POISSON GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 51
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

# QUANTIZATION OF THE COTANGENT BUNDLE VIA THE TANGENT GROUPOID

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**1. Introduction.** The problem of quantizing a given classical system remained not completely solved for more than eighty years. After Van Hove theorem (see [8] for a more recent exposition and other no-go theorems) it is known that no complete result can be obtained, and hence a great number of different approaches appeared. The most important ones, from the mathematical point of view, have been geometric quantization and deformation quantization. Deformation quantization was introduced in [1] as a generalization of previous ideas of Moyal using the deformation theory of structures proposed by Gerstenhaber. Unlike geometric quantization, that can be applied only to some special cases (there are conditions on the symplectic form of the manifold to define a first step, the prequantization, and no canonical way of constructing the second part, the polarization) it has been proved [6, 7] that any symplectic manifold admits a deformation quantization construction. Nonetheless, from the practical point of view, it has an intrinsic problem: the deformation of structures has to be constructed on an algebra of formal power series of functions in the deformation parameter. This implies that, in the general theory, there is no way of controlling the convergence of the series we are handling when we set the deformation parameter to be the physical value of Planck constant $\hbar$.

One solution proposed to this problem is the introduction of a norm on the space of functions, in such a way that the deformation of the algebra of functions becomes a deformation of a $C^*$-algebra structure into another (the $C^*$-algebra of classical observables is mapped on the $C^*$-algebra of quantum compact operators). This is *grosso modo* the notion of strict quantization due to Rieffel summarized in a recent work [13]. Rieffel stated there a series of problems related to the construction, in some general situations,

2000 *Mathematics Subject Classification*: Primary 81S10; Secondary 58H05.

Research partially supported by DGES project, PB96–0717. Thanks are given to all other coworkers of [2] for enlightening and fruitful discussions.

The paper is in final form and no version of it will be published elsewhere.

of strict quantizations. In principle, the definition of a strict quantization is analogous to the concept of asymptotic morphisms (see, for instance, [5]). Namely:

DEFINITION 1. Let $M$ be a Poisson manifold. By a *strict quantization* of $M$ we mean a dense $*$-subalgebra $A$ of $C_\infty(M)$ which is carried into itself by the Poisson bracket, together with a closed subset $I$ of the real line containing 0 as a non-isolated point, and for each $\hbar \in I$ a linear map $Q_\hbar$ (usually preserving the involution) of $A$ into a $C^*$-algebra $\bar{A}_\hbar$ which is generated by the range of $Q_\hbar$, such that for $\hbar = 0$ the map $Q_0$ is the canonical inclusion of $A$ into $C_\infty(M)$, and such that: i) The maps $\hbar \to Q_\hbar(a)$ define the structure of a continuous field of $C^*$-algebras on the family $\{\bar{A}_\hbar\}$. ii) For $f, g \in A$ we have

$$\|(Q_\hbar(f)Q_\hbar(g) - Q_\hbar(g)Q_\hbar(f))/\hbar - iQ_\hbar(\{f, g\})\|_\hbar \to 0$$

as $\hbar \to 0$.

There is an analogous concept, strict deformation quantization, in which the set of functions is kept, deforming only the norm as well as the product.

In the present work we exploit some ideas by Connes [4] and Landsman [10] to construct a quantization procedure for a cotangent bundle. Actually we report and generalize the construction presented in a recent work [2] to a bigger set of quantizations. We will see that this construction allows us to define a strict quantization in a very large set of cases. We will also study some physical properties of the corresponding quantizations, and we will see that in some particular cases, the construction can be used to define a strict deformation quantization.

The paper is divided into two parts: a brief exposition of the geometrical structures we are handling (the tangent and normal groupoids) which covers the first section, and the use of these groupoids to define a strict quantization, in sections two (the general case) and three (a short comment about the construction of a strict deformation quantization in exponential manifolds).

**2. The tangent groupoid: its geometrical structure.** We will suppose, in the following, that the reader is familiar with the basic notions of groupoid theory, such as the definition itself, some particular cases we are interested in, as those of topological and smooth groupoids, and the concept of convolution algebra. A quick reference for all these concepts may be found at [4]. We would just like to make a remark about this last point. Though the canonical way of introducing the concept of convolution algebra of a smooth groupoid is via its half densities, we are going to work always on a Riemannian manifold $M$, where of course we have the usual measure $d\mu_g(p)$. This will allow us to define the convolution directly on the set of functions of the groupoid.

The case we are interested in is a particular case of a smooth groupoid proposed by Connes [4]. The tangent groupoid is introduced as union of two groupoids where we define a topology. Let us do it carefully:

Let $G = M \times M \times ]0,1[ \cup TM \times [0]$ be a groupoid over $G^0 = M \times [0,1[$ such that:

- The inclusion map $G^0 \to G$ is defined as:

$$(x, \epsilon) \to (x, x, \epsilon) \in M \times M \times ]0,1[ \text{ for } x \in M, \quad \epsilon > 0$$
$$(x, 0) \to x \in M \subset TM \text{ as the zero section, for } \epsilon = 0$$

- The range and source maps are defined as:

$$\begin{aligned} r(x,y,\epsilon) &= (x,\epsilon) \text{ for } x \in M,\ \epsilon > 0 \\ r(z,X_z) &= (x,0) \text{ for } z \in M, \quad X_z \in T_zM \\ s(x,y,\epsilon) &= (y,\epsilon) \text{ for } y \in M,\ \epsilon > 0 \\ s(z,X_z) &= (x,0) \text{ for } z \in M, \quad X_z \in T_zM \end{aligned}$$

- And finally the composition map is defined as:

$$\begin{aligned} (x,y,\epsilon)\cdot(y,z,\epsilon) &= (x,z,\epsilon) \text{ for } \epsilon > 0 \text{ and } x,y,z \in M \\ (z,X_z)\cdot(z,Y_z) &= (z,X_z+Y_z) \text{ for } z \in M \text{ and } X_z, Y_z \in T_zM \end{aligned}$$

This tangent groupoid is a disjoint union of two smooth ones ($G_1 = M \times M \times ]0,1[$ and $G_2 = TM$) and it can be endowed with a suitable topology in such a way that $G_2$ becomes the boundary of the whole set $G$, and $G_1$ as an open subset of $G$, by specifying the following convergence for sequences: Every convergent sequence on $G_1$ or $G_2$ will converge in our joined space. In addition [4], a sequence of elements of $\{(x_n, y_n, \epsilon_n) | (x_n, y_n, \epsilon_n) \in G_1\}$ converges to an element of the tangent bundle $(z, X_z)$ if, on a local chart $U_\mu$,

$$x_n^\mu \to z^\mu, \quad y_n^\mu \to z^\mu, \quad \frac{x_n^\mu - y_n^\mu}{\epsilon_n} \to X_z^\mu \ .$$

This condition does not depend on the choice of the local chart, because given two different overlapping charts $U_i$ and $U_j$ and a sequence of points in $M \times M$, if the transition function is $\Psi_{ij}$ we can write:

$$\Psi_{ij}(y_n^\mu) = \Psi_{ij}(x_n^\mu + \epsilon_n X_{x_n}^\mu + o(\epsilon^2)) = \Psi_{ij}(x_m^\mu) + \epsilon_n (\Psi_{ij})_*(X_{x_n}^\mu) + o(\epsilon^2) \ ,$$

and thus: $\frac{x_n^\mu - y_n^\mu}{\epsilon_n} = (\Psi_{ij})_*(X_{x_n}^\mu)$, which proves the assertion.

We recently proved [2] Connes assertion [4] that the tangent groupoid can be endowed with a differentiable structure in such a way that it is a smooth groupoid.

The two main points to be proved carefully were first, the differentiable structure defined on $G$, $G^0$ and $G^{(2)}$, and second the smoothness of the product (the other points are much easier). The proof is based on the definition of a bijection between an open set in $M \times M$ containing the diagonal and an open set in $TM \times [0,1[$, such that the differentiable structure in $TM \times [0,1[$ is transported into the groupoid. We can proceed similarly with the space of pairs, and we use it to work with the product.

**2.1.** *The normal groupoid.* The tangent groupoid structure is very similar to that of *normal groupoid*, which can be constructed in a completely analogous way, just replacing the tangent bundle by the normal bundle with respect to the injection of the base in the pair groupoid (the injection which defines the groupoid structure for $G_1$ above). The idea is thus to use again the bijection between an open set in $M \times M \times [0,1[$ and $NM \times [0,1[$ via the tubular neighbourhood theorem (see [9] for a detailed exposition). Actually this is the most logical way of working with these structures, since the tubular neighbourhood theorem establishes a bijection between an open set of the normal bundle with respect to an injection $i : M \mapsto S$, and an open set in $S$ containing the image of $M$ (see, for instance [11]). In order to establish the bijection with the tangent bundle, a further mapping between the normal and the tangent bundle is needed. Since the classical information is contained in $TM$ we can say that the geometrical structure of the construction is encoded in the normal bundle, and we will see that the physical content will be encoded in the mapping between the normal and the tangent bundle, which will fix the quantization properties.

The use of the normal groupoid allows us to study the smoothness in a more general way. In fact, given any groupoid $G$, and its base $i : G^0 \to G$, one can define the normal groupoid $\mathcal{N}_{G^0}^G$ to $G$ as the union of $G\times ]0,1[$ over $G^0\times ]0,1[$ and $N^iG^0$ over $G^0$, with the same topological content as the tangent groupoid had: the normal bundle is the boundary of the union of groupoids. The topological and differentiable structure are obtained as in the previous case, and then [12], if $G$ is a smooth groupoid, then $\mathcal{N}_{G^0}^G$ is also a smooth groupoid.

The proof requires some subtle points for the smoothness of the product; the interested reader is referred to [2] or [3]. The proof requires the use of the normal bundle considered as a quotient manifold, with its fibers being equivalence classes of vectors, without the usual choice of the fibers as the orthogonal complement to the tangent to the injection with respect to the Riemannian structure we have.

**2.2.** *Summary of the geometrical construction.* We have built up two similar smooth groupoids, the normal and the tangent groupoids, with the only difference being the boundary of the differentiable manifold which is the whole groupoid. The tubular neighbourhood theorem allows us to establish a bijection between an open set $U \subset N^iM\times[0,1[$ containing the zero section, and an open set $V \subset M \times M\times]0,1[$ containing the diagonal, given by $\Psi(\exp_{\imath(p)}([Y_i(p)]),\epsilon) = (p,[Y_p],\epsilon)$, where $[Y_i(p)]$ means the class of vectors defining the fiber of the normal bundle at the point $i(p)$.

In order to establish the same mapping for the tangent groupoid we need a bundle isomorphism between the tangent and the normal ones. Given the diagonal injection which defines the inclusion of the base in the pair groupoid $i : M \to M \times M$ with $i(p) = (p,p)$, the tangent to this map obviously defines the inclusion of the tangent bundle $TM$ in $T(M \times M)$ as $Ti(p,X_p) = (p,p,X_p,X_p)$. Thus, the normal fibers are defined on the points $i(M)$ to be the class of vectors of $T_{i(p)}(M \times M)$ modulo the vectors above $((i_*)_p(T_pM))$. Any choice of the representative of the class fixes the description of the normal bundle. Given the Riemannian structure $g$ we have (we can lift the Riemannian metric from $M$ to $M \times M$, just fixing two identical copies of $g$ on the product space), the most usual choice is the orthogonal complement of $(i_*)_p(T_pM)$, which in this case is $[Y_p] = (Y_p,-Y_p)$, but any other choice would produce an equivalent construction.

As an isomorphism between $TM$ and $N^iM$ we can choose $\Phi : TM \to N^iM$ such that $\Phi(p,X_p) = (i(p),\phi_1(X_p),\phi_2(X_p))$, where $\phi_i : T_pM \to T_pM$ are linear, but in order to ensure that the point $\Phi(p,X_p)$ belongs to the normal bundle, the map $\phi_1-\phi_2$ must be invertible, and then the mapping is a bundle isomorphism. Thus, the composition $\sigma = \Phi\circ\Psi$ defines the transformation between an open set in $TM$ containing the zero section and the open set $V \in M \times M$ containing the diagonal, which was needed in the first section.

We should also introduce in this framework the real parameter $\epsilon$: we set the previous mapping between the normal and the tangent bundles to act actually on the product:

$$\Phi_\epsilon : TM \times [0,1[\to N^iM \times [0,1[\,, \quad \Phi_\epsilon(p,X_p,\epsilon) = (i(p),\phi_1(X_p,\epsilon),\phi_2(X_p,\epsilon),\epsilon)\,, \qquad (1)$$

with $\phi_1(X_p,0) = \phi_2(X_p,0) = 0$.

**3. The tangent groupoid as a method of quantization.** In the previous section we have seen that there is a diffeomorphism between an open subset either of the normal or of the tangent groupoids containing the zero section and an open set in $M \times M$ which contains the diagonal. Let us consider a classical system with configuration space $M$. There are two algebra structures in the space of observables, $C^\infty(T^*M)$: the pointwise algebra and the Poisson algebra. When our base manifold $M$ is endowed with a Riemannian structure, we can define a Fourier transform $\mathcal{F} : C^\infty(TM) \to C^\infty(T^*M)$ given by:

$$f(p, P) = \mathcal{F}(\hat{f}) = \frac{1}{(2\pi)^n} \int_{T_pM} d\mu_g(p) e^{ig(P, X_p)} \hat{f}(p, X_p) \tag{2}$$

where the measure is defined through the metric as $d\mu_g(p) = \sqrt{g(p)}\, dX_p$.

In a remarkable paper by Landsman [10], one can find the previous definition as well as the main lines of the following exposition, for some particular cases. Another important source for the following is again Connes' paper, and his definition of asymptotic morphisms of $C^*$-algebras [5].

Once we have transported our classical observables to the tangent bundle, we can look again on the tangent groupoid. There we have a diffeomorphism onto an open set in $M \times M \times ]0,1[$, whose pull-back obviously maps the functions of both spaces. The notion of continuity included in the groupoid, implies that a continuous function on the tangent groupoid is defined by a pair of functions, one in $TM$ (which could be a classical function), and another one in $M \times M \times ]0,1[$, which we will try to relate with a quantum operator, by considering it as a kernel of a compact operator acting on the Hilbert space $(\mathcal{L}^2(M), d\mu_g)$. If we regard the convolution algebra of the groupoid we see that our idea works, since the convolution of the functions in $G_2$, $k(x,z) = \int_M d\mu_g(y) k_1(x,y) k_2(y,z)$, reproduces the product of kernels in a Hilbert space. Thus the action of the operators will be written

$$(A\Omega)(x) = \int_M d\mu_g(y) k_A(x,y) \Omega(x).$$

Summarizing, we see that if we take the Fourier transform of a classical observable, and we apply to it the diffeomorphism of tangent groupoid $\Phi_\epsilon$ we get a function on $M \times M \times \{\epsilon\}$ that can be regarded as a quantum kernel. This is our definition of a quantization mapping: $Q_\epsilon(f)(x,y) = \epsilon^{-n} (\Phi_\epsilon)^{-1*}(\hat{f})((\Phi_\epsilon)^{-1}(x,y))$, which we will write in the following as: $Q_\epsilon(f)(e_p^{\phi_1(X_p,\epsilon)}, e_p^{\phi_2(X_p,\epsilon)}) = \epsilon^{-n} \hat{f}(p, X_p)$.

**3.1.** *Ambiguities in the quantization.* An interesting question arising now is the point of the freedom in the choice of the quantization of the classical functions using our approach, which depends on the properties we ask for to our mapping.

PROPOSITION 1. *The ambiguity of the most general quantization in these scheme is equal to:*

$$\mathrm{End}(TM \times [0,1]) \oplus \mathrm{GL}(TM \times [0,1])$$

*where we use* $\mathrm{End}(TM \times [0,1])$ *to denote the set of mappings defined on the bundle, and* $\mathrm{GL}(TM \times [0,1])$ *to denote the linear and invertible subset of it.*

Proof. The proof is completely trivial. We know that $(\phi_1, \phi_2)$ establishes an isomorphism between the normal bundle $N^iM$ and the tangent bundle $TM$. This implies that, given a vector $X_p$ at the point $p \in M$, the image $(\phi_1(X_p, \epsilon), \phi_2(X_p, \epsilon)$ must belong to the fiber of the normal bundle at the point $\Delta(p) = (p, p)$, and therefore the only condition to be fulfilled is $\phi_1(X_p, \epsilon) \neq \phi_2(X_p, \epsilon)$, which implies the assertion above.

If we ask now the quantization to verify some properties, the ambiguity will, of course, decrease. As an example, we can impose four different conditions: strictness, semitraciality, reality and traciality.

**3.2.** *Strict quantization.* The analysis we present now serves as a generalization of Landsman's analysis for the case of Weyl quantization in arbitrary Riemannian manifolds to be discussed below.

The main point is the study of the norm of the operator on the Hilbert space $A = \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\})$. The proof of the strictness property lies in showing that the norm of this operator vanishes. This fact can be studied using the following theorem:

Theorem 2. *Given a bounded operator A, and a normalized vector v, the following relation holds:* $2 \sup\langle v|A|v\rangle \geq ||A||$.

Landsman's argument for Weyl quantization uses its reality property, searching for normalized vectors of particular types. This result is much more general.

The procedure is thus as follows: we choose a normalized state of the Hilbert space, $\Omega(x)$, and compute the corresponding matrix element:

$$\int_{M\times M} \Omega^*(x_1)A(x_1, x_2)\Omega(x_2)d\mu_g(x_1)d\mu_g(x_2), \tag{3}$$

We can use the diffeomorphism between the open set containing the diagonal of $M \times M$ and $TM \times [0, 1[$, and try to compute the expression above in terms of classical objects, with the addition of the parameter $\epsilon$. This implies a change in the previous integral $M \times M$ to $TM \times [0, 1[$.

This can be understood as a change of variables via the diffeomorphism $Q_\hbar$. The Jacobian of the transformation is computed easily when we take coordinates in each manifold (of course via a partition of unity for the integral):

$$\int_{M\times M} \Omega^*(x_1)A(x_1, x_2)\Omega(x_2)d\mu_g(x_1)d\mu_g(x_2) = \int_{TM} \bar{\Omega}^*(\gamma)A(\gamma, \gamma')\Omega(\gamma')J(\gamma, \gamma')$$

where $\gamma$ and $\gamma'$ represent the geodesics associating the pair $(x_1, x_2)$ with a point $(x, \dot{x})$ in $TM$, and the Jacobian is $J(\gamma, \gamma') = \epsilon^{-n} \frac{\sqrt{g(x_1)g(x_2)}}{g(x)} \det H$ where of course $x_1 = e_x^{\phi_1(\dot{x},\epsilon)}$ and $e_2 = e_x^{\phi_2(\dot{x},\epsilon)}$. We will also write it in terms of the tangent bundle coordinates $J(\gamma, \gamma') = J(p, X, \epsilon)$ where $(\gamma, \gamma') = \Phi(p, X, \epsilon)$.

The matrix $H$ is given by:

$$H = \begin{pmatrix} \dfrac{\partial x_1^\mu}{\partial x^\nu} & \dfrac{\partial x_1^\mu}{\partial \dot{x}^\nu} \\ \dfrac{\partial x_2^\mu}{\partial x^\nu} & \dfrac{\partial x_2^\mu}{\partial \dot{x}^\nu} \end{pmatrix} . \tag{4}$$

Landsman analyzed the determinant above using that each entry can be considered as the coordinate of a Jacobi field in the Riemannian manifold $M$. We have done a less detailed study of it, but enough to prove that for the norm of $A$, the integral vanishes.

For the particular case of $A$, it is obvious that we will need further transformations if we want to keep the previous classical framework Let us now rewrite the expression of the norm of $A(x_1, x_2)$ in terms of the new coordinates. As $A(x_1, x_2)$ is:

$$\left(\int_M Q_\hbar(f)(x_1,x_3)Q_\hbar(g)(x_3,x_2) - Q_\hbar(g)(x_1,x_3)Q_\hbar(f)(x_3,x_2)\right) - Q_\hbar(\{f,g\})(x_1,x_2)$$

we need not only two points $(x_1, x_2)$, which are fixed, but another point $x_3$ running over $M$. The change of coordinates above is therefore not complete, and we need also a second vector, say $\dot{x}'$ in such a way that $x_3 = e_x^{\epsilon\dot{x}'}$. Obviously if we want to fix $x$, the new vector will have to run on $T_xM$ to define the integral.

This transformation is properly defined since we know that any point in $V \subset M \times M$ is in the domain of the exponential at any point of the diagonal. Thus the submanifold defined by the deplacement of $x_3$ is contained in the domain of the exponential from the point $x$.

We can transport the integral in $M \times M \times M$ in (3) to another integral defined on $TM \oplus TM$. The final expression is:

$$\|A\| = \int_{TM\oplus TM} \Omega^*(\alpha)J(x,\dot{x},\epsilon)I(x,\dot{x},\dot{x}',\epsilon)\Omega(\beta)d\mu_g(x)d\mu_g(\dot{x})d\mu_g(\dot{x}') \tag{5}$$

where $I(x, \dot{x}, \dot{x}', \epsilon)$ is given by

$$J(\alpha,\beta)\left(\int_M Q_\hbar(f)(\alpha,\alpha')Q_\hbar(g)(\alpha',\beta) - Q_\hbar(g)(\alpha,\alpha')Q_\hbar(f)(\alpha',\beta)\right) - Q_\hbar(\{f,g\})(\alpha,\beta)$$

where $J(\alpha, \beta) = J(x, \dot{x}, \epsilon)$ is the Jacobian corresponding to the mentioned change of coordinates.

PROPOSITION 3. *For a suitable choice of $(\phi_1, \phi_2)$ the function $I(x, \dot{x}, \dot{x}', \epsilon)$ vanishes asymptotically, i.e.* $\lim_{\epsilon\to 0} I(x, \dot{x}, \dot{x}', \epsilon) = 0$

We will now fix as our basis the one associated to the normal coordinate system centered at $x_0$ (this is a basis for $M$ and it is trivially extended to our case). This has the advantage of giving a very simple expression for our points. The expression of the points is then: $x_i^\mu = x_0^\mu + \phi_i(\dot{x}, \epsilon)^\mu$.

Now, we perform a Taylor expansion of the kernels around the points $(e_p^{\phi_1(Z,\epsilon)}, e_p^{\phi_2(Z,\epsilon)})$ and $(e_p^{\phi_1(V,\epsilon)}, e_p^{\phi_2(V,\epsilon)})$, and we remark that the coefficients in the expansion depend on $\epsilon$ through the functions $\phi_1$ and $\phi_2$. What we have to require of these mappings is precisely that the development yields to a vanishing zeroth order term and a linear term equal to the Poisson bracket (written on the tangent bundle coordinates via the Fourier transform). A straightforward but tedious analysis leads to the following conclusions:

- There is a suitable pair of points, $Z = (\phi_1 - \phi_2)^{-1}(\dot{x}' - \phi_1(\dot{x}))$ $\quad V = (\phi_1 - \phi_2)^{-1}(\dot{x}' + \phi_1(\dot{x}))$, such that the zeroth order term of the development of $Q_\hbar(f)Q_\hbar(g)$ asymptotically goes to the point-wise algebra of functions on the cotangent bundle (therefore to the convolution product on the tangent one). Thus, the antisymmetric part which appears in (3) vanishes.

- The condition of strict quantization fixes the linear $\epsilon$ term in $\phi_1$ and $\phi_2$ to be the corresponding one to the generalization of Weyl's quantization, i.e. $\phi_1(\epsilon, X) = \epsilon X/2 + \varphi_1(X,\epsilon)\,, \phi_2(\epsilon, X) = -\epsilon X/2 + \varphi_2(X,\epsilon)$, where the dependence on $\epsilon$ of $\varphi_i$ is at least quadratic.

In any case, the conclusion we get from this result is the following:

PROPOSITION 4. *Any quantization scheme with the previous dependence (i.e. any mapping between the normal and the tangent bundles) in this framework verifies Rieffel's strict quantization axioms (i.e. it defines an asymptotic morphism).*

**3.3.** *Semitraciality.* The condition of semitraciality implies that the quantization of the unit function on $T^*M$ has to be the unit kernel. Let us translate this condition to our language. We know that the quantization mapping is defined as: $Q_\epsilon(f)(e_p^{\phi_1(X,\epsilon)}, e_p^{\phi_2(X,\epsilon)}) = \epsilon^{-n}\hat{f}(p,X)$, with $(\phi_1-\phi_2)\in \mathrm{GL}(TM)$.

Thus the function corresponding to the unit function in $T^*M$ will be the delta function in $TM$. The kernel will read: $Q_\epsilon(1)(e_p^{\phi_1(X,\epsilon)}, e_p^{\phi_2(X,\epsilon)}) = \epsilon^{-n}\delta(X)$.

To define the action of the unit kernel, we require that:

$$\int d\mu(x)d\mu(y)\psi(x)\delta(x,y)\phi(y) = \int d\mu(x)\psi(x)\phi(x)$$

Now, we check the relation to be fulfilled:

$$\int d\mu(x_1)d\mu(x_2)\psi(x_1)Q_\epsilon(1)(x_1,x_2)\phi(x_2)$$
$$= \int d\mu(p)d\mu(X)J(p,X,\epsilon)\psi(p,X,\epsilon)\delta(X/\epsilon)\phi(p,X,\epsilon)$$

and then

$$\int d\mu(x_1)d\mu(x_2)\psi(x_1)Q_\epsilon(1)(x_1,x_2)\phi(x_2) = \int d\mu(p)\psi(p)J(p,0,\epsilon)\phi(p)\ .$$

Thus the kernel constructed on the unit function behaves as the kernel of the unity if and only if $J(p,0,\epsilon)=1$. We have to compute the Jacobian at $X=0$. The factor of the metrics is trivially equal to one, since

$$\frac{\sqrt{g(x_1(p,0,\epsilon))g(x_2(p,0,\epsilon))}}{g(p)} = \frac{g(p)}{g(p)} = 1.$$

Hence, the only point to be proved is that $\epsilon^{-n}\det H(x,0,\epsilon) = 1$. The determinant is easily computed, because

$$H = \begin{pmatrix} \delta^\mu_\nu & (D_{\dot{x}}\phi_1)^\mu_\nu \\ \delta^\mu_\nu & (D_{\dot{x}}\phi_2)^\mu_\nu \end{pmatrix} \qquad (6)$$

and then $\det H = \det(D_{\dot{x}}\phi_1 - D_{\dot{x}}\phi_2)$. Thus we have proved:

THEOREM 5. *The condition to be fulfilled by a quantization of this framework to be semitracial is that:*

$$\det(D_{\dot{x}}\phi_1 - D_{\dot{x}}\phi_2) = \epsilon^n$$

*and therefore, the group of all possible semitracial quantizations becomes* $\mathrm{End}(TM \times [0,1[) \oplus \mathrm{SL}(TM \times [0,1[)$.

**3.4.** *Reality.* The condition of reality implies that the kernels corresponding to real functions in the cotangent bundle are self-adjoint. When one translates this condition on the kernels, it reads: $k(y,x) = (k(x,y))^*$.

Let us translate this condition onto the functions on the tangent bundle:

$$Q_\epsilon(f)(e_p^{\phi_2(\dot{x},\epsilon)}, e_p^{\phi_1(\dot{x},\epsilon)}) = (Q_\epsilon(f)(e_p^{phi_1(\dot{x},\epsilon)}, e_p^{\phi_2(\dot{x},\epsilon)}))^* = \epsilon^{-n}\hat{f}^*(p,\dot{x}) = \epsilon^{-n}\hat{f}(p,-\dot{x}),$$

which implies that $Q_\epsilon(f)(e_p^{\phi_2(\dot{x},\epsilon)}, e_p^{\phi_1(\dot{x},\epsilon)}) = Q_\epsilon(f)(e_p^{\phi_1(-\dot{x},\epsilon)}, e_p^{\phi_2(-\dot{x},\epsilon)})$, and thus we conclude that the reality condition implies that: $\phi_1(X,\epsilon) = -\phi_2(X,\epsilon)$.

**3.5.** *Traciality.* A quantization is said to be tracial when the trace on the quantum operators can be computed as the trace on the cotangent bundle.

The construction proposed by Landsman that we are going to generalize requires the introduction of the Wigner function,$W_{\Omega_1,\Omega_2} \in C^\infty(TM)$, defined by: $\langle \Omega_1 | Q_\epsilon(f)\Omega_2\rangle = \int f W_{\Omega_1,\Omega_2}$. It is quite simple to see that we can define this function to be:

$$W_{\Omega_1,\Omega_2}(q,p) = \int d\mu(X) e^{-ip^\mu X_\mu} J(x,X,\epsilon)\Omega_1^*(\gamma(x,\phi_1(X,\epsilon)))\Omega_2(\gamma(x,\phi_2(X,\epsilon)))$$

where the Jacobian is the function we have computed above.

Let us now formulate the tracial property in the context of Wigner functions. It reads: $\int W_{\Omega_1}^* W_{\Omega_2} = |\langle \Omega_1|\Omega_2\rangle|^2$. The property is not fulfilled by the quantization we have defined, because when we calculate $\epsilon^{-2n}\int W_{\Omega_1}^* W_{\Omega_2}$ we find that it is given by

$$\int J^2(x,\dot{x},\epsilon)\Omega_1^*\left(\gamma\left(x,\phi_1(\dot{x},\epsilon)\right)\right)\Omega_1\left(\gamma\left(x,\phi_2(\dot{x},\epsilon)\right)\right)\Omega_2^*\left(\gamma\left(x,\phi_1(\dot{x},\epsilon)\right)\right)\Omega_2\left(\gamma\left(x,\phi_2(\dot{x},\epsilon)\right)\right).$$

This shows that the problem comes from the factor of the Jacobian: it should appear without the square power in order to provide the change of coordinates into the desired expression (see above). This is not surprising since we know that even in $\mathbb{R}^{2n}$ it is quite difficult to find a quantization procedure verifying this property.

The solution, proposed by Landsman for the Weyl case, is generalizable for any other mapping between the normal and the tangent bundles. The idea is simply:

i) Change of the quantization mapping: the expression in $Q_\epsilon(f)$ takes the square root of the Jacobian, i.e.: $Q_\epsilon(f)(e_p^{\phi_1(X,\epsilon)}, e_p^{\phi_2(X,\epsilon)}) = \epsilon^{-n} f(p,X) J^{-\frac{1}{2}}(x,X,\epsilon)$

ii) Change of the Wigner function: in order to keep the condition true, we have to modify also the previous definition. If we replace the Jacobian by its square root, it is simple to see that the new definition is suitable for the quantization mapping above. Thus, the final expression for the Wigner function will be:

$$W_{\Omega_1,\Omega_2}(q,p) = \int d\mu(X) e^{-ip^\mu X_\mu} J^{1/2}(x,X,\epsilon)\Omega_1^*(\gamma(x,\phi_1(X),\epsilon))\Omega_2(\gamma(x,\phi_2(X,\epsilon))).$$

Once we have analyzed the situation, we can better understand why traciality is such a restrictive condition when considered directly on $\mathbb{R}^n$. Since in a direct analysis we do not know all its geometric meaning, only those quantizations for which the Jacobian is equal to one for the $\mathbb{R}^n$ case will produce a tracial quantization mapping. Nonetheless any of those which are not directly tracial may be turned into it by applying the procedure above.

## 4. Can we define a deformation quantization using the groupoid?

**4.1.** *The problem in the general case.* In the preceding sections we have analyzed the possibility of defining a mapping with some given properties, between the algebra of classical functions and an algebra of functions that we identify as the algebra of kernels of operators in a suitable Hilbert space.

Nevertheless, it would be interesting to find a similar construction which provides a strict deformation quantization, in the sense we have described in the introduction.

*A priori*, it seems quite easy: we have to use the diffeomorphism in both directions:

$$f \times_\epsilon g = \sigma^{-1}\left(\sigma(f)\cdot\sigma(g)\right)$$

with the convolution product $\cdot$.

But the problem is much more complicated, since the convolution product of kernels (which is the natural one in quantum mechanics) is not defined in $V$ where our kernels live, but in all $M \times M$. Therefore, the convolution product of two quantized functions that belong to $C^\infty(V)$ will no longer belong to it, but to the global algebra of kernels $C^\infty(M\times M)$. Thus, the inverse mapping is not well defined and the deformed product can not, in general, be obtained in this framework.

Where does the problem with the convolution product actually lie? The answer is easy again: given two kernels, they are functions defined on the open set $V$, and as $C^\infty$ functions they go to zero on the "boundary", i.e. given any real number $\delta$ there is a compact set, contained in $V$, and on whose boundary the values of the functions are lower than $\delta$. But the convolution makes that given a point $(z_1, z_2)$ arbitrarily near the "boundary" the product function will be: $k(z_1,z_2) = \int d\mu(y) Q(f)(z_1,y)Q(g)(y,z_2)$, where the points $(z_1, y)$ and $(y, z_2)$ are interior points of $V$, giving, in general, a non-vanishing contribution to the integral unless $z_1$ or $z_2$ belong to the respective boundary factors of an open set $V$ which should be factorizable. In such a case, the integral above would be zero. The problem is that the only open factorizable set $V$ that moreover contains completely the diagonal is the whole manifold $M \times M$.

**4.2.** *The case of exponential manifolds.* It is evident that the problem we have presented above comes from the fact that the tubular neighbourhood theorem that we use to construct our mapping establishes the desired diffeomorphism between an open set in $TM$ (though this point can be solved using the compressibility of hermitian bundles, see [11]) and another open set $V \in M \times M$ containing the diagonal. It is not difficult to see that if $V$ is properly contained in $M \times M$ there is no way of defining the deformed product on the tangent bundle (and hence on the cotangent one). We would need to find a case where $V = M \times M$, and this happens only for an exponential manifold $M$. The construction of the product is exactly the algorithm we suggested before, but now with well behaved maps. Moreover, the main properties to be checked were already studied when strictness was analyzed.

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