Computing Role Assignments of Proper Interval Graphs in Polynomial Time*

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Abstract. A homomorphism from a graph G to a graph R is locally surjective if its restriction to the neighborhood of each vertex of G is surjective. Such a homomorphism is also called an R-role assignment of G. Role assignments have applications in distributed computing, social network theory, and topological graph theory. The Role Assignment problem has as input a pair of graphs (G, R) and asks whether G has an R-role assignment. This problem is NP-complete already on input pairs (G, R) where R is a path on three vertices. So far, the only known non-trivial tractable case consists of input pairs (G, R) where G is a tree. We present a polynomial time algorithm that solves Role Assignment on all input pairs (G, R) where G is a proper interval graph. Thus we identify the first graph class other than trees on which the problem is tractable. As a complementary result, we show that the problem is Graph Isomorphism-hard on chordal graphs, a superclass of proper interval graphs and trees.

1 Introduction

Graph homomorphisms form a natural generalization of graph colorings: there is a homomorphism from a graph G to the complete graph on k vertices if and only if G is k-colorable. A *homomorphism* from a graph $G = (V_G, E_G)$ to a graph $R = (V_R, E_R)$ is a mapping $r : V_G \to V_R$ that maps adjacent vertices of G to adjacent vertices of R, i.e., $r(u)r(v) \in E_R$ whenever $uv \in E_G$. A homomorphism r from G to R is *locally surjective* if the following is true for every vertex u of G: for every neighbor y of $r(u)$ in R, there is a neighbor v of u in G with $r(v) = y$. We also call such an r an *R-role assignment*. See Figure 1 for an example.

Role assignments originate in the theory of social behavior [7, 19]. A role graph R models roles and their relationships, and for a given society we can ask if its individuals can be assigned roles such that relationships are preserved: each person playing a particular role has exactly the roles prescribed by the

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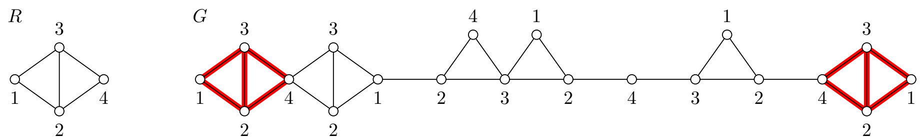

Fig. 1. A graph R and a proper interval graph G with an R-role assignment.

model among its neighbors. In this way, a large network of individuals can be compressed into a smaller network that still gives some description of the large network. Role assignments are also useful in the area of distributed computing, in which one of the fundamental problems is to arrive at a final configuration where all processors have been assigned unique identities. Chalopin et al. [4] show that, under a particular communication model, this problem can be solved on a graph G representing the distributed system if and only if G has no R-role assignment for a graph R with fewer vertices than G. Role assignments are useful in topological graph theory as well, where a main question is which graphs G allow role assignments to planar graphs R [21].

The Role Assignment problem has as input a pair of graphs (G, R) and asks whether G has an R-role assignment. It is NP-complete on arbitrary graphs G, even when R is any fixed connected bipartite graph on at least three vertices [10]. Hence, for polynomial time solvability, our only hope is to put restrictions on G. So far, the only known non-trivial graph class that gives tractability is the class of trees: Role Assignment is polynomial time solvable on input pairs (G, R) where G is a tree and R is arbitrary [11]. Are there other graph classes on which Role Assignment can be solved in polynomial time?

We show that Role Assignment can be solved in polynomial time on input pairs (G, R) where G is a proper interval graph and R is arbitrary. Our work is motivated by the above question and continues the research direction of Sheng [23], who characterizes proper interval graphs that have an R-role assignment for some fixed role graphs R with a small number of vertices. Proper interval graphs, also known as unit interval graphs or indifference graphs, are widely known due to their many theoretical and practical applications [3, 14, 22]. By our result, they form the first graph class other than trees on which Role Assignment is shown to be polynomial time solvable. To obtain our algorithm we prove structural properties of clique paths of proper interval graphs related to role assignments. This enables us to give an additional result, namely a polynomial time algorithm for the problem of deciding whether there exists a graph R with fewer vertices than a given proper interval graph G such that G has an R-role assignment. Recall that this problem stems from the area of distributed computing [4]. It is co-NP-complete in general [5]. Finally, to indicate that Role Assignment might remain hard on larger graph classes, we show that it is Graph Isomorphism-hard for input pairs (G, R) where G belongs to the class of chordal graphs, a superclass of both proper interval graphs and trees.

2 Preliminaries

All graphs considered in this paper are undirected, finite and simple, i.e., without loops or multiple edges. A graph is denoted $G = (V_G, E_G)$, where V_G is the set of vertices and E_G is the set of edges. We will use the convention that $n = |V_G|$ and $m = |E_G|$. For a vertex u of G, $N_G(u) = \{v \mid uv \in E_G\}$ denotes the set of *neighbors* of u, also called the *neighborhood* of u. The *degree* of a vertex u is $\deg_G(u) = |N_G(u)|$. A graph $H = (V_H, E_H)$ is a *subgraph* of G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. For $U \subseteq V_G$, the graph $G[U] = (U, \{uv \in E_G \mid u, v \in U\})$ is called the subgraph of G *induced* by U. A graph is *complete* if it has an edge between every pair of vertices. A set of vertices $A \subseteq V_G$ is a *clique* if $G[A]$ is complete. A clique is *maximal* if it is not a proper subset of any other clique.

An *isomorphism* from a graph G to a graph H is a bijective mapping $f : V_G \to V_H$ such that for any two vertices $u, v \in E_G$, we have $uv \in E_G$ if and only if $f(u)f(v) \in E_H$. We say that G is *isomorphic* to H and write $G \simeq H$.

Let u and v be two vertices of a graph G. Then a *path* between u and v is a sequence of distinct vertices $P = u_1u_2 \cdots u_p$ starting at $u_1 = u$ and ending at $u_p = v$, where each pair of consecutive vertices u_i, u_{i+1} forms an edge of G. If uv is an edge as well we obtain a *cycle*. Sometimes we fix an orientation of P. In that case we write $u_i \overrightarrow{P} u_j = u_iu_{i+1} \cdots u_j$ and $u_j \overleftarrow{P} u_i = u_ju_{j-1} \cdots u_i$ to denote the subpath from u_i to u_j, or from u_j to u_i, respectively. The *length* of a path or cycle is its number of edges. The set of vertices of a path or cycle P is denoted by V_P. A graph is *connected* if there is a path between every pair of vertices. A *connected component* of G is a maximal connected subgraph of G.

Let $A_1, \ldots, A_p$ be a sequence of sets. For $i = 1, \ldots, p$, we use shorthand notation $A_{\leq i} = A_1 \cup \cdots \cup A_i$ and $A_{\geq i} = A_i \cup \cdots \cup A_p$.

2.1 Chordal, Interval, and Proper Interval Graphs

A graph isomorphic to the graph $K_{1,3} = (\{a, b_1, b_2, b_3\}, \{ab_1, ab_2, ab_3\})$ is called a *claw* with *center* a and *leaves* b_1, b_2, b_3. A graph is called *claw-free* if it does not have a claw as an induced subgraph. An *asteroidal triple (AT)* in a graph G is a set of three mutually nonadjacent vertices u_1, u_2, u_3 such that G contains a path P_{ij} from u_i to u_j with $P_{ij} \cap N_G(u_k) = \emptyset$ for all distinct $i, j, k \in \{1, 2, 3\}$. A graph is called *AT-free* if it does not have an AT.

A graph is *chordal* if it contains no induced cycle of length at least 4. A graph is an *interval graph* if intervals of the real line can be associated with its vertices such that two vertices are adjacent if and only if their corresponding intervals overlap. Interval graphs are a subclass of chordal graphs: a chordal graph is an interval graph if and only if it is AT-free [17].

The following characterization of interval graphs is also well known. Let G be a connected graph with maximal cliques $K_1, \ldots, K_p$ and let $\mathcal{K}_v$ denote the set of maximal cliques in G containing vertex $v \in V_G$. Then G is an interval graph if and only if G has a path decomposition that is a *clique path* [12], i.e., a path $P = K_1 \cdots K_p$ such that for each $v \in V_G$ the set $\mathcal{K}_v$ induces a connected subpath in P. We say that the maximal cliques of G are the *bags* of P. A bag K_i

introduces a vertex u of G if $u \in K_i$ for $i = 1$ or $u \in K_i \setminus K_{i-1}$ for some $i \geq 2$. In that case, by the definition of a clique path, u is not in a bag K_h with $h \leq i-1$. If $u \in K_i$ for $i = p$ or $u \in K_i \setminus K_{i+1}$ for some $i \leq p-1$, then we say that K_i *forgets* u. Note that every bag introduces at least one vertex, and forgets at least one vertex. Because G is connected, we also observe that each bag, except K_1, contains at least one vertex from a previous bag. We denote the index of the bag in P that introduces a vertex u (the *first* bag in which u appears) by $f_P(u)$ and the index of the bag that forgets u (the *last* bag in which u appears) by $l_P(u)$. We say that u *transcends* a vertex v in P if $f_P(u) < f_P(v)$ and $l_P(v) < l_P(u)$. A clique path has at most n bags, and can be constructed in linear time (see e.g. [12]).

An interval graph is *proper interval* if it has an interval representation in which no interval is properly contained in any other interval. An interval graph is a proper interval graph if and only if it is claw-free [22]. Equivalently, a chordal graph is a proper interval graph if and only if it is AT-free and claw-free. Chordal graphs, interval graphs, and proper interval graphs can all be recognized in linear time, and have at most n maximal cliques (see e.g. [3, 14]). The following theorem will be used heavily in our proofs.

Theorem 1 ([15]). *A connected chordal graph is a proper interval graph if and only if it has a unique clique path in which no vertex transcends any other vertex.*

Two adjacent vertices u and v of a graph G are *twins* if $N_G(u) \cup \{u\} = N_G(v) \cup \{v\}$. Let G be a connected proper interval graph with clique path $P = K_1 \cdots K_p$. Note that two vertices u and v of G are twins if and only if $f_P(u) = f_P(v)$ and $l_P(u) = l_P(v)$. We partition V_G into sets of twins. A vertex that has no twin appears in its twin set alone. We order the twin sets with respect to P, and label them $T_1, \ldots, T_S$, in such a way that $i < j$ if and only if for all $u \in T_i, v \in T_j$, either $f_P(u) < f_P(v)$, or $f_P(u) = f_P(v)$ and $l_P(u) < l_P(v)$. We call $T_1, \ldots, T_s$ the *ordered twin sets* of G. The following observation immediately follows from this definition and the definition of a clique path. Hence, this observation is even valid for interval graphs that are not proper.

Observation 1 *Let G be a connected proper interval graph with clique path $P = K_1 \ldots K_p$ and ordered twin sets $T_1, \ldots, T_s$. Then for $h = 1, \ldots, s-1$, there exists a bag that contains twin sets T_h and T_{h+1}. Furthermore, if a bag contains twin sets T_b and T_c with $b < c$ then it contains twin sets $T_{b+1}, \ldots, T_{c-1}$ as well.*

2.2 Role Assignments

If r is a homomorphism from G to R and $U \subseteq V_G$, then we write $r(U) = \bigcup_{u \in U} r(u)$. Recall that r is an R-role assignment of G if $r(N_G(u)) = N_R(r(u))$ for every vertex u of G. Graph R is called a *role graph* and its vertices are called *roles*. Throughout the paper, we use n and m to refer to the number of vertices and edges of G. We frequently make use of the following two known results.

Observation 2 ([10]) *Let G be a graph and let R be a connected graph such that G has an R-role assignment. Then each vertex $x \in V_R$ appears as a role of some vertex $u \in V_G$, i.e., $r(u) = x$. Furthermore, if $|V_G| = |V_R|$ then $G \simeq R$.*

Lemma 1 ([10]). *Let G and R be two graphs such that G has an R-role assignment r, and let $x, y \in V_R$ be roles connected by a path $z_1 \cdots z_\ell$ in R, with $x = z_1$ and $y = z_\ell$. Then for each $u \in V_G$ with $r(u) = x$ there exists a vertex $v \in V_G$ and a path $t_1 \cdots t_\ell$ in G, with $u = t_1$ and $v = t_\ell$, such that $r(t_i) = z_i$ for $i = 1, \ldots, \ell$.*

Our first result, given in Theorem 2, shows that chordal graphs, interval graphs, and proper interval graphs are closed under role assignments, and it is needed in Section 3. We postpone its proof to the journal version of this paper. Note that, for each of the three statements in Theorem 2, the reverse implication is not valid. In order to see this let G be the 6-cycle and R be the 3-cycle.

Theorem 2. *Let G be a graph and let R be a connected graph such that G has an R-role assignment.*
(i) If G is a chordal graph then R is a chordal graph.
(ii) If G is an interval graph then R is an interval graph.
(iii) If G is a proper interval graph then R is a proper interval graph.

3 Role Assignments on Proper Interval Graphs

We start with the following key result. Note that this result is easy to verify for paths.

Theorem 3. *Let G and R be two connected proper interval graphs such that G has an R-role assignment r. Let P and P' be the clique paths of G and R, respectively. Then the bags of P and P' can be ordered such that $P = K_1 \cdots K_p$ and $P' = L_1 \cdots L_q$, with $q \leq p$, and $r(K_i) = L_i$, for $i = 1, \ldots, q$.*

Proof. By the definition of a role assignment, $|V_G| \geq |V_R|$ holds. Assume first that $|V_G| = |V_R|$. Then, as a result of Observation 2, G and R are isomorphic. By Theorem 1 the clique paths of G and R are unique. Hence the ordering of the bags in each path is unique up to reversal. We can try each direction for one of the paths, and the statement of the theorem holds.

For the rest of the proof, assume that $|V_G| > |V_R|$. Then at least one vertex of R is the role of more than one vertex of G. Let x be such a role. Then there exist vertices u and u' in G with $r(u) = r(u') = x$. Assume $l_P(u) = h$ and $f_P(u') = i$, where we may assume that $h < i$ because K_h and K_i are cliques, and vertices with the same role can not be adjacent. Let x be chosen in such a way that every vertex in $K_{\leq i-1}$ has a unique role, i.e., $|r(K_{\leq i-1})| = |K_{\leq i-1}|$.

Claim 1. Every vertex of R occurs as a (unique) role of a vertex of $K_{\leq i-1}$.

We prove this claim by contradiction. Suppose there is a role y that does not occur as a role of a vertex in $K_{\leq i-1}$. As a result of Observation 2, there exists a vertex v in G with $r(v) = y$. Let $f_P(v) = j$. Since y does not appear as a role on $K_{\leq i-1}$, we find that $j \geq i$. We may choose v such that there is no vertex in $K_{\leq j-1}$ with role y. Because K_j is a clique, we find that v is the only vertex of K_j with role y.

Let $Q' = z_1 \cdots z_\ell$, with $x = z_1$ and $y = z_\ell$, be a shortest path between x and y in R. By Lemma 1 we find that G contains a path $Q_1 = t_1 \cdots t_\ell$ with $u = t_1$, such that $r(t_i) = z_i$ for $i = 1, \ldots, \ell$. Since Q' is a shortest path from x to y in R, and there is no other vertex in K_j with role y, our choice of v implies that we may assume that $v = t_\ell$.

By the same reasoning we find a path $Q_2 = t'_1 \cdots t'_\ell$, with $u' = t'_1$ and $v = t'_\ell$, such that $r(t'_i) = z_i$ for $i = 1, \ldots, \ell$. Hence Q_1 and Q_2 are two paths with $r(V_{Q_1}) = r(V_{Q_2}) = V_{Q'}$ and $|V_{Q_1}| = |V_{Q_2}| = |V_{Q'}|$. Consequently, u is not on Q_2 and u' is not on Q_1. However, since $l_P(u) = h < f_P(u') = i \leq f_P(v) = j$ and K_i, K_j are cliques, we find that Q_1 contains a neighbor w of u'.

Suppose $i = j$. Then u' and v are neighbors in G, and consequently, xy is an edge of R. This means that u and v are neighbors in G. Hence, there is a bag in P containing both of them. This means that $h = l_P(u) \geq f_P(v) = j$. However, this is not possible since $h < i \leq j$.

Suppose $i < j$. Then $w = t_2$ as otherwise r maps the path $u'w\overrightarrow{Q_1}t_\ell$ to a path from x to y in R that is shorter than Q'. By the same reasoning, we find that w is the only neighbor of u' on Q_1. Since Q_1 is a shortest path and $uu' \notin E_G$, this means that G contains an induced claw with center t_2 and leaves u, u', t_3, which contradicts the assumption that G is a proper interval graph. This completes the proof of Claim 1.

By Claim 1 we find that $r(K_{\leq i-1}) = V_R$, and consequently, as $|r(K_{\leq i-1})| = |K_{\leq i-1}|$, we obtain $|K_{\leq i-1}| = |V_R|$. Let r' be the restriction of r to $K_{\leq i-1}$.

Claim 2. r' is an R-role assignment of $G[K_{\leq i-1}]$.

We prove Claim 2 as follows. Suppose r' is not an R-role assignment of $G[K_{\leq i-1}]$. Because r is a homomorphism from G to R, we find that r' is an homomorphism from $G[K_{\leq i-1}]$ to R. Hence, there must exist a vertex $t \in K_{\leq i-1}$ and vertices $z, z' \in V_R$ with $r'(t) = r(t) = z$, $zz' \in E_R$ and $z' \notin r'(N_G(t))$. Since r is an R-role assignment of G, we find that $z' \in r(N_G(t))$. Hence $l_P(t) \geq i + 1$. Consequently, as $t \in K_{\leq i-1}$, we find that t belongs to K_i. We proceed as follows. Since $r(K_{\leq i-1}) = V_R$, there exists a vertex $t' \in K_{\leq i-1}$ with $r'(t') = r(t') = z'$. By definition of r, we find that t' has a neighbor s in G with $r(s) = z$. Because t has no neighbor with role z', we find that t and t' are not adjacent in G. Hence $s \neq t$ holds. Since every vertex of $K_{\leq i-1}$ has a unique role and vertex $t \in K_{\leq i-1}$ already has role z, we find that $s \notin K_{\leq i-1}$. This means that K_i does not only contain t but also contains t'. However, since K_i is a clique, t and t' must be adjacent. With this contradiction we have completed the proof of Claim 2.

Due to Claim 2 and the aforementioned observation that $|K_{\leq i-1}| = |V_R|$, we may apply Observation 2 and obtain that $G[K_{\leq i-1}]$ is isomorphic to R. By Theorem 1, the clique paths of $G[K_{\leq i-1}]$ and R are unique. Hence, $i = q + 1$, and the statement of the theorem follows. □

Note that Theorem 3 is not valid for interval graphs, which can be seen with the following example. Let G be the path $u_1u_2u_3u_4$ to which we add a vertex u_5 with edge u_2u_5 and a vertex u_6 with edge u_3u_6. Let $P = K_1 \cdots K_5$ be a clique path of G with $K_1 = \{u_1, u_2\}$, $K_2 = \{u_2, u_5\}$, $K_3 = \{u_2, u_3\}$, $K_4 = \{u_3, u_6\}$ and $K_5 = \{u_3, u_4\}$. Let R be the 4-vertex path 1234. The unique clique path of R is $P' = L_1L_2L_3$ with $L_1 = \{1, 2\}$, $L_2 = \{2, 3\}$ and $L_3 = \{3, 4\}$. However, we find that G has an R-role assignment r with $r(u_1) = r(u_5) = 1$, $r(u_2) = 2$, $r(u_3) = 3$, and $r(u_4) = r(u_6) = 4$.

Also note that we can apply Theorem 3 twice depending on the way the bags in the clique path of the proper interval graph G are ordered. This leads to a rather surprising corollary that might be of independent interest.

Corollary 1. *Let G be a connected proper interval graph with clique path $P = K_1 \cdots K_p$. If G has an R-role assignment and R is connected, then $R \simeq G[K_{\leq i}]$ and $R \simeq G[K_{\geq p-i+1}]$, for some $1 \leq i \leq p$.*

As an illustration of Corollary 1 we have indicated the two copies of R in G with bold edges in Figure 1. Due to Theorem 2 we do not need to restrict R to be a proper interval graph in the statement of the above corollary. Hence for any two connected graphs G and R, where G is proper interval with $|V_G| > |V_R|$, if G has an R-role assignment then G contains two (not necessarily vertex-disjoint) induced subgraphs isomorphic to R.

Theorem 3 only shows what an R-role assignment r of a proper interval graph G looks like at the beginning and end of the clique path of G. To derive our algorithm, we need to know the behavior of r in the middle bags as well. We therefore give the following result, which is valid when R has at least three maximal cliques and the number of maximal cliques in G is not too small. Its proof is postponed to the journal version of this paper. The special cases when R has just one or two maximal cliques or G has few maximal cliques will be dealt with separately in the proof of Theorem 4.

Lemma 2. *Let G be a connected proper interval graph with clique path $P = K_1 \cdots K_p$. Let R be a connected proper interval graph with clique path $P' = L_1 \cdots L_q$ and ordered twin sets $X_1, \ldots, X_t$. Let r be an R-role assignment of G with $r(K_q) = L_q$. Let T be the subset of K_q that consists of all vertices with roles in X_t. Then the following holds if $q \geq 3$ and $p \geq 2q + 1$.*

(i) If there is a vertex in T not in K_{q+1}, then there exists an index $i \geq q+1$ such that $K_{\geq q+1} \setminus K_{\leq q} \subseteq K_{\geq i}$ and the restriction of r to $K_{\geq i}$ is an R-role assignment of $G[K_{\geq i}]$ with $r(K_i) = L_q$. Furthermore, if $i > q+1$ then $r(K_h) \subseteq X_t$ for $h = q+1, \ldots, i-1$.

(ii) If all vertices in T are in K_{q+1}, then there exists an index $i \geq q + 1$ such that $T = K_{\leq i-1} \cap K_i$ and $T \cap K_{i+1} = \emptyset$, and the restriction of r to $K_{\geq i}$ is an R-role assignment of $G[K_{\geq i}]$ with $r(K_i) = L_q$.

Let G and R be two connected proper interval graphs with clique paths $P = K_1 \cdots K_p$ and $P' = L_1 \cdots L_q$, respectively. A mapping $r : \; K_{\leq i} \to V_R$ for some $1 \le i \le p$ is a *starting R-role assignment* of $G[K_{\le i}]$ if for all $u \in K_{\le i} \setminus K_{i+1}$ we have that $r(N_G(u)) = N_R(r(u))$, and for all $u \in K_{\le i} \cap K_{i+1}$ we have that $r(N_G(u)) \subseteq N_R(r(u))$. Note that a starting R-role assignment of $G[K_{\le i}]$ is an R-role assignment of G if and only if $i = p$.

Let $1 \le i \le p$, and let r be a starting R-role assignment of $G[K_{\le i}]$. We say that $v \in K_{\le i} \cap K_{i+1}$ is *missing* role $x \in V_R$ if x is a neighbor of $r(v)$, and x is not a role of a neighbor of v in $K_{\le i}$. Let $X_1, \ldots, X_t$ be the ordered twin sets of R. We denote the set of missing roles of v that are in X_c by $M_c(v)$. We say that r can be *finished by* r^* if r^* is an R-role assignment of G with $r^*(u) = r(u)$ for all $u \in K_{\le i}$.

The following lemma is important for our algorithm.

Lemma 3. *Let G and R be two connected proper interval graphs. Let G have clique path $P = K_1 \cdots K_p$, and let R have ordered twin sets $X_1, \ldots, X_t$. Let $r : K_{\le i} \to V_R$ be a starting R-role assignment of $G[K_{\le i}]$ for some $1 \le i \le p$. Then $K_{\le i} \cap K_{i+1}$ does not contain two vertices u, v such that $M_c(u) \setminus M_c(v) \neq \emptyset$ and $M_c(v) \setminus M_c(u) \neq \emptyset$ for some $1 \le c \le t$.*

Proof. In order to derive a contradiction, assume that such vertices u and v exist. Note that u and v are adjacent, because both of them belong to bag K_{i+1}. Let $x \in M_c(u) \setminus M_c(v)$ and $y \in M_c(v) \setminus M_c(u)$. Because u misses x and $x \in X_c$, we find that $r(u)$ is adjacent to all roles in $X_c \setminus \{r(u)\}$. Hence $r(u)$ is adjacent to $y \in X_c$, unless $r(u) = y$. However, the latter case is not possible, because in that case v, being adjacent to u, would not miss y. So, indeed $r(u)$ and y are adjacent. From $y \in M_c(v) \setminus M_c(u)$ we then deduce that u already has a neighbor $w \in K_{\le i}$ with role $r(w) = y$. Since v misses y and R contains no self-loop, we find that $r(v) \neq y$, and consequently $w \neq v$. Since v misses y, the edge uw must be in a bag before v got introduced. Hence, we obtain $f_P(u) < f_P(v)$. Analogously, we get $f_P(v) < f_P(u)$. This is not possible, and we have proven Lemma 3. □

We are now ready to present our main result.

Theorem 4. Role Assignment *can be solved in polynomial time on input pairs (G, R) where G is a proper interval graph and R is an arbitrary graph.*

Proof. First we give an algorithm with running time $\mathcal{O}(n^3)$ that takes as input a *connected* proper interval graph G and a *connected* graph R, and decides whether G has an R-role assignment.

If $|V_R| > n$ or R is not a proper interval graph, then we know by respectively Observation 2 and Theorem 2 that the answer is `NO`. These conditions can be checked in linear time, as explained in the preliminaries. Thus we assume that $|V_R| \le n$ and R is a proper interval graph.

Let G have clique path $P = K_1 \cdots K_p$. Recall that P can be constructed in linear time. Let R have clique path $P' = L_1 \cdots L_q$ and ordered twin sets $X_1, \ldots, X_t$. Because $|V_R| \le n$, we find that $q \le p$ and that we can compute P' and the ordered twin sets in $\mathcal{O}(|V_R| + |E_R|) = \mathcal{O}(n^2)$ time. Since Lemma 2 applies only when $q \ge 3$, we distinguish between the cases where $q = 1$, $q = 2$, and $q \ge 3$.

Case 1. $q = 1$**.** Then R is a complete graph. By Theorem 3, we find that $|K_1| = |L_1|$ must hold, and we give each vertex in K_1 a different role. This yields a starting R-role assignment r of $G[K_1]$.

Suppose $i \geq 1$ and that we have extended r to a starting R-role assignment of $G[K_{\leq i}]$. By Lemma 3 we can order the vertices in $K_i \cap K_{i+1}$ as $u_1, \ldots, u_b$ such that $M_1(u_a) \subseteq M_1(u_{a+1})$ for $a = 1, \ldots, b-1$. We assign different roles to the vertices of $K_{i+1} \setminus K_i$, where we first use the roles of $M_1(u_a)$ before using any roles of $M_1(u_{a+1})$ for $a = 1, \ldots, b-1$. If we have used all the roles and there are still vertices in K_{i+1} with no role yet, we output `NO`. Otherwise we must verify if the resulting mapping is a starting R-role assignment of $G[K_{\leq i+1}]$ by checking if all vertices in $K_{i+1} \setminus K_{i+2}$ have neighbors with all the required roles. If this is not the case, we output `NO`, because any R-role assignment is a starting role assignment of $G[K_{\leq i+1}]$. If this is the case, we stop if $i+1 = p$, because a starting R-role assignment of $G[K_{\leq p}] = G$ is an R-role assignment of G; otherwise we repeat the above procedure with $i := i + 1$.

It is clear that this algorithm is correct. It runs in $\mathcal{O}(n^3)$ time, because ordering the vertices in $K_i \cap K_{i+1}$ takes $\mathcal{O}(n^2)$ time and there are $\mathcal{O}(n)$ bags.

Case 2. $q = 2$**.** The algorithm for this case uses similar arguments as above (but in a more advanced way). Due to space restrictions we postpone its proof.

Case 3. $q \geq 3$**.** First suppose $p \leq 2q$. By Theorem 3, both $G[K_{\leq q}]$ and $G[K_{\geq p-q+1}]$ must be isomorphic to R and have an R-role assignment, in case G has an R-role assignment. Because $p \leq 2q$, every vertex of G is in $K_{\leq q} \cup K_{\geq p-q+1}$. Hence, there are just four possibilities of assigning roles to vertices of G, namely two possibilities for $K_{\leq q}$ combined with two possibilities for $K_{\geq p-q+1}$. We check if one of them leads to an R-role assignment of G. Verifying whether a mapping $V_G \rightarrow V_R$ is an R-role assignment of G can be done in $\mathcal{O}(n^3)$ time by considering each vertex and checking if it has the desired roles occurring in its neighborhood.

Suppose $p \geq 2q + 1$. We first check if $G[K_{\leq q}]$ is isomorphic to R. This can be done in linear time [18]. If $G[K_{\leq q}]$ is not isomorphic to R then we output `NO` due to Theorem 3. Suppose $G[K_{\leq q}] \simeq R$ and that without loss of generality we have a starting R-role assignment r of $G[K_{\leq q}]$ with $r(K_i) = L_i$ for $i = 1, \ldots, q$. We now check whether we are in situation (i) or (ii) of Lemma 2. Then in both situations we can determine in $\mathcal{O}(n)$ time the desired index i and afterwards we continue with the graph $G[K_{\geq i}]$ unless we found no starting R-role assignment of $G[K_{\leq i}]$; in that case we output `NO`. The total running time of this procedure is $\mathcal{O}(n^3)$.

We have thus presented and proved the correctness of an algorithm with running time $\mathcal{O}(n^3)$ for testing whether a connected proper interval graph G has an R-role assignment for a connected graph R. If G is disconnected then we run the algorithm on each connected component separately. The total running time is still $\mathcal{O}(n^3)$. It remains to study the case when R is disconnected. In this case we cannot assume that $|V_R| \leq |V_G|$. Let c_R be the number of connected components of R. By the definition of a role assignment, G has an R-role assignment if and only if each connected component of G has an R'-role assignment for some connected component R' of R. Hence we can run our algorithm on every pair of connected components of G and R. This gives a total running time $\mathcal{O}(n^3 \cdot c_R)$, which is clearly polynomial. □

Recall that the problem of testing if a graph G has an R-role assignment for some smaller graph R is co-NP-complete in general [5]. Theorem 4 together with Corollary 1 has the following consequence.

Corollary 2. *There exists a polynomial time algorithm that has as input a proper interval graph G and that tests whether there exists a graph R with $|V_R| < |V_G|$ such that G has an R-role assignment.*

Proof. Let G be a proper interval graph on n vertices. First assume that G is connected. Let $P = K_1 \dots K_p$ be the clique path of G. Recall that $p \leq n$. By Corollary 1 we find that G only has an R-role assignment if $R \simeq G[K_{\leq i}]$ for some $1 \leq i \leq p$. This means that we need to apply the $\mathcal{O}(n^3)$ time algorithm for connected proper interval graphs of Theorem 4 at most $p \leq n$ times. Hence we find that testing whether G has an R-role assignment for some graph R with $|V_R| < |V_G|$ takes $\mathcal{O}(n^4)$ time.

Now assume that G is disconnected. Let $G_1, \dots, G_a$ with $a \geq 2$ be the connected components of G. For $j = 1, \dots, a$ we define $n_j = |V_{G_j}|$. As long as $j \leq a - 1$ we do as follows. We consider G_j and check if G_j has an R_j-role assignment for some role graph R_j with $|V_{R_j}| \leq n_j$. If so, then we replace connected component G_j by connected component R_j in G, i.e., we output $R = G_1 \oplus \dots G_{j-1} \oplus R_j \oplus G_j \oplus \dots \oplus G_a$, where $\oplus$ denotes the disjoint union operation on graphs. Suppose not. Then we consider G_{j+1}. If $j = a$ and we did not find a suitable role graph R in this way, then we output NO. Because we need $\mathcal{O}(n_j^4)$ time for each G_j and $n = n_1 + \dots + n_a$, the total running time of this algorithm is $\mathcal{O}(n^4)$, which is polynomial, as desired. □

As a consequence, we have in fact a stronger result: given a proper interval graph G, we can list in polynomial time all graphs R (up to isomorphism) with $|V_R| < n$ such that G has an R-role assignment.

4 Complementary Results and an Open Question

A homomorphism r from a graph G to a graph R is *locally injective* if $|r(N_G(u))| = |N_G(u))|$ for every $u \in V_G$, and r is *locally bijective* if $r(N_G(u)) = N_R(r(u))$ and $|r(N_G(u))| = |N_G(u))|$ for every $u \in V_G$. Locally injective homomorphisms, also called *partial coverings*, have applications in frequency assignment [8] and telecommunication [9]. Locally bijective homomorphisms are also called *coverings* and have applications in topological graph theory [20] and distributed computing [1, 2]. The corresponding decision problems, called Partial Cover and Cover respectively, are NP-complete for arbitrary G even when R is fixed to be the complete graph on four vertices [9, 16].

In this section, to give a complete picture, we study the computational complexity of all three locally constrained homomorphisms on chordal, interval, and proper interval graphs. Our findings can be summarized in the table below, where the three problems have input (G, R) and the left column indicates the graph class that G belongs to. In the table, R is assumed to be an arbitrary graph.

	Partial Cover	Cover	Role Assignment
Chordal	NP-complete	GI-complete	GI-hard
Interval	NP-complete	Polynomial	?
Proper Interval	NP-complete	Polynomial	Polynomial

We start with the following result, which allows us to conclude several of the entries in the above table, and which can be viewed as interesting on its own.

Theorem 5. *Let G be a chordal graph and let R be a connected graph. Then there exists a locally bijective homomorphism from G to R if and only if every connected component of G is isomorphic to R.*

Proof. If G is disconnected then we consider each connected component of G separately. Assume that G is connected. If G is isomorphic to R, then the identity mapping from G to R is our desired locally bijective homomorphism.

For the reverse implication, suppose that there exists a locally bijective homomorphism r from G to R. Because any locally bijective homomorphism is also locally surjective, we can apply Theorem 2 in order to find that R is chordal. For the same reason we can apply Observation 2 in order to find that each vertex in R appears as a role of at least one vertex in G. We claim that each vertex in R appears as a role of exactly one vertex in G. In order to derive a contradiction, suppose there exists a vertex $x \in V_R$ such that $r^{-1}(x)$ has size at least two.

Let v and v' be two different vertices of G belonging to $r^{-1}(x)$. Let P be a shortest path from v to v' in G. Because P is shortest, P is an induced path. From the definition of a locally bijective homomorphism we deduce the following two statements. Firstly, because two vertices with the same role cannot be adjacent, we find that $|V_P| \neq 2$. Secondly, because a vertex has no two neighbors with the same role, we find that $|V_P| \neq 3$. Hence, P is an induced path with $|V_P| \geq 4$. This, together with $r(v) = r(v') = x$, means that $r(P)$ forms an induced cycle D in R with $|V_D| = |V_P| - 1$. Because R is chordal, D must consist of three vertices, say $D = xyzx$. Consequently, $|V_P| = 4$ holds.

Let C be the connected component of $G[r^{-1}(x) \cup r^{-1}(y) \cup r^{-1}(z)]$ that contains v and v'. By definition of a locally bijective homomorphism, every vertex is of degree two in D. This means that D is an induced cycle in G. Because every vertex of P belongs to D, and $|V_P| = 4$, we find that $|V_D| \geq 4$. This contradicts our assumption that G is chordal. We conclude that indeed each vertex in R appears as a role of exactly one vertex in G. This means that r is an isomorphism between G and R, and we find that $G \simeq R$, as desired. □

It is known that Graph Isomorphism is Graph Isomorphism-complete even for pairs (G, R) where G and R are chordal graphs [18]. This implies together with Theorem 5 that Cover is Graph Isomorphism-complete for pairs

(G, R) where G and R are chordal graphs. On the other hand, Cover is polynomial time solvable on interval graphs, and hence also on proper interval graphs, since isomorphism between two interval graphs can be checked in polynomial time [18]. Because every locally bijective homomorphism is locally surjective, we can use Theorem 2 to deduce that these three results stay valid for input pairs (G, R) where only G is required to be chordal and R may be an arbitrary graph. This explains the three corresponding entries in the table.

Unfortunately, as indicated in the table, the problem Partial Cover remains NP-complete even on pairs (G, R) where G is a proper interval graph (and R is an arbitrary graph). To see this, observe that a complete graph G allows a locally injective homomorphism to an arbitrary graph R if and only if R contains G as a subgraph. This gives a reduction from the well-known NP-complete problem Clique (cf. [13]).

We present one more complexity result on the Role Assignment problem. This result explains a corresponding entry in the table after applying Theorem 2. It shows that, unless Graph Isomorphism is polynomial time solvable, we do not have hope of solving Role Assignment in polynomial time on chordal graphs.

Theorem 6. Role Assignment *is* Graph Isomorphism*-hard on input pairs (G, R) where G and R are chordal graphs.*

Proof. As we argued above, Cover is Graph Isomorphism-complete on input pairs (G, R) where both G and R are chordal graphs. It is not hard to see that we may also assume that G and R are connected and have the same number of vertices. We give a polynomial time reduction from Cover to Role Assignment. Let G and R be two connected role graphs with $|V_G| = |V_R|$. We claim that G allows a locally bijective homomorphism to R if and only if G allows a locally surjective homomorphism to R.

Suppose G allows a locally bijective homomorphism r to R. Because any locally bijective homomorphism is locally surjective by definition, r is a locally surjective homomorphism from G to R. To prove the reverse implication, suppose G allows a locally surjective homomorphism to R. Recall that $|V_G| = |V_R|$. Then we use Observation 2 to deduce that $G \simeq R$. Hence, G allows a locally bijective homomorphism to R, namely the identity mapping. This completes the reduction and the proof. □

Just as for Role Assignment, we denote the problems Cover and Partial Cover as R-Cover and R-Partial Cover, respectively, if R is fixed, i.e., not a part of the input. In that case we obtain the following result.

Proposition 1. *For any fixed R, the problems R-*Role Assignment*, R-*Cover*, and R-*Partial Cover *can be solved in linear time on chordal graphs.*

Proof. We first observe that a homomorphism from G to R maps the vertices in a clique of G to different vertices of R. Hence, in order to get a `YES` answer, a largest clique in G can have at most $|V_R|$ vertices. We compute the number of vertices in a largest clique of G in linear time. If this number is greater than $|V_R|$, we output NO. Otherwise, because the treewidth of a chordal graph is equal to the number of vertices in a largest clique minus 1, we find that G has treewidth bounded by $|V_R|$, which is a constant, as R is fixed. Since all three problems are expressible in monadic second order logic, linear time solvability follows from a well-known result of Courcelle [6]. □

We conclude with the following two open questions resulting from the table.

1. Is Role Assignment NP-complete on input pairs (G, R) when G is a chordal graph?

2. What is the computational complexity of Role Assignment on input pairs (G, R) when G is an interval graph?

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