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A new characterization of P_6-free graphs[☆,☆☆]

Pim van 't Hof, Daniël Paulusma[*]

Department of Computer Science, Durham University, Science Laboratories, South Road, Durham DH1 3LE, England, United Kingdom

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ABSTRACT

We study P_6-free graphs, i.e., graphs that do not contain an induced path on six vertices. Our main result is a new characterization of this graph class: a graph G is P_6-free if and only if each connected induced subgraph of G on more than one vertex contains a dominating induced cycle on six vertices or a dominating (not necessarily induced) complete bipartite subgraph. This characterization is minimal in the sense that there exists an infinite family of P_6-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. Our characterization of P_6-free graphs strengthens results of Liu and Zhou, and of Liu, Peng and Zhao. Our proof has the extra advantage of being constructive: we present an algorithm that finds such a dominating subgraph of a connected P_6-free graph in polynomial time. This enables us to solve the HYPERGRAPH 2-COLORABILITY problem in polynomial time for the class of hypergraphs with P_6-free incidence graphs.

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1. Introduction

All graphs in this paper are undirected, finite, and simple, i.e., without loops and multiple edges. Furthermore, unless specifically stated otherwise, all graphs are non-trivial, i.e., contain at least two vertices. For undefined terminology we refer to [9].

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. We say that the graph G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. A subgraph G' of G is an *induced subgraph* of G if G' contains all the edges $xy \in E$ with $x, y \in V'$; we say that V' *induces* the subgraph G'. We write $G[V']$ to denote the subgraph of G induced by V'. A subset $S \subseteq V$ is called a *clique* if $G[S]$ is a complete graph. A clique S is called *maximal* if for every proper superset S' of S the graph $G[S']$ is not complete. We write P_k, C_k, K_k to denote the path, cycle and complete graph on k vertices, respectively. A graph $G = (V, E)$ is called *bipartite* if V can be partitioned into two classes V_1, V_2, called the *partition classes* of G, such that every edge of G connects a vertex in V_1 with a vertex in V_2. A bipartite graph G is called *complete* if every two vertices from different partition classes of G are adjacent. A set $U \subseteq V$ *dominates* a set $U' \subseteq V$ if any vertex $v \in U'$ either lies in U or has a neighbor in U. We also say that U *dominates* $G[U']$. A subgraph H of G is a *dominating subgraph* of G if $V(H)$ dominates G.

A graph G is called H-*free* for some graph H if G does not contain an induced subgraph isomorphic to H. For any family $\mathcal{F}$ of graphs, let $Forb(\mathcal{F})$ denote the class of graphs that are F-free for every $F \in \mathcal{F}$. We consider the class $Forb(\{P_t\})$ of graphs that do not contain an induced path on t vertices. Note that $Forb(\{P_2\})$ is the class of graphs without any edge and $Forb(\{P_3\})$ is the class of graphs all components of which are complete graphs.

The class of P_4-free graphs (or cographs) has been studied extensively (cf. [6]). Wolk [19,20] shows that a graph G is P_4-free and C_4-free if and only if each connected induced subgraph of G contains a dominating vertex (see also Theorem 11.3.4

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* Corresponding author. Tel.: + 44 0 191 33 41723; fax: + 44 0 191 33 41701.
E-mail addresses: pim.vanthof@durham.ac.uk (P. van 't Hof), daniel.paulusma@durham.ac.uk (D. Paulusma).

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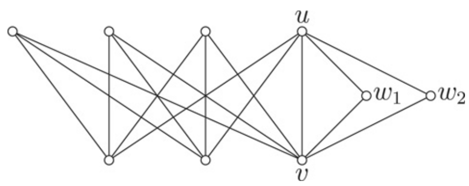

Fig. 1. An example of a TECB graph.

in [6]). We show in Section 3 that we can slightly generalize this theorem to obtain the following characterization of P_4-free graphs.

Theorem 1. *A graph G is P_4-free if and only if each connected induced subgraph of G contains a dominating induced C_4 or a dominating vertex.*

There are many other characterizations of the class of P_4-free graphs in the literature. We mention one by Bacsó and Tuza [1], who show that a graph G is P_4-free if and only if in every connected induced subgraph G' of G, all maximal cliques dominate G'.

Apart from characterizing the class of P_4-free graphs, Bacsó and Tuza also characterize the class $Forb(\{P_5, C_5\})$ in [1]. There, they prove that a graph G is P_5-free and C_5-free if and only if each connected induced subgraph of G contains a dominating clique. The same result has been independently proven by Cozzens and Kelleher [8]. Liu and Zhou [15] improve this by obtaining the following characterization of P_5-free graphs.

Theorem 2 ([15]). *A graph G is P_5-free if and only if each connected induced subgraph of G contains a dominating induced C_5 or a dominating clique.*

A graph G is called *triangle extended complete bipartite (TECB)* if it is a complete bipartite graph or if it can be obtained from a complete bipartite graph F by adding some extra vertices $w_1, \dots, w_r$ and edges w_iu, w_iv for $1 \le i \le r$ to exactly one edge uv of F (see Fig. 1 for an example).

The following characterization of P_6-free graphs is due to Liu, Peng and Zhao [16].

Theorem 3 ([16]). *A graph G is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) TECB graph.*

If we consider graphs that are not only P_6-free but also triangle-free, then we have one of the main results in [15].

Theorem 4 ([15]). *A triangle-free graph G is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating complete bipartite graph.*

A characterization of $Forb(\{P_t\})$ for $t \ge 7$ is given in [2]: $Forb(\{P_t\})$ is the class of graphs for which each connected induced subgraph has a dominating subgraph of diameter at most $t - 4$.

Our results

Section 4 contains our main result.

Theorem 5. *A graph $G = (V, E)$ is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph. Moreover, we can find such a dominating subgraph of G in $\mathcal{O}(|V|^3)$ time.*

This theorem strengthens Theorems 3 and 4 in two different ways. Firstly, Theorem 5 shows that we may omit the restriction "triangle-free" in Theorem 4 and that we may replace the class of TECB graphs by its proper subclass of complete bipartite graphs in Theorem 3. Secondly, in contrast to the proofs of Theorems 3 and 4, the proof of Theorem 5 is constructive: we provide an algorithm for finding a desired dominating subgraph of a P_6-free graph $G = (V, E)$ in $\mathcal{O}(|V|^3)$ time. Note that we cannot use some brute force approach to obtain such a polynomial time algorithm, since a dominating complete bipartite graph might have arbitrarily large size.

To illustrate the incremental technique used to construct the $\mathcal{O}(|V|^3)$ time algorithm in the proof of Theorem 5, we use the same technique to give constructive proofs of Theorems 1 and 2 in Section 3. We point out that the proof of Theorem 2 in [15] is non-constructive, and to our knowledge there is no constructive proof of Theorem 1 in the literature. In particular, we present an $\mathcal{O}(|V|^3)$ time algorithm that finds a dominating induced C_5 or a dominating clique of a P_5-free graph $G = (V, E)$. Note that we cannot use a brute force approach to find a dominating clique of a P_5-free graph, as such a clique might have arbitrarily large size. Bacsó and Tuza [1], and independently Cozzens and Kelleher [8], present a polynomial time algorithm that finds a dominating clique of a connected graph without an induced P_5 or C_5. When run on a P_5-free graph G that does contain a C_5, the algorithms in [1,8] either find a dominating clique of G or terminate and state that G contains an induced C_5. We point out that the algorithm in our proof of Theorem 2 always finds a dominating clique or a *dominating* induced C_5 in a P_5-free graph.

As mentioned before, Section 4 contains a constructive proof of Theorem 5. We also show that the characterization in Theorem 5 is minimal in the sense that there exists an infinite family of P_6-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. We would like to mention that the algorithm used to prove Theorem 5 also works for an arbitrary (not necessarily P_6-free) graph G: in that case the algorithm either finds a dominating subgraph as described in Theorem 5 or finds an induced P_6 in G. We end Section 4 by characterizing the class of graphs for which each connected induced subgraph has a dominating induced C_6 or a dominating *induced* complete bipartite subgraph, again by giving a constructive proof. This class consists of graphs that, apart from P_6, have exactly one more forbidden induced subgraph (the so-called *net*). This generalizes a result in [3].

As an application of our main result, we consider the Hypergraph 2-Colorability problem in Section 5. It is well-known that this problem is NP-complete in general (cf. [11]). We prove that for the class of hypergraphs with P_6-free incidence graphs the problem becomes polynomially solvable. Moreover, we show that for any 2-colorable hypergraph H with a P_6-free incidence graph, we can find a 2-coloring of H in polynomial time.

Section 6 contains the conclusions, discusses a number of related results in the literature and mentions open problems.

2. An outline of our algorithms

In this section we outline the on-line approach used in the proofs of Theorems 1, 2 and 5. In each of the proofs we describe an algorithm that finds a desired dominating set D of an input graph $G = (V, E)$. The algorithm first establishes a *connected order* $\pi = x_1, \ldots, x_{|V|}$ of V, i.e., an order $\pi = x_1, \ldots, x_{|V|}$ of the vertices of G such that $G_i := G[\{x_1, \ldots, x_i\}]$ is connected for $i = 1, \ldots, |V|$. It then processes the vertices of G one-by-one in a vertex-incremental way, i.e., by adding the next vertex in the order π in every step. Assuming that in an earlier step the algorithm has found a desired dominating subgraph D_{i-1} of G_{i-1}, it adds the next vertex x_i and extends the previously found solution. If the set D_{i-1} dominates G_i, the algorithm sets $D_i := D_{i-1}$ and continues with the next step. Otherwise, it uses the set D_{i-1} plus one or more extra vertices of G_i to find a desired dominating set D_i of G_i. We show that such a transformation can be done in polynomial time by making use of a so-called *minimizer*. Since the algorithm only performs $|V|$ steps, the total running time stays polynomial. For computational complexity purposes, we represent a graph $G = (V, E)$ by its *adjacency matrix*, i.e., the $|V| \times |V|$ matrix $A = (a_{ij})$ with rows and columns indexed by the vertices of V such that $a_{uv} = 1$ if $uv \in E$ and $a_{uv} = 0$ otherwise.

In order to explain the concept of a minimizer we need the following terminology for a graph $G = (V, E)$. Let $w \in V$ and $D \subseteq V$. Let $N_G(w)$ denote the set of neighbors of w in G. We write $N_D(w) := N_G(w) \cap D$ and $N_G(D) := \cup_{u \in D} N_G(u) \setminus D$. If no confusion is possible, we write $N(w)$ instead of $N_G(w)$, and $N(D)$ instead of $N_G(D)$. A vertex $v' \in V \setminus D$ is called a *D-private neighbor* of a vertex $v \in D$ if $N_D(v') = \{v\}$. We say that D is *connected* if $G[D]$ is connected.

Let u, v be a pair of adjacent vertices in a dominating set D of a graph G such that $\{u, v\}$ dominates D. Note that this means D is connected. We call a dominating set $D' \subseteq D$ of G a *minimizer of D for uv* if $\{u, v\} \subseteq D'$ and each vertex of $D' \setminus \{u, v\}$ has a D'-private neighbor in G. It is important to note that a minimizer D' is connected (because u and v are adjacent vertices in D' and $\{u, v\}$ dominates D') as we will use this property in our algorithms. The following lemma states that we can obtain a minimizer D' from D in polynomial time.

Lemma 1. *Let D be a dominating set of a graph $G = (V, E)$, and let $u, v \in D$ be a pair of adjacent vertices such that $\{u, v\}$ dominates $G[D]$. We can find a minimizer of D for uv in $\mathcal{O}(|V|^2)$ time.*

Proof. Let $D = \{w_1, \ldots, w_q\}$ with $w_1 = u$ and $w_2 = v$. Below we explain how we can obtain a minimizer D' of D for uv in $\mathcal{O}(|V|^2)$ time. We define $S_i := \emptyset$ for $i = 1, \ldots, q$. For each $x \in V$ we compute in $\mathcal{O}(|V|)$ time the index p such that $x \in N(w_p)$ but $x \notin N(w_1) \cup \cdots \cup N(w_{p-1})$ and set $S_p := S_p \cup \{x\}$. Note that such an index p always exists as $uv \in E$, $\{u, v\}$ dominates D, and D dominates G. We then obtain $\mathcal{S} = \{S_1, \ldots, S_q\}$ in $\mathcal{O}(|V|^2)$ total time. By construction, the nonempty sets in $\mathcal{S}$ form a partition of V. It is clear that $S_1 \neq \emptyset$ as $v \in S_1$ and $S_2 \neq \emptyset$ as $u \in S_2$. We will not include a vertex w_i with $S_i = \emptyset$ in D', since all its neighbors will be in $S_1 \cup \cdots \cup S_{i-1}$.

We cannot define D' as the set $\{w_k \in D \mid S_k \neq \emptyset\}$, as there might be a vertex w_i with $S_i \neq \emptyset$ whose neighbors in S_i are all adjacent to vertices in $\{w_k \in D \mid S_k \neq \emptyset \text{ and } k \geq i+1\}$. Hence we define a set $R := \emptyset$ and do as follows for $i = q, q-1, \ldots, 3$ (so for indices in decreasing order). We set $T_i := \emptyset$. For each $x \in S_i$ we check in $\mathcal{O}(|V|)$ time if there exists an index $p' \geq i+1$ such that $x \in N(w_{p'})$ and $T_{p'} \neq \emptyset$. If so, then $R := R \cup \{x\}$. Otherwise, we set $T_i := T_i \cup \{x\}$. After setting $T_1 := S_1$ and $T_2 := S_2$ we then obtain $\mathcal{T} = \{T_1, \ldots, T_q\}$ in $\mathcal{O}(|V|^2)$ total time.

We define $D' := \{w_i \in D \mid T_i \neq \emptyset\}$. By definition, D' is a minimizer of D for uv if D' dominates G, $\{u, v\} \subseteq D'$, and each vertex of $D' \setminus \{u, v\}$ has a D'-private neighbor in G. Clearly, $\{u, v\} \in D'$ as $T_1 = S_1 \neq \emptyset$ and $T_2 = S_2 \neq \emptyset$. By construction, each vertex in $T_i \subseteq S_i$ is a neighbor of w_i and the nonempty sets in $\mathcal{T}$, together with R in case $R \neq \emptyset$, form a partition of the set of vertices in $S_1 \cup \cdots \cup S_p = V$. Hence, D' dominates $V \setminus R$. Let $x \in R$. We claim that x is adjacent to a vertex in D'. Suppose $x \in S_h$. Because $x \in R$, we have $h \geq 3$. By our algorithm, $x \in N(w_j)$ for some w_j with $j \geq h+1$ and $T_j \neq \emptyset$. By definition, $w_j \in D'$. Hence, D' dominates G.

We claim that for every $i \geq 3$ each vertex in T_i with $w_i \in D'$ is a D'-private neighbor of w_i in G. This can be seen as follows. First suppose $x \in T_i$ for some $w_i \in D'$ with $i \geq 3$ is a neighbor of some vertex in $\{w_1, \ldots, w_{i-1}\} \cap D'$. Then $x \notin S_i$, and consequently $x \notin T_i$ as $T_i \subseteq S_i$, a contradiction. Now suppose that x is a neighbor of some vertex in $\{w_{i+1}, \ldots, w_q\} \cap D'$. Then x is the neighbor of some vertex w_j with $j \geq i+1$ and $T_j \neq \emptyset$ by definition of D'. Then, by our algorithm, $x \in R$ instead of

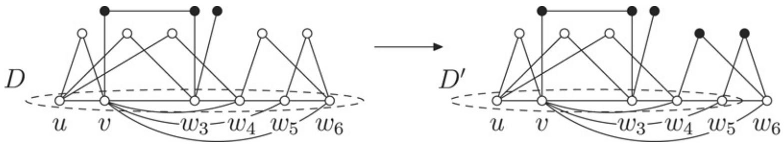

Fig. 2. A dominating set D and a minimizer D' of D for uv.

$x \in T_i$, a contradiction. Hence, all vertices of T_i are D'-private neighbors of w_i in G. We conclude that D' is indeed a minimizer of D for uv. □

We point out that a connected dominating set D of a graph G may have several minimizers for the same edge depending on the order in which its vertices are considered.

Example. Consider the graph G and its connected dominating set D in the left-hand side of Fig. 2. All D-private neighbors are colored black. Note that u and v are two adjacent vertices in D and that $\{u, v\}$ dominates D. That means we can find a minimizer of D for uv by applying the algorithm described in the proof of Lemma 1.

Suppose we choose the order $w_1 = u, w_2 = v, w_3, w_4, w_5, w_6$. We first find nonempty sets S_1, S_2, S_3, S_4, S_5 and we conclude that S_6 is empty. Then we find that T_6 is empty, and that T_5, T_4, T_3 as well as $T_2 = S_2$ and $T_1 = S_1$ are nonempty. The set $D' := \{u, v, w_3, w_4, w_5\}$ is a minimizer of D for uv. The right-hand side of Fig. 2 shows the graph G and the minimizer D' of D for uv: every vertex in $D' \setminus \{u, v\}$ has a black colored D'-private neighbor. Note that u does not have a D'-private neighbor but v does.

If we choose the order $w'_1 = u, w'_2 = v, w'_3 = w_3, w'_4 = w_6, w'_5 = w_4, w'_6 = w_5$ in the algorithm described in the proof of Lemma 1, then we find a different minimizer of D for uv. We first find that S'_1, S'_2, S'_3, S'_4 are nonempty and that S'_5, S'_6 are empty. Then we find that T'_6, T'_5 are empty and T'_4, T'_3, T'_2, T'_1 are nonempty. This means that $D'' := \{u, v, w_3, w_6\}$ is a minimizer of D for uv. Note that every vertex of D'' (including u) has a D''-private neighbor.

3. Finding connected dominating subgraphs in P_4-free and P_5-free graphs

We now use the technique described in Section 2 to prove Theorem 1.

Theorem 1. *A graph G is P_4-free if and only if each connected induced subgraph of G contains a dominating induced C_4 or a dominating vertex.*

Proof. If G is not P_4-free, then G contains an induced subgraph isomorphic to P_4, and that subgraph has no dominating induced C_4 nor a dominating vertex. So to prove Theorem 1, it suffices to prove that if G is a connected P_4-free graph, then we can find a dominating induced C_4 or a dominating vertex of G.

Let $G = (V, E)$ be a connected P_4-free graph with connected order $\pi = x_1, \ldots, x_{|V|}$. Let $D_1 := \{x_1\}$. Suppose $i \geq 2$. Assume that D_{i-1} induces a dominating C_4 in G_{i-1} or is a dominating vertex of G_{i-1}. We write $x := x_i$. If $x \in N(D_{i-1})$, then we set $D_i := D_{i-1}$. Suppose otherwise. We show how we can use D_{i-1} to find a suitable dominating set D_i of G_i, which suffices to prove Theorem 1.

Since π is connected, G_i contains a vertex y (not in D_{i-1}) adjacent to x.

Case 1. *D_{i-1} induces a dominating C_4 in G_{i-1}.*

We write $G[D_{i-1}] = c_1c_2c_3c_4c_1$. Without loss of generality, assume that y is adjacent to c_1. Then y must also be adjacent to c_2 (respectively c_4), as otherwise xyc_1c_2 (respectively xyc_1c_4) is an induced P_4, contradicting the P_4-freeness of G_i. In fact, y must be adjacent to c_3 as well, since otherwise xyc_2c_3 would be an induced P_4 in G_i. If y dominates G_i, then we choose $D_i := \{y\}$. Otherwise, let z be a vertex of G_i not adjacent to y. Since D_{i-1} dominates z, z must be adjacent to at least one vertex c_k of D_{i-1}. The path xyc_kz and P_4-freeness of G_i imply that z must be adjacent to x. Hence the set $C := \{x, y, c_k, z\}$ induces a C_4 in G_i. We claim that C also dominates G_i, which means we can choose $D_i := C$. Suppose C does not dominate G_i, and let z' be a vertex not dominated by C. Since D_{i-1} dominates G_{i-1}, z' must be adjacent to at least one vertex c_ℓ in $D_{i-1} \setminus \{c_k\}$. Then $z'c_\ell yx$ is an induced P_4 in G_i, a contradiction.

Case 2. *D_{i-1} is a dominating vertex of G_{i-1}.*

We write $D_{i-1} = \{d\}$. If y dominates G_i, then we choose $D_i := \{y\}$. Otherwise, let z be a vertex of G_i not adjacent to y. Since d dominates z, D_i contains the path $xydz$. The P_4-freeness of G_i implies that z must be adjacent to x. Note that $\{x, y, d, z\}$ dominates G_i, since d dominates every vertex in G_{i-1}. Since $\{x, y, d, z\}$ also induces a C_4 in G_i, we can choose $D_i := \{x, y, d, z\}$. □

Note that we can easily find a dominating vertex or a dominating induced C_4 of a P_4-free graph $G = (V, E)$ in $\mathcal{O}(|V|^4)$ time by using a brute force approach. Using the technique described in the proof of Theorem 1, we can find such a subgraph in $\mathcal{O}(|V|^2)$ time, as transforming a set D_{i-1} to D_i takes $\mathcal{O}(|V|)$ time and there are $|V| - 1$ of such transformations. Note that finding a minimizer was not necessary here.

We now present an algorithmic proof of Theorem 2, again using the techniques described in Section 2.

Theorem 2. *A graph G is P_5-free if and only if each connected induced subgraph of G contains a dominating induced C_5 or a dominating clique.*

Proof. If G is not P_5-free, then G contains an induced subgraph isomorphic to P_5, and that subgraph has no dominating induced C_5 nor a dominating clique. So to prove Theorem 2, it suffices to prove that if G is a connected P_5-free graph, then we can find a dominating induced C_5 or a dominating clique of G.

Let $G = (V, E)$ be a connected P_5-free graph with connected order $\pi = x_1, \ldots, x_{|V|}$. Let $D_1 := \{x_1\}$. Suppose $i \geq 2$. Assume that D_{i-1} induces a dominating C_5 in G_{i-1} or is a dominating clique of G_{i-1}. We write $x := x_i$. If $x \in N(D_{i-1})$, then we set $D_i := D_{i-1}$. Suppose otherwise. We show how we can find a suitable dominating set D_i of G_i from D_{i-1}.

Since π is connected, G_i contains a vertex y (not in D_{i-1}) adjacent to x.

Case 1. *D_{i-1} induces a dominating C_5 in G_{i-1}.*

We write $G[D_{i-1}] = c_1c_2c_3c_4c_5c_1$. Since D_{i-1} dominates G_{i-1} and $y \in V(G_{i-1})$, y must be adjacent to D_{i-1}. Without loss of generality, we assume that y is adjacent to c_1. Obviously, $D^1 := D_{i-1} \cup \{y\}$ dominates G_i. Suppose c_3 has a D^1-private neighbor c_3'. Then $c_3'c_3c_4c_5c_1$ is an induced P_5 in G_i, a contradiction. Hence c_3 has no D^1-private neighbor and $D^2 := D^1 \setminus \{c_3\}$ dominates G_i. Similarly, c_4 has no D^2-private neighbor c_4', since otherwise $c_4'c_4c_5c_1c_2$ would be an induced P_5. So $D^3 := D^2 \setminus \{c_4\} = \{c_1, c_2, c_5, y\}$ still dominates G_i.

Suppose c_2 does not have a D^3-private neighbor. Then $D^4 := \{y, c_1, c_5\}$ dominates G_i. If c_5 has no D^4-private neighbor, then $\{y, c_1\}$ dominates G_i and is a clique of G_i, so we choose $D_i := \{y, c_1\}$. Suppose c_5 has a D^4-private neighbor c_5'. Since $c_5'c_5c_1yx$ is a path on five vertices, we must have $yc_5 \in E(G_i)$ or $xc_5' \in E(G_i)$. If $yc_5 \in E(G_i)$, then $\{y, c_1, c_5\}$ is a clique of G and we choose $D_i := \{y, c_1, c_5\}$. In case $xc_5' \in E(G_i)$, we can choose $D_i := \{x, y, c_1, c_5, c_5'\}$ since $\{x, y, c_1, c_5, c_5'\} \supset D^4$ dominates G_i and induces a C_5 in G.

So we may without loss of generality assume that c_2 has a D^3-private neighbor c_2'. Suppose $yc_2 \notin E(G_i)$. Since $xyc_1c_2c_2'$ is a P_5 in G_i, we must have $xc_2' \in E(G_i)$. Let $D := \{x, y, c_1, c_2, c_2'\}$. We claim that D dominates G_i. Suppose, for contradiction, that there exists a vertex $z_1 \notin N_{G_i}(D)$. Since z_1 must be adjacent to D^3, we have $z_1c_5 \in E(G_i)$. But then $z_1c_5c_1c_2c_2'$ is an induced P_5 in G_i, a contradiction. Hence, D dominates G_i. Since $G[D]$ is isomorphic to C_5, we can choose $D_i = D$.

Suppose $yc_2 \in E(G_i)$. If c_5 has no D^3-private neighbor, then $\{y, c_1, c_2\}$ dominates G_i and we choose $D_i := \{y, c_1, c_2\}$. Assume that c_5 has a D^3-private neighbor c_5''. Using similar arguments as before, we may assume that y is adjacent to c_5. Note that the path $c_2'c_2yc_5c_5''$ cannot be induced in G_i, so c_2' must be adjacent to c_5''. Let $D' := \{c_2', c_2, y, c_5, c_5''\}$. We claim that D' dominates G_i. Suppose D' does not dominate G_i. Then there exists a vertex $z \notin N_{G_i}(D')$. Recall that $D^3 = \{c_1, c_2, c_5, y\}$ dominates G_i. Hence z must be adjacent to c_1. But then $zc_1c_2c_2'c_5''$ induces a P_5 in G_i, a contradiction. Hence D' dominates G_i. Since $G[D']$ is isomorphic to C_5, we can choose $D_i := D'$.

Case 2. *D_{i-1} is a dominating clique of G_{i-1}.*

Let y be adjacent to $d_1 \in D_{i-1}$. Since $\{y, d_1\}$ dominates $D_{i-1} \cup \{y\}$, we can compute a minimizer D of $D_{i-1} \cup \{y\}$ for yd_1 by Lemma 1. If y is adjacent to all vertices in $D \setminus \{y\}$, then D is a clique and we choose $D_i := D$. Suppose otherwise. Let d_2 be not adjacent to y. By the definition of a minimizer, d_2 has a D-private neighbor d_2'. Since the path $xyd_1d_2d_2'$ cannot be induced in G_i, we have $xd_2' \in E(G_i)$. We claim that the induced cycle $C := xyd_1d_2d_2'x$ dominates G_i. Suppose C does not dominate G_i. Then there exists a vertex $z \notin N_{G_i}(C)$. Since D dominates G_{i-1}, z must be adjacent to a vertex $d \in D \setminus \{d_1, d_2, y\}$. Then $zdd_2d_2'x$ is an induced P_5 in G_i, a contradiction. Hence we can choose $D_i := V(C)$. □

A closer analysis of the proof of Theorem 2 shows that Case 1 takes $\mathcal{O}(|V|)$ time, while Case 2 takes $\mathcal{O}(|V|^2)$ time if we apply Lemma 1 to compute the minimizer. Since there are $|V| - 1$ transformations, we find the following corollary. Note that we cannot use some brute force approach to find such a subgraph, since a dominating clique might have arbitrarily large size.

Corollary 1. *We can find a dominating induced C_5 or a dominating clique of a connected P_5-free graph $G = (V, E)$ in $\mathcal{O}(|V|^3)$ time.*

4. Finding connected dominating subgraphs in P_6-free graphs

In this section we present a constructive proof of our main result, Theorem 5. Let G be a connected P_6-free graph. We say that D is a *type 1 dominating set* of G if D dominates G and $G[D]$ is an induced C_6. We say that D is a *type 2 dominating set* of G *defined by* $A(D)$ and $B(D)$ if D dominates G and $G[D]$ contains a spanning complete bipartite subgraph with partition classes $A(D)$ and $B(D)$. In the proof of Theorem 5 we present an algorithm that finds a type 1 or type 2 dominating set of G in polynomial time by using the incremental technique described in Section 2.

Theorem 5. *A graph $G = (V, E)$ is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph. Moreover, we can find such a dominating subgraph of G in $\mathcal{O}(|V|^3)$ time.*

Proof. If a graph is not P_6-free, it contains an induced P_6 which contains neither a dominating induced C_6 nor a dominating complete bipartite graph. So to prove Theorem 5, it suffices to prove that if $G = (V, E)$ is a connected P_6-free graph, then we can find a type 1 or type 2 dominating set of G in $\mathcal{O}(|V|^3)$ time.

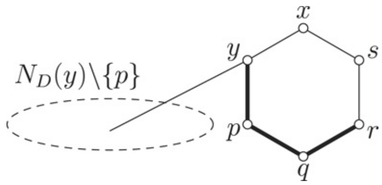

Fig. 3. The graph $G[D^1]$.

Let $G = (V, E)$ be a connected P_6-free graph with connected order $\pi = x_1, \ldots, x_{|V|}$. Let $D_2 := \{x_1, x_2\}$. Suppose $i \geq 3$. Assume that D_{i-1} is a type 1 or type 2 dominating set of G_{i-1}. We write $x := x_i$. If $x \in N(D_{i-1})$, which we can check in $\mathcal{O}(|V|)$ time, then we set $D_i := D_{i-1}$. Suppose otherwise. We show how we can use D_{i-1} to find D_i in $\mathcal{O}(|V|^2)$ time. Since the total number of iterations is $|V| - 2$, we then find a desired dominating subgraph of $G_{|V|} = G$ in $\mathcal{O}(|V|^3)$ time.

Since π is connected, G_i contains a vertex y (not in D_{i-1}) adjacent to x. We can find such a vertex in $\mathcal{O}(|V|)$ time.

Case 1. *D_{i-1} is a type 1 dominating set of G_{i-1}.*

We write $G[D_{i-1}] = c_1c_2c_3c_4c_5c_6c_1$. We claim that $D := N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i, which means that $D_i := D$ is a type 2 dominating set of G_i that is defined by $A(D_i) := \{y\}$ and $B(D_i) := \{x\} \cup N_{D_{i-1}}(y)$ and that can be obtained in $\mathcal{O}(|V|)$ time. Suppose D does not dominate G_i, and let $z \in V(G_i)$ be a vertex not dominated by D. Since D_{i-1} dominates G_{i-1}, we may without loss of generality assume that $yc_1 \in E(G_i)$.

Suppose $yc_4 \in E(G_i)$. Note that z is dominated by G_{i-1}. Without loss of generality, assume that z is adjacent to c_2. Consequently, y is not adjacent to c_2. Since z is not adjacent to any neighbor of y and the path $zc_2c_1yc_4c_5$ cannot be induced in G_i, either z or y must be adjacent to c_5. If $zc_5 \in E(G_i)$, then $xyc_4c_5zc_2$ is an induced P_6 in G_i. Hence $zc_5 \notin E(G_i)$ and $yc_5 \in E(G_i)$. In case $zc_6 \in E(G_i)$ we obtain an induced path $xyc_5c_6zc_2$ on six vertices, and in case $zc_6 \notin E(G_i)$ we obtain an induced path $zc_2c_1c_6c_5c_4$. We conclude $yc_4 \notin E(G_i)$.

Suppose y is not adjacent to any vertex in $\{c_3, c_5\}$. Since G_i is P_6-free and $xyc_1c_2c_3c_4$ is a P_6 in G_i, y must be adjacent to c_2. But then $xyc_2c_3c_4c_5$ is an induced P_6 in G_i, a contradiction. Hence y is adjacent to at least one vertex in $\{c_3, c_5\}$, say $yc_5 \in E(G_i)$. By symmetry (using c_5, c_2 instead of c_1, c_4) we find $yc_2 \notin E(G_i)$.

Suppose z is adjacent to c_2. The path $zc_2c_1yc_5c_4$ on six vertices and the P_6-freeness of G_i imply $zc_4 \in E(G_i)$. But then $c_2zc_4c_5yx$ is an induced P_6. Hence $zc_2 \notin E(G_i)$. Also $zc_4 \notin E(G_i)$ as otherwise $zc_4c_5yc_1c_2$ would be an induced P_6, and $zc_3 \notin E(G_i)$ as otherwise $zc_3c_2c_1yx$ would be an induced P_6. Then z must be adjacent to c_6 yielding an induced path $zc_6c_1c_2c_3c_4$ on six vertices. Hence we may choose $D_i := D$.

Case 2. *D_{i-1} is a type* 2 *dominating set of G_{i-1}.*

Since D_{i-1} dominates G_{i-1}, we may assume that y is adjacent to some vertex $a \in A(D_{i-1})$. Let $b \in B(D_{i-1})$. Note that a and b are adjacent vertices in $D_{i-1} \cup \{y\}$ and that $\{a, b\}$ dominates $D_{i-1} \cup \{y\}$. Hence we can find a minimizer D of $D_{i-1} \cup \{y\}$ for ab in $\mathcal{O}(|V|^2)$ time by Lemma 1. By definition, D dominates G_i. Also, $G[D]$ contains a spanning (not necessarily complete) bipartite graph with partition classes $A \subseteq A(D_{i-1})$, $B \subseteq B(D_{i-1}) \cup \{y\}$. Note that we have $y \in D$, because x is not adjacent to D_{i-1} and therefore x is a D-private neighbor of y. Since y might not have any neighbors in B but does have a neighbor (vertex a) in A, we chose $y \in B$.

Claim 1. *If $G[D]$ contains an induced P_4 starting in y and ending in some $r \in A$, then we can find a type* 1 *or a type* 2 *dominating set D_i of G_i in $\mathcal{O}(|V|^2)$ time.*

We prove Claim 1 as follows. Suppose $ypqr$ is an induced path in $G[D]$ with $r \in A$. Since D is a minimizer of $D_{i-1} \cup \{y\}$ for ab and $r \in D \setminus \{a, b\}$, r has a D-private neighbor s by definition. We already identified s when running the algorithm of Lemma 1. Since $xypqrs$ is a path on six vertices and $x \notin N(D_{i-1})$ holds, x must be adjacent to s. We first show that $D^1 := N_D(y) \cup \{x, y, q, r, s\}$, obtained in $\mathcal{O}(|V|)$ time, dominates G_i. See Fig. 3 for an illustration of the graph $G[D^1]$. Suppose D^1 does not dominate G. Then there exists a vertex $z \in N(D) \setminus N(D^1)$. Note that $G[(D \setminus \{y\}) \cup \{z\}]$ is connected because the edge ab makes $D \setminus \{y\}$ connected and $\{a, b\}$ dominates D. Let P be a shortest path in $G[(D \setminus \{y\}) \cup \{z\}]$ from z to a vertex $p_1 \in N_D(y)$ (possibly $p_1 = p$). Since $z \notin N(D^1)$ and $p_1 \in D^1$, we have $|V(P)| \geq 3$. This means that $Pyxs$ is an induced path on at least six vertices, unless $r \in V(P)$, since r is adjacent to s. However, if $r \in V(P)$, then the subpath $z\overrightarrow{P}r$ of P from z to r has at least three vertices, because $z \notin N(D^1)$. This means that $z\overrightarrow{P}rsxy$ contains an induced P_6, a contradiction. Hence D^1 dominates G_i.

To find a type 1 or type 2 dominating set D_i of G_i, we transform D^1 into D_i in $\mathcal{O}(|V|^2)$ time as follows. Suppose q has a D^1-private neighbor q'. Then $q'qpyxs$ is an induced P_6 in G_i, a contradiction. Hence q has no D^1-private neighbor and the set $D^2 := D^1 \setminus \{q\}$ still dominates G_i. Similarly, r has no D^2-private neighbor r', since otherwise $r'rsxyp$ would be an induced P_6 in G_i. So the set $D^3 := D^2 \setminus \{r\}$ also dominates G_i. Now suppose s does not have a D^3-private neighbor. We can check this in $\mathcal{O}(|V|^2)$ time. Then the set $D^3 \setminus \{s\}$ dominates G_i. In that case, we find a type 2 dominating set D_i of G_i defined by $A(D_i) := \{y\}$ and $B(D_i) := N_D(y) \cup \{x\}$. Assume that we found a D^3-private neighbor s' of s in G_i. Let $D^4 := D^3 \cup \{s'\}$.

Suppose $N_D(y) \setminus \{p\}$ contains a vertex p_2 that has a D^4-private neighbor p_2'. Then $p_2'p_2yxss'$ is an induced P_6, contradicting the P_6-freeness of G_i. Hence we can remove all vertices of $N_D(y) \setminus \{p\}$ from D^4, and the resulting set $D^5 := \{p, y, x, s, s'\}$ still dominates G_i. We claim that $D^6 := D^5 \cup \{q\}$ is a type 1 dominating set of G_i. Clearly, D^6 dominates G_i, since $D^5 \subseteq D^6$.

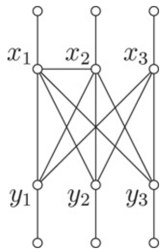

Fig. 4. The graph F_3.

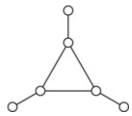

Fig. 5. The net.

Since $qpyxss'$ is a P_6 and $qpyxs$ is induced, q must be adjacent to s'. Hence D^6 is a type 1 dominating set of G_i, and we choose $D_i := D^6$. This proves Claim 1.

Let $A_1 := N_A(y)$ and $A_2 := A \setminus A_1$. Let $B_1 := N_B(y)$ and $B_2 := B \setminus (B_1 \cup \{y\})$. We can obtain these sets in $\mathcal{O}(|V|)$ time. Since $a \in A_1$, we have $A_1 \neq \emptyset$. If $A_2 = \emptyset$, then we define a type 2 dominating set D_i of G_i by $A(D_i) := A$ and $B(D_i) := B$. Suppose $A_2 \neq \emptyset$. Note $|B| \geq 2$, because $\{b, y\} \subseteq B$. If $B_2 = \emptyset$, then we define D_i by $A(D_i) := A \cup \{y\}$ and $B(D_i) := B_1 = B \setminus \{y\}$. Suppose $B_2 \neq \emptyset$. We check in $\mathcal{O}(|V|^2)$ time if $G[A_1 \cup A_2]$ contains a spanning complete bipartite graph with partition classes A_1 and A_2. If so, we define D_i by $A(D_i) := A_1$ and $B(D_i) := A_2 \cup B$. Otherwise we have found two non-adjacent vertices $a_1 \in A_1$ and $a_2 \in A_2$. Let $b^* \in B_2$. Then $ya_1b^*a_2$ is an induced P_4 starting in y and ending in a vertex of A. By Claim 1, we can find a type 1 or type 2 dominating set D_i of G_i in $\mathcal{O}(|V|^2)$ extra time. This finishes the proof of Theorem 5. □

The characterization in Theorem 5 is minimal due to the existence of the following family $\mathcal{F}$ of P_6-free graphs. For each $i \geq 2$, let $F_i \in \mathcal{F}$ be the graph obtained from a complete bipartite subgraph with partition classes $X_i = \{x_1, \ldots, x_i\}$ and $Y_i = \{y_1, \ldots, y_i\}$ by adding the edge x_1x_2 as well as for each $h = 1, \ldots, i$ a new vertex x'_h adjacent only to x_h and a new vertex y'_h adjacent only to y_h (see Fig. 4 for the graph F_3).

Note that each F_i is P_6-free and that the smallest connected dominating subgraph of F_i is $F_i[X_i \cup Y_i]$, which contains a spanning complete bipartite subgraph. Also note that none of the graphs F_i contain a dominating *induced* complete bipartite subgraph due to the edge x_1x_2.

We conclude this section by characterizing the class of graphs for which each connected induced subgraph contains a dominating induced C_6 or a dominating *induced* complete bipartite subgraph. Again, we will show how to find these dominating induced subgraphs in polynomial time by using the incremental technique outlined in Section 2. The *net* is the graph on six vertices depicted in Fig. 5.

Theorem 6. *A graph $G = (V, E)$ is in Forb($\{P_6, net\}$) if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating induced complete bipartite graph. Moreover, we can find such a dominating subgraph of G in $\mathcal{O}(|V|^4)$ time.*

Proof. Neither the graph P_6 nor the net has a dominating induced C_6 or a dominating induced complete bipartite subgraph. Hence to prove Theorem 6, it suffices to show that if G is a connected graph in *Forb*($\{P_6, net\}$), then we can find a dominating induced C_6 or a dominating induced complete bipartite subgraph of G in $\mathcal{O}(|V|^4)$ time.

Let $G = (V, E)$ be a connected graph in *Forb*($\{P_6, net\}$) with connected order $\pi = x_1, \ldots, x_{|V|}$. Recall that we write $G_i := G[\{x_1, \ldots, x_i\}]$, and note that $G_i \in$ *Forb* ($\{P_6, net\}$) for every i. For every $2 \leq i \leq n$ we want to find a dominating set D_i of G_i that either induces a C_6 or a complete bipartite subgraph in G_i. Let $D_2 := \{x_1, x_2\}$. Suppose $i \geq 3$. Assume that D_{i-1} induces a dominating C_6 or a dominating complete bipartite subgraph in G_{i-1}. We show how we can use D_{i-1} to find D_i in $\mathcal{O}(|V|^3)$ time. Since the total number of iterations is $|V| - 2$, we find a desired dominating subgraph of $G_{|V|} = G$ in $\mathcal{O}(|V|^4)$ time. We write $x := x_i$ and check in $\mathcal{O}(|V|)$ time if $x \in N(D_{i-1})$. If so then we set $D_i := D_{i-1}$. Suppose otherwise. Since π is connected, G_i contains a vertex y (not in D_{i-1}) adjacent to x. We can find y in $\mathcal{O}(|V|)$ time. We first prove a useful claim.

Claim 1. *If $N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i, then we can find a dominating induced C_6 or a dominating induced complete bipartite subgraph of G in $\mathcal{O}(|V|^3)$ time.*

We prove Claim 1 as follows. Suppose $D^* := N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i. We check whether $G[D^*]$ is complete bipartite in $\mathcal{O}(|V|^2)$ time. If so, then we choose $D_i := D^*$ and we are done. Otherwise y has a neighbor u in D_{i-1} with $N_{D^*}(u) \setminus \{y\} \neq \emptyset$. If u has no D^*-private neighbor, which we can check in $O(|V|^2)$ time, then we remove u from D^* and perform the same check in the smaller set $D^* \setminus \{u\}$. Let u' be a D^*-private neighbor of u in G_i. Let $v \in N_{D^*}(u) \setminus \{y\}$. Then u' is adjacent to any D^*-private neighbor v' of v, as otherwise $G[\{u, v, y, u', v', x\}]$ is isomorphic to the net. So we find that

$D^1 := (D^* \setminus N_{D^*}(u)) \cup \{y, u'\}$ dominates G_i. If u' does not have a D^1-private neighbor, then we remove u' from D^1, check if y is adjacent to two neighbors in the smaller set $D^1 \setminus \{u'\}$ and repeat the above procedure which runs in $\mathcal{O}(|V|^3)$ total time. Let u'' be a D^1-private neighbor of u'. Suppose $N_{D^1}(y) = \{x, u\}$. Then $D^1 = \{x, y, u, u'\}$. If x does not have a D^1-private neighbor, then we choose $D_i := \{y, u, u'\}$. If x has a D^1-private neighbor x', then the P_6-freeness of G_i implies that x' is adjacent to u'', and we choose $D_i := \{x', x, y, u, u', u''\}$.

Suppose $N_{D^1}(y) \setminus \{x, u\} \neq \emptyset$, say y is adjacent to some vertex $t \in D^1 \setminus \{x, u\}$. If t does not have a D^1-private neighbor, then we remove t from D^1 and check if y is adjacent to some vertex in the smaller set $D^1 \setminus \{x, u, t\}$. Let t' be a D^1-private neighbor of t. Then the path $u''u'uytt'$ is an induced P_6 of G_i, unless u'' is adjacent to t'. However, in that case $xyuu'u''t'$ is an induced P_6. This contradiction finishes the proof of Claim 1.

Case 1. *D_{i-1} induces a dominating C_6 in G_{i-1}.*

Since D_{i-1} is a type 1 dominating set of G_{i-1}, we know from the corresponding Case 1 in the proof of Theorem 5 that $D := N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i. We can find D in $\mathcal{O}(|V|)$ time. By Claim 1, we can find a dominating induced C_6 or a dominating induced complete bipartite subgraph of G in $\mathcal{O}(|V|^3)$ extra time.

Case 2. *D_{i-1} induces a dominating complete bipartite subgraph in G_{i-1}.*

Let $A(D_{i-1})$ and $B(D_{i-1})$ denote the partition classes of D_{i-1}. Note that both $A(D_{i-1})$ and $B(D_{i-1})$ are independent sets. Since D_{i-1} dominates G_{i-1}, we may without loss of generality assume that y is adjacent to some vertex $a \in A(D_{i-1})$. Let $b \in B(D_{i-1})$. Note that a and b are adjacent vertices in $D_{i-1} \cup \{y\}$ and that $\{a, b\}$ dominates $D_{i-1} \cup \{y\}$. Hence we can find a minimizer D of $D_{i-1} \cup \{y\}$ for ab in $\mathcal{O}(|V|^2)$ time by Lemma 1. By definition, D dominates G_i. Also, $G[D]$ contains a spanning (not necessarily complete) bipartite graph with partition classes $A \subseteq A(D_{i-1})$ and $B \subseteq B(D_{i-1}) \cup \{y\}$. Note that $y \in D$, because x is not adjacent to D_{i-1} and therefore x is a D-private neighbor of y, and consequently, $y \in B$ because y is adjacent to $a \in A$ and y might not have any neighbors in B. Let $A_1 := N_A(y)$ and $A_2 := A \setminus A_1$. Let $B_1 := N_B(y)$ and $B_2 := B \setminus (B_1 \cup \{y\})$. We can obtain these sets in $\mathcal{O}(|V|)$ time. Since $a \in A_1$, we have $A_1 \neq \emptyset$.

Suppose $G[D]$ contains an induced P_4 starting in y and ending in a vertex in A. Just as in the proof of Theorem 5 we can obtain in $\mathcal{O}(|V|^2)$ time a dominating C_6 of G_i or else we find that $N_D(y) \cup \{x, y\}$, and consequently $N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i. In the first case, we choose D_i to be the obtained dominating induced C_6. In the second case, we can find a dominating induced C_6 or a dominating induced complete bipartite subgraph of G in $\mathcal{O}(|V|^3)$ extra time by Claim 1. So we may assume that $G[D]$ does not contain such an induced P_4. This means that at least one of the sets A_2, B_2 is empty, as otherwise we find an induced path yab_2a_2 for any $a_2 \in A_2$ and $b_2 \in B_2$. We may without loss of generality assume that $A_2 = \emptyset$. Otherwise, in case $B_2 = \emptyset$, we obtain $B = B_1$, which means that y is adjacent to b, so we can reverse the role of A and B. If $B_2 = \emptyset$, then we find that $A_1 \cup B_1 \cup \{y\} \subset N_{D_{i-1}}(y) \cup \{x, y\}$ dominates G_i, and we are done in $\mathcal{O}(|V|^3)$ extra time as a result of Claim 1. So $B_2 \neq \emptyset$. Let $b_2 \in B_2$.

We claim that $D^2 := A_1 \cup B_2 \cup \{x, y\}$ dominates G_i. Suppose otherwise. Then there exists a vertex b_1' adjacent to some vertex $b_1 \in B_1$ but not adjacent to D^2. Then $G[\{y, a, b_1, x, b_2, b_1'\}]$ is isomorphic to the net, a contradiction. Hence D^2 dominates G_i. From D^2 we construct D_i in $\mathcal{O}(|V|^2)$ extra time as follows. If x does not have a D^2-private neighbor, then we can choose $D_i := D^2 \setminus \{x\}$, since $G[D^2 \setminus \{x\}]$ is a complete bipartite graph with partition classes A_1 and $B_2 \cup \{y\}$. Suppose x has a D^2-private neighbor x'. If b_2 does not have a D^2-private neighbor, then we remove b_2 from D^2, and check whether B_2 contains another vertex. If not, then we can choose $D_i := A_1 \cup \{x, y\}$, since $G[A_1 \cup \{x, y\}]$ is a complete bipartite graph with partition classes $A_1 \cup \{x\}$ and $\{y\}$. Suppose b_2 has a D^2-private neighbor b_2'. Then the path $x'xyab_2b_2'$ is a path on six vertices, so we must have $x'b_2' \in E$.

We claim that $D^3 := \{x', x, y, a, b_2, b_2'\}$ dominates G_i. Suppose otherwise. Then there exists a vertex c' adjacent to some vertex c in $A_1 \cup B_2$ but not adjacent to a vertex in D^3. Suppose $c \in A_1$. Then $c'cb_2b_2'x'x$ is an induced P_6. Suppose $c \in B_2$. Then $c'cayxx'$ is an induced P_6. So D^3 dominates G_i. Since D^3 also induces a C_6 in G_i, we may choose $D_i := D^3$. This finishes the proof of Theorem 6. □

Bacsó, Michalak and Tuza [3] prove (non-constructively) that a graph G is in *Forb* $(\{C_6, P_6, net\})$ if and only if each connected induced subgraph of G contains a dominating induced complete bipartite graph. Note that Theorem 6 immediately implies this result.

5. The Hypergraph 2-Colorability problem

A *hypergraph* H is a pair $(Q, \mathcal{S})$ consisting of a set $Q = \{q_1, \ldots, q_m\}$, called the *vertices* of H, and a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of nonempty subsets of Q, called the *hyperedges* of H. With a hypergraph $(Q, \mathcal{S})$ we associate its *incidence graph* I, which is a bipartite graph with partition classes Q and $\mathcal{S}$, where for any $q \in Q, S \in \mathcal{S}$ we have $qS \in E(I)$ if and only if $q \in S$. For any $S \in \mathcal{S}$, we write $H - S := (Q, \mathcal{S} \setminus S)$. A *2-coloring* of a hypergraph $H = (Q, \mathcal{S})$ is a partition (Q_1, Q_2) of Q such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$.

The Hypergraph 2-Colorability problem asks whether a given hypergraph has a 2-coloring. This problem, also known as Set Splitting, is NP-complete, even when restricted to hypergraphs for which every hyperedge has size at most 3 (cf. [11]). The Hypergraph 2-Colorability problem becomes polynomially solvable when restricted to hypergraphs for which every hyperedge has size at most 2, since that problem is equivalent to the 2-Colorability problem for graphs, i.e., to checking whether a given graph is bipartite. We now present another class of hypergraphs for which the Hypergraph 2-Colorability problem becomes polynomially solvable. Let $\mathcal{H}_6$ denote the class of hypergraphs with P_6-free incidence graphs.

Theorem 7. *The* Hypergraph 2-Colorability *problem restricted to* $\mathcal{H}_6$ *is polynomially solvable. Moreover, for any 2-colorable hypergraph* $H = (Q, \mathcal{S}) \in \mathcal{H}_6$ *with* $|Q| = m$ *and* $|\mathcal{S}| = n$*, we can find a 2-coloring of* H *in* $\mathcal{O}((m+n)^3)$ *time.*

Proof. Let $H = (Q, \mathcal{S}) \in \mathcal{H}_6$ with $|Q| = m$ and $|\mathcal{S}| = n$, and let I be the P_6-free incidence graph of H. We assume that I is connected, as otherwise we just proceed component-wise.

Claim 1. *We may without loss of generality assume that* $\mathcal{S}$ *does not contain two sets* S_i, S_j *with* $S_i \subseteq S_j$.

We prove Claim 1 as follows. Suppose $S_i, S_j \in \mathcal{S}$ with $S_i \subseteq S_j$. We show that H is 2-colorable if and only if $H - S_j$ is 2-colorable. Clearly, if H is 2-colorable then $H - S_j$ is 2-colorable. Suppose $H - S_j$ is 2-colorable. Let (Q_1, Q_2) be a 2-coloring of $H - S_j$. By definition, $S_i \cap Q_1 \neq \emptyset$ and $S_i \cap Q_2 \neq \emptyset$. Since $S_i \subseteq S_j$, we also have $S_j \cap Q_1 \neq \emptyset$ and $S_j \cap Q_2 \neq \emptyset$, so (Q_1, Q_2) is a 2-coloring of H. This proves Claim 1.

Note that we can reach the situation mentioned in Claim 1 in $\mathcal{O}(m^2n^2)$ time. By Theorem 5, we can find a type 1 or type 2 dominating set D of I in $\mathcal{O}((m+n)^3)$ time. Below we show how we use such a dominating set to find a 2-coloring of H in $\mathcal{O}(m+n)$ extra time, assuming H has 2-coloring. Since I is bipartite, $I[D]$ is bipartite. Let A and B be the partition classes of $I[D]$. Since I is connected, we may without loss of generality assume $A \subseteq Q$ and $B \subseteq \mathcal{S}$. Let $A' := Q \setminus A$ and $B' := \mathcal{S} \setminus B$. We distinguish two cases.

Case 1. *D is a type* 1 *dominating set of* I.

We write $I[D] = q_1S_1q_2S_2q_3S_3q_1$, so $A = \{q_1, q_2, q_3\}$ and $B = \{S_1, S_2, S_3\}$. Suppose $A' = \emptyset$, so $Q = \{q_1, q_2, q_3\}$. Obviously, H has no 2-coloring. Suppose $A' \neq \emptyset$ and let $q' \in A'$. Since D dominates I, q' has a neighbor, say S_1, in B. If S_2 and S_3 both have no neighbors in A', then $q'S_1q_2S_2q_3S_3$ is an induced P_6 in I, a contradiction. Hence at least one of them, say S_2, has a neighbor in A'.

We claim that the partition (Q_1, Q_2) of Q with $Q_1 := A' \cup \{q_1\}$ and $Q_2 := \{q_2, q_3\}$ is a 2-coloring of H. We have to check that every $S \in \mathcal{S}$ has a neighbor in both Q_1 and Q_2. Recall that S_1 has neighbors q_1 and q_2 and S_3 has neighbors q_1 and q_3. Hence S_1 has a neighbor in both Q_1 and Q_2, and the same holds for S_3. Since S_2 is adjacent to q_2 and has a neighbor in A', S_2 also has a neighbor in both Q_1 and Q_2. It remains to check the vertices in B'. Let $S \in B'$. Since D dominates I and I is bipartite, S has at least one neighbor in A. Suppose S has exactly one neighbor, say q_1, in A. Then $Sq_1S_1q_2S_2q_3$ is an induced P_6 in I, a contradiction. Hence S has at least two neighbors in A. The only problem occurs if S is adjacent to q_2 and q_3 but not to q_1. However, since S_2 is adjacent to q_2 and q_3, S must have a neighbor in A' due to Claim 1. Hence (Q_1, Q_2) is a 2-coloring of H.

Case 2. *D is a type* 2 *dominating set of* I.

Suppose $A' = \emptyset$. Then $|B| = 1$ as a result of Claim 1. Let $B = \{S\}$ and $q \in A$. Since S is adjacent to all vertices in A, we find that $B' = \emptyset$ as a result of Claim 1. Hence H has no 2-coloring if $|A| = 1$, and H has a 2-coloring $(\{q\}, A \setminus \{q\})$ if $|A| \geq 2$. Suppose $A' \neq \emptyset$. We claim that (A, A') is a 2-coloring of H. This can be seen as follows. By definition, each vertex in $\mathcal{S}$ is adjacent to a vertex in A. Suppose $|B| = 1$ and let $B = \{S\}$. Since S dominates Q and $A' \neq \emptyset$, S has at least one neighbor in A'. Suppose $|B| \geq 2$. Since every vertex in B is adjacent to all vertices in A, every vertex in $\mathcal{S}$ must have a neighbor in A' as a result of Claim 1. □

6. Conclusions

The key contributions of this paper are the following. We presented a new characterization of the class of P_6-free graphs, which strengthens results of Liu and Zhou [15] and Liu, Peng and Zhao [16]. We used an algorithmic technique to prove this characterization. Our main algorithm efficiently finds for any given connected P_6-free graph a dominating subgraph that is either an induced C_6 or a (not necessarily induced) complete bipartite graph. Besides these main results, we also showed that our characterization is "minimal" in the sense that there exists an infinite family of P_6-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. We also characterized the class $Forb(\{P_6, net\})$ in terms of connected dominating subgraphs, thereby generalizing a result of Bacsó, Michalak and Tuza [3].

Our main algorithm can be useful to determine the computational complexity of decision problems restricted to the class of P_6-free graphs. To illustrate this, we applied this algorithm to prove that the Hypergraph 2-Colorability problem is polynomially solvable for the class of hypergraphs with P_6-free incidence graphs. Are there any other decision problems for which the algorithm is useful? In recent years, several authors studied the classical k-Colorability problem for the class of P_ℓ-free graphs for various combinations of k and ℓ [14,17,18]. The 3-Colorability problem is proven to be polynomially solvable for the class of P_6-free graphs [17]. Hoàng et al. [14] show that for all fixed $k \geq 3$ the k-Colorability problem becomes polynomially solvable for the class of P_5-free graphs. They pose the question whether there exists a polynomial time algorithm to determine if a P_6-free graph can be 4-colored. We do not know yet if our main algorithm can be used for simplifying the proof of the result in [17] or for solving the open problem described above. We leave these questions for future research.

The next class to consider is the class of P_7-free graphs. Recall that a graph G is P_7-free if and only if each connected induced subgraph of G contains a dominating subgraph of diameter at most three [2]. Using an approach similar to the one described in this paper, it is possible to find such a dominating subgraph in polynomial time. However, a more important question is whether this characterization of P_7-free graphs can be narrowed down. Also determining the computational complexity of the Hypergraph 2-Colorability problem for the class of hypergraphs with P_7-free incidence graphs is still an open problem.

Finally, a natural problem for a given graph class deals with its recognition. We are not aware of any recognition algorithms for (even bipartite or triangle-free) P_7-free graphs that have a better running time than the trivial algorithm that checks for every 7-tuple of vertices whether they induce a path. This might be another interesting direction for future research, considering the following results on recognition of (subclasses of) P_6-free graphs. Fouquet [10] presents a cubic recognition algorithm for the class of P_6-free graphs (in an internal report). Giakoumakis and Vanherpe [12] show that bipartite P_6-free graphs can be recognized in linear time. They do this by extending the techniques developed in [7] for linear time recognition of P_4-free graphs (also see [13]) and by using a characterization of P_6-free graphs in terms of canonical decomposition trees (which is not related to our characterization) from [10]. Brandstädt, Klembt and Mahfud [5] show that triangle-free P_6-free graphs have bounded clique-width. The recognition algorithm they obtain from this result runs in quadratic time. Since the class of P_6-free graphs has unbounded clique-width (cf. [4]), their technique cannot be applied to find a quadratic recognition algorithm for the class of P_6-free graphs.

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