k_v, k_e, C, \delta, w, c)$ such that $U \subseteq W$ and $D \subseteq L$. By [Observation 2](#), D has no edges incident to u or v . Also $u, v \notin U$, because $u, v \in S$. Notice that $\delta(x) \geq 2$ by the **S-neighbour rule**. If (U', D') is an efficient solution for $(G', k_v, k_e, C, \delta', w', c')$, then D' has no edges e incident to z , because $w'(e) > k_e$. Similarly, $z \notin U'$ because $w'(z) > k_v$. Also note that $\delta'(x) \geq 1$ because of the **S-neighbour rule**. We obtain that (U', D') is a solution for the original instance.

We exhaustively apply **S-contraction rule 1**. Note that S is an independent set in the obtained instance.

Stopping rule. If G has two components that contain vertices of \overline{W} , then return a no-answer and stop. Suppose \overline{W} contains a vertex v which is isolated in G . In this case if $w(V(G) \setminus \{v\}) \leq k_v$ and $c(V(G) \setminus \{v\}) \leq C$, then return a $(V(G) \setminus \{v\}, \emptyset)$ as a solution and stop, otherwise, return a no-answer and stop.

Clearly, if G has two components that contain vertices of \overline{W} , then one of these components should be deleted. By [Claim A](#), we know that if there is a solution then there must be a minimal cost solution that does not delete any vertices of \overline{W} . This contradiction means that there is no solution. If $v \in \overline{W}$ is an isolated vertex of G , then because $d_G(v) \geq \delta(v)$, it follows that $\delta(v) = d_G(v)$ and we conclude that $(V(G) \setminus \{v\}, \emptyset)$ must be a solution. Therefore, the **stopping rule** is safe.

Assume that we do not stop at this stage. Then we obtain the instance $(G, k_v, k_e, C, \delta, w, c)$ of the problem and sets W, L such that the sets $S = \{v \in V(G) \mid d_G(v) = \delta(v)\} \setminus W$ and $T = \{v \in V(G) \mid d_G(v) > \delta(v)\} \setminus W$ form a partition of \overline{W} (note

that these sets may be empty), S is an independent set, no vertex of \overline{W} is isolated in G , and $L \cap E(S) = \emptyset$. Also for any $v \in S$, $N_G(v) \subseteq T$, by the **set adjustment rule**.

By **Claim A**, it is safe to modify the weights as follows.

Weight adjustment rule. Set $w(v) = k_v + 1$ for $v \in V(G) \setminus W$ and set $w(e) = k_e + 1$ for $e \in E(G) \setminus L$.

Our next aim is to bound the size of S . In the proof of **Theorem 1** we simply deleted the vertices of S and adjusted δ appropriately. Here we need to preserve connectivity. Hence, we delete vertices only if this does not destroy connectivity and we use contractions otherwise.

S-deletion rule. If, for a vertex $v \in S$, one of the following is fulfilled

- $d_G(v) = 1$,
- $d_G(v) = 2$ and for $\{x, y\} = N_G(v)$, $xy \in E(G) \setminus L$ or
- there is a vertex $u \in S$ such that $u \neq v$ and $N_G(v) \subseteq N_G(u)$,

then delete v and set $\delta(x) = \delta(x) - 1$ for $x \in N_G(v)$; if $\delta(x) < 0$, then return a no-answer and stop.

S-contraction rule 2. If $v \in S$, then let $u \in N_G(v)$ and let $\Delta = d_G(u) - \delta(u)$. For every $vx \in E(v) \setminus \{vu\}$ such that $ux \in E(G)$, delete vx , add a vertex z adjacent to v and x , set $\delta(z) = 2$, $w(z) = k_v + 1$, $w(zx) = w(zv) = k_e + 1$ and $c(z) = c(zx) = c(zv) = 0$ and add z to S . Then contract uv in the obtained graph and set $\delta(y) = d_G(y) + \Delta$, $w(y) = k_v + 1$ and $c(y) = 0$ for the vertex y obtained from u and v .

The above two rules are safe, because $N_G(v) \subseteq T$ and the vertices of T are not included in any solutions.

We apply these rules exhaustively. First we apply the **S-deletion rule** whenever it is possible. Then we apply the **S-contraction rule 2**. Notice that the **S-contraction rule** creates new vertices that are obtained by subdividing the edges of $E(v)$ and they are placed in S . Therefore, it may happen that we can again apply the **S-deletion rule**, and in this case we do so. Finally, we get the graph G with the following properties:

- (i) for any $v \in S$, $d_G(v) = 2$ and for $\{x, y\} = N_G(v)$, $x, y \in V(L)$, and
- (ii) for any distinct $u, v \in S$, $N_G(u) \neq N_G(v)$ (by the **S-deletion rule**). In particular, this means that $|S| \leq (2|L|)^2$.

Let $W' = W \cup V(L) \cup S$ and $T' = T \setminus V(L)$. Clearly, W' and T' form a partition of $V(G)$ (one of the sets could be empty). Notice that $|W'| \leq |W| + 3|L| = (k_v + k_e)^{O(\alpha^2)}$. Now our aim is to bound the size of T' .

T'-deletion rule. If there are two distinct $u, v \in T'$ such that $N_G(u) \cap W' = N_G(v) \cap W'$, $d_G(u) - \delta(u) = d_G(v) - \delta(v)$ and v is an isolated vertex of $G[T']$, then delete v and set $\delta(x) = \max\{0, \delta(x) - 1\}$ for $x \in N_G(u)$.

To see that the **T'-deletion rule** is safe, it is sufficient to recall that $\delta(v) \neq d_G(v)$ because we already applied the **vertex deletion rule**. Hence, $|N_G(v) \cap W'| > d_G(v) - \delta(v)$, so in any solution u and v have common adjacent vertices that are not deleted. Because $E(v) \cap L = \emptyset$ and $E(u) \cap L = \emptyset$, the edges of $E(u)$ and $E(v)$ cannot be deleted. Therefore, we maintain connectivity by the **T'-deletion rule**. It is straightforward to verify that the **T'-deletion rule** is safe with respect to degree restrictions.

T'-contraction rule. If there are two distinct $u, v \in T'$ such that $N_G(u) \cap W' = N_G(v) \cap W'$, $d_G(u) - \delta(u) = d_G(v) - \delta(v)$ and u and v are in the same component of $G[T']$, do the following.

- For each $vx \in E(v)$ such that $x \notin T'$, delete vx and set $\delta(x) = \max\{0, \delta(x) - 1\}$.
- Let $y \in N_G(v)$ in the obtained graph and let $\Delta = d_G(y) - \delta(y)$. For every $x \in N_G(v) \cap N_G(y)$, set $\delta(x) = \max\{0, \delta(x) - 1\}$. Contract yv to a vertex z and set $\delta(z) = d_G(z) + \Delta$, $w(z) = k_v + 1$, $c(z) = 0$ and let $w(zx) = k_e + 1$, $c(zx) = 0$ for every $x \in N_G(z)$. Add z to T' .

To show that the **T'-contraction rule** is safe, again recall that $\delta(v) \neq |N_G(v) \cap W'|$ because we already applied the **vertex deletion rule**. Hence, in any solution, u and v have common adjacent vertices in W' that are not deleted. Because $E(T') \cap L = \emptyset$, the edges of $E(T')$ cannot be deleted. Therefore, we do not destroy connectivity by the **T'-contraction rule**. It is straightforward to verify that the **T'-contraction rule** is safe with respect to degree restrictions.

We exhaustively apply the above two rules. First, we apply the **T'-deletion rule** if possible. Then we apply the **T'-contraction rule** and if after the application of this rule we again can apply the **T'-deletion rule**, we do so.

For $i = 0, 1, 2$, let $T_i = \{v \in T' : |N_G(v) \cap W'| = i\}$, and $T_{\geq 3} = \{v \in T' : |N_G(v) \cap W'| \geq 3\}$. Because we exhaustively applied the **vertex deletion rule**, we have that $T_0 = T_1 = \emptyset$. By **Lemma 1**, $|T_{\geq 3}| \leq 2|N_G(T')| - 4 \leq 2|W'| - 4$ (or $T_{\geq 3}$ is empty).

Therefore we have that $G[T']$ has at most $2|W'|$ components that contain vertices of $T_{\geq 3}$. It remains to evaluate $|T_2|$. Because of the **vertex deletion rule**, for any $v \in T_2$, $d_G(v) - \delta(v) = 1$ as otherwise we would either stop or delete the neighbours of v in W . Any two distinct $u, v \in T_2$ such that $N_G(u) \cap W' = N_G(v) \cap W'$ belong to distinct components of $G[T']$ by the **T' -deletion rule** and the **T' -contraction rule**. There are at most $\binom{|W'|}{2}$ such components that are isolated vertices of $G[T']$ and there are at most $\binom{|W'|}{2}|T_{\geq 3}|$ vertices in T_2 that belong to the components that contain vertices of $T_{\geq 3}$, and the total number of such vertices is at most $\binom{|W'|}{2}(2|W'|)$. Let T'_2 denote the set of remaining vertices of T_2 . Observe that each component of $G[T'_2]$ is a component of $G[T']$ and has at least two vertices of T_2 . Moreover, for any two vertices u and v in the same component of $G[T'_2]$, $N_G(u) \cap W' \neq N_G(v) \cap W'$. Let G' be the graph obtained from G by contracting the edges of $G[T'_2]$. Each component of $G[T'_2]$ is contracted into a single vertex. Let Z denote the set of vertices of G' obtained from the components of $G[T'_2]$. The set Z is independent and for each $v \in Z$, $d_{G'}(v) \geq 3$. By Lemma 1, $|Z| \leq 2|N_{G'}(Z)| - 4 \leq 2|W'| - 4$ (or Z is empty). Hence, $G[T'_2]$ has at most $2|W'|$ components. Because each component has at most $\binom{|W'|}{2}$ vertices, $|T'_2| \leq \binom{|W'|}{2}(2|W'|)$. Hence, $|T_2| \leq \binom{|W'|}{2}(4|W'| + 1)$. We have that $|V(G)| = |W'| + |T'| = |W'| + |T_0| + |T_1| + |T_2| + |T_{\geq 3}| = O(|W'|^3)$. Since W' has $(k_v + k_e)^{O(\alpha^2)}$ vertices, we obtain that the obtained graph G has size $k^{O(1)}$ where $k = k_v + k_e$, i.e. we have a polynomial kernel.

To complete the proof, it remains to observe that the construction of the normalized instance can be done in polynomial time by Lemma 3, the construction of W and L can be done in polynomial time by Lemma 6, and all the subsequent reduction rules can be applied in polynomial time. \square

4. Conclusions

We proved that DPGGD and DCPGGD are NP-complete but allow polynomial kernels when parameterized by $k_v + k_e$. These problems generalize the DEGREE CONSTRAINED EDITING(S) problem and its connected variant for $S = \{\text{ed}, \text{vd}\}$; this can be seen, for instance, by testing all possible pairs k_v, k_e with $k_v + k_e = k$ or by a slight adjustment of our algorithms. Note that by setting $k_v = 0$ or $k_e = 0$ we obtain the same results for $S = \{\text{ed}\}$ and $S = \{\text{vd}\}$, respectively (for $S = \{\text{ed}\}$ this is not so surprising: we recall that the problem DEGREE CONSTRAINED EDITING($\{\text{ed}\}$) is polynomial-time solvable for general graphs even if the vertices and edges have costs [27]).

Several open problems remain. We note that graph modification problems that permit edge additions are less natural to consider for planar graphs, because the class of planar graphs is not closed under edge addition. However, we could allow other, more appropriate, operations such as edge contractions and vertex dissolutions when considering planar graphs. Belmonte et al. [1] considered the setting in which only edge contractions are allowed and obtained initial results for general graphs that extend the work of Mathieson and Szeider [27] on DEGREE CONSTRAINED EDITING(S) in this direction.

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