





Line-of-Sight Probability for Channel Modeling in 3-D Indoor Environments

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Abstract—A three-dimensional (3-D) analytical indoor line-of-sight (LOS) probability model is given in this letter. According to the indoor layout of a specific building, the proposed closed-form model can be used to compute the LOS probability in a 3-D environment. The model is validated in both single rooms and a typical office building by Monte Carlo simulations. Moreover, the coverage rate in a typical office building with a 3-D ultra-dense small cells network is evaluated based on the proposed LOS probability model. The numerical results show that the proposed model matches the simulations well in terms of both the LOS probability and the coverage rate.

Index Terms—Indoor wireless network, line-of-sight (LOS) probability, three-dimensional (3-D) space.

I. INTRODUCTION

FOR wireless communications, the blockage effects caused by the obstacles intersecting the propagation path between the transmitter and the receiver may attenuate the signal significantly. Especially on the millimeter-wave (mmWave) bands, which are the candidate frequency bands in 5G applications, the signals are more vulnerable to the blockage effects resulted from the propagation characteristics [1]. In the indoor environment, the blockage effects become more pronounced due to the complicated layouts of buildings. A practical approach to express the blockage effects in the network performance evaluation is to distinguish the line-of-sight (LOS) and the non-line-of-sight (NLOS) transmitter–receiver (T–R) links and compute their path loss by different path loss exponents (PLEs) separately [2], [3]. Therefore, the probability that the signal reaches the user equipment (UE) with a LOS path in a given propagation environment, i.e., the LOS probability, is critical and has been incorporated in various network performance evaluation schemes [4], [5].

In the literature, the indoor LOS probability models are either empirical or plan-based. The empirical models are obtained from the measured data and then applied to different scenarios

assuming that they are generally valid [6], [7]. The plan-based models are derived according to the indoor layout of each practical building and present an improved accuracy by inducing the information of the specific propagation environment [8], [9]. The above-mentioned models have been derived assuming that both the UEs and the base stations (BSs) have the same and fixed heights, i.e., only apply to the two-dimensional (2-D) single-floor scenarios and corresponding propagation models.

As the indoor data traffic increases rapidly [10], [11], the BSs are densely and flexibly deployed indoors to satisfy the diversified user demands. Consequently, the overall network performance evaluation for a multifloor building should be conducted in the three-dimensional (3-D) physical space, since the cross-floor connection/interference exists. The work in [12] shows that the characteristics of the actual network deployment in urban buildings are identical to a 3-D homogeneous Poisson point process (PPP). Thus, the network performance metrics, such as the coverage rate, can be analytically derived by employing the 3-D homogeneous PPP distributed BSs to represent the practical network deployment. This approach has been used in the 3-D network performance evaluation [12], [13]. However, the indoor blockage effects are either ignored [12] or represented by certain idealistic assumptions, such as the equally spaced walls [13]. Since the 2-D models are invalid in a 3-D space, an exact 3-D LOS probability model is desired to address the blockage effects in a specific building and improve the accuracy of the network performance evaluation.

In this letter, we propose an approach to compute the LOS probability for practical buildings in the 3-D space. The LOS probability of a typical T–R link can then be derived analytically given the dimensions of a building. The derived analytical model is validated in both single rooms and a typical office building by Monte Carlo simulations. Assuming a one-tier dense homogeneous network, the coverage rate in the building is also computed based on the proposed model. Taking both the 2-D empirical models in [6] and [7] and the 2-D plan-based model in [8] as the benchmarks, the numerical results of the LOS probability and the coverage rate in a typical office building are compared and analyzed.

II. ANALYTICAL MODEL OF LOS PROBABILITY

A. System Model

In an indoor environment, the walls, floors, and ceilings, which are mainly constructed with heavy concrete, attenuate signals significantly. Therefore, a typical room is represented by a rectangular cuboid and the ceiling, floor, and walls of this room are denoted by the faces of the cuboid. A building

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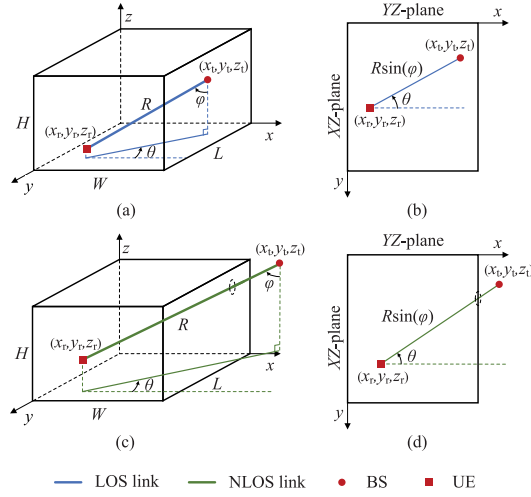


Fig. 1. (a) LOS link in a typical room of a building. (b) Projection of the LOS link on the xy plane. (c) NLOS link in a typical room of a building. (d) Projection of the NLOS link on the xy plane.

consisting of rooms with different dimensions can be considered as a structure composed of cuboids. The UEs are assumed to be randomly and uniformly distributed in the building. The link length R [m] is the Euclidean distance between the BS and the UE with the range $(0, +\infty)$. The LOS probability in the building is defined as the probability that a signal propagates from the BS to the UE without any presence of walls, floors, or ceilings. The dimensions of a room are given by the length L [m], the width W [m], and the height H [m], which are perpendicular to the xz plane, yz plane, and xy plane, respectively. Without losing generality, we assume that $H < W < L$ for each room. In this section, we first derive the LOS probability model for a single room and then derive the LOS probability model for a practical building.

B. 3-D LOS Probability in a Single Room

In this section, based on the system model described above, we present the 3-D LOS probability model for a single room. As shown in Fig. 1, a T-R link is considered as a NLOS link if it is intersected by an arbitrary face of the room. Otherwise, it is a LOS link. The locations of the UE and the BS are given by the coordinates (x_t, y_t, z_t) and (x_r, y_r, z_r) , respectively, in the 3-D Cartesian coordinate system. The length of the projections of the T-R link on the x-axis, y-axis, and z-axis are denoted by x , y , and z , which can be obtained by $x = |x_t - x_r|$, $y = |y_t - y_r|$, and $z = |z_t - z_r|$, respectively. Here, we define two angles, φ and θ , to describe the direction of each link in the 3-D space. As shown in Fig. 1, θ denotes the angle of the projection on the xy plane, while φ is the angle on the plane defined by the T-R link and its projection on the xy plane.

Theorem 1: The joint probability density function (PDF) of the length of the projections and the angles for a T-R link in a room is given by

$$f(x, y, z, \varphi, \theta) = \begin{cases} \frac{4}{\pi^2 H W L}, & (x, y, z, \varphi, \theta) \in \Omega \\ 0, & \text{else} \end{cases} \quad (1)$$

where $\Omega = \{(x, y, z, \varphi, \theta) | (0 < x < W) \wedge (0 < y < L) \wedge (0 < z < H) \wedge (0 < \theta < \frac{\pi}{2}) \wedge (0 < \varphi < \frac{\pi}{2})\}$ and “ \wedge ” denotes the logical conjunction.

Proof: According to the definitions in the system model, we have the distributions $\theta \sim U(0, \frac{\pi}{2})$, $\varphi \sim U(0, \frac{\pi}{2})$, $x \sim U(0, W)$, $y \sim U(0, L)$, and $z \sim U(0, H)$, where $U(\cdot)$ denotes the uniform distribution. The distributions of the angles (φ, θ) and the length of the projections (x, y, z) of the T-R link are all independent from each other. Thus, the joint PDF is the product of their distributions.

Theorem 2: The LOS probability for a T-R link with link length R in a 3-D room with the dimension $H \times W \times L$ can be derived by

$$\text{Pr}_{\text{LOS}}(R) = P_0 + \int_{\varphi_3}^{\varphi_4} \mathcal{H}_1(\varphi) d\varphi - \int_{\varphi_2}^{\varphi_4} \mathcal{H}_2(\varphi) d\varphi \quad (2)$$

where

$$P_0 = \frac{1}{3\pi^2 H W L}$$

$$\times \begin{cases} 3H(2\pi LW + R^2)(\varphi_3 - \varphi_1) \\ + 12HR(L + W)(\cos(\varphi_3) - \cos(\varphi_1)) \\ + R \sin(\varphi_1)(3HR \cos(\varphi_1) + 6\pi LW + R^2) \\ - R \sin(\varphi_3)(3HR \cos(\varphi_3) + 6\pi LW + R^2) \\ + R^2 \cos(2\varphi_3)[R \sin(\varphi_3) - 3(L + W)] \\ - R^2 \cos(2\varphi_1)[R \sin(\varphi_1) - 3(L + W)] \end{cases} \quad (3)$$

$$\mathcal{H}_1(\varphi) = \frac{2(H - R \cos(\varphi))}{\pi^2 H W L}$$

$$\times \begin{cases} 2W \sin(\varphi) \sqrt{R^2 - L^2 \csc^2(\varphi)} - 2WR \sin(\varphi) \\ + 2LW \arcsin\left(\frac{L \csc(\varphi)}{R}\right) - L^2 \end{cases} \quad (4)$$

$$\mathcal{H}_2(\varphi) = \frac{2(H - R \cos(\varphi))}{\pi^2 H W L}$$

$$\times \begin{cases} -2L \sin(\varphi) \sqrt{R^2 - W^2 \csc^2(\varphi)} - 2WR \sin(\varphi) \\ + 2LW \arccos\left(\frac{W \csc(\varphi)}{R}\right) + R^2 \sin^2(\varphi) + W^2 \end{cases} \quad (5)$$

The angles are defined by

$$\varphi_1 = \arccos\left[\min\left(\frac{H}{R}, 1\right)\right] \quad (6)$$

$$\varphi_2 = \begin{cases} \arcsin\left[\min\left(\frac{W}{R}, 1\right)\right], & R < D_1 \\ \varphi_1, & \text{else} \end{cases} \quad (7)$$

$$\varphi_3 = \begin{cases} \arcsin\left[\min\left(\frac{L}{R}, 1\right)\right], & R < D_2 \\ \varphi_1, & \text{else} \end{cases} \quad (8)$$

$$\varphi_4 = \begin{cases} \arcsin\left[\min\left(\frac{D_3}{R}, 1\right)\right], & R < D_4 \\ \varphi_1, & \text{else} \end{cases} \quad (9)$$

where

$$D_1 = \sqrt{H^2 + W^2}, \quad D_2 = \sqrt{H^2 + L^2} \quad (10)$$

$$D_3 = \sqrt{W^2 + L^2}, \quad D_4 = \sqrt{H^2 + W^2 + L^2}.$$

When $R < W$, the last two terms in (2) equal to zero. Otherwise, the LOS probability can be approximated by a closed-form expression as

$$\text{Pr}_{\text{LOS}}(R) \triangleq P_0 + \frac{1}{6} \times \begin{cases} (\varphi_4 - \varphi_3) [\mathcal{H}_1(\varphi_3) + \mathcal{H}_1(\varphi_4) + 4\mathcal{H}_1\left(\frac{\varphi_3 + \varphi_4}{2}\right)] \\ - (\varphi_4 - \varphi_2) [\mathcal{H}_2(\varphi_2) + \mathcal{H}_2(\varphi_4) + 4\mathcal{H}_2\left(\frac{\varphi_2 + \varphi_4}{2}\right)] \end{cases} \quad (11)$$

Proof: See Appendix A.

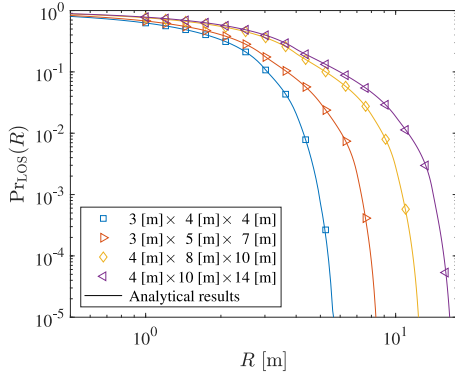


Fig. 2. LOS probability in single rooms versus link length. Analytical results and Monte Carlo simulations in different types of single rooms. The results generated by simulations are denoted by the markers.

The numerical validation of Theorem 2 is performed by the Monte Carlo simulations. As shown in Fig. 2, the analytical results in (11) are in good agreements with the simulations performed for different room sizes.

C. 3-D LOS Probability in a Multifloor Building

In this section, the 3-D LOS probability in a typical building is given. The volume for the i th room in a building is represented by V_i , then the total volume for a building consisting of N_B rooms is given by $V_B = \sum_{i=1}^{N_B} V_i$. Since UEs are uniformly deployed in the building, the probability for a UE located in the i th room is $p_i = \frac{V_i}{V_B}$.

Lemma 1: The LOS probability for a T-R link with the length R in a building with N_B rooms can be denoted by

$$\Pr_B(R) = \sum_{i=1}^{N_B} p_i \Pr_{i, \text{LOS}}(R) \quad (12)$$

where $N_B \in \mathbb{N}_1$, p_i is the probability that the UE end of this link is located in the i th room of the building and $\Pr_{i, \text{LOS}}(R)$ denotes the LOS probability in the i th room.

Proof: Define C_i as the scenario that the UE end of a T-R link is located in the i th room and S as the scenario where the link is a LOS link. Then, the overall LOS probability in a building can be computed by summing up the conditional LOS probabilities given that C_i happens and weighted by p_i , as

$$\Pr_B(R) = \sum_{i=1}^{N_B} p_i \Pr[S|C_i]. \quad (13)$$

The conditional probability can be derived by Theorem 2, and then (12) is obtained.

The LOS probability given by (12) is validated next in a typical office building. Fig. 3 shows a building geometry where each floor of this five-floor building consists of 40 rooms with dimensions $10 \text{ m} \times 10 \text{ m} \times 3 \text{ m}$ and two corridors with dimensions $100 \text{ m} \times 5 \text{ m} \times 3 \text{ m}$, as described by the WINNER II A1 scenario [7]. Two commonly used 2-D empirical LOS probability models, the mixed office model in [6, Table 7.4.2-1] and the WINNER II A1 model in [7, Table 4-7], and the 2-D plan-based exact LOS probability model in [8] are also employed to predict the LOS probability in this scenario as the benchmarks. As can be seen in Fig. 4, the proposed 3-D model fits the simulations well. The 2-D

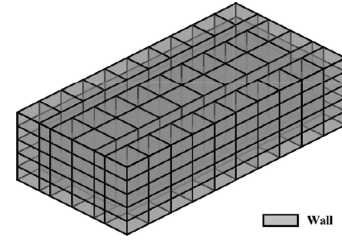


Fig. 3. A five-floor building with WINNER II A1 layout for each floor.

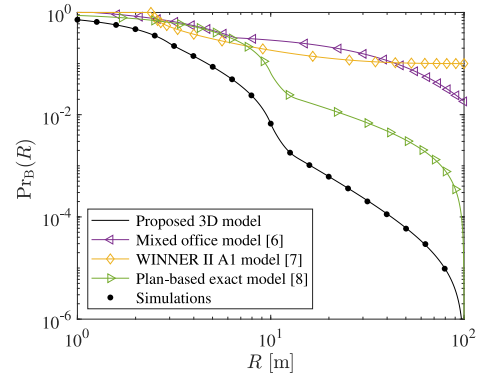


Fig. 4. LOS probability in a building versus link length. Analytical results and simulations in a typical office building shown in Fig. 3.

empirical models show poor agreements with the simulations, since the specific layout of the building is not included. The 2-D plan-based exact model is implemented by inserting the dimensions of the 2-D floor plan of the WINNER II A1 scenario. Despite the similar trend to the simulations, this model overestimates the LOS probability due to the missing direction information in the 3-D space. The 2-D models only address the probability of intersections on the horizontal direction but ignore the probability that the T-R link is obstructed by the two faces parallel to the xy plane. Therefore, the proposed 3-D model can obtain the LOS probability accurately for a practical building, while the 2-D models lose accuracy when directly applied in a 3-D scenario.

III. NETWORK PERFORMANCE EVALUATION BASED ON 3-D LOS PROBABILITY MODEL

A. Network Model

A one-tier downlink dense homogeneous network is studied in a typical office building, as shown in Fig. 3. We assume that the locations of BSs are given by a 3-D homogeneous PPP with density λ_B [BSSs \cdot km $^{-3}$] in the infinite 3-D Euclidean space \mathbb{R}^3 . The path gain over a link length R is expressed as $R^{-\alpha_L}$ and $R^{-\alpha_N}$ for the LOS link and the NLOS link, where α_L and α_N are the PLEs for LOS and NLOS scenarios, respectively. The PLEs α_L and α_N include the impacts of the structural materials and should be adjusted according to the propagation environment. The BSs are assumed to operate in the same frequency band with the same transmit power. Each UE in the building is associated with the BS providing the strongest signal, while signals received from other BSs are considered as interference. The effects of the small-scale fading and the thermal noise are ignored in this interference-limited network.

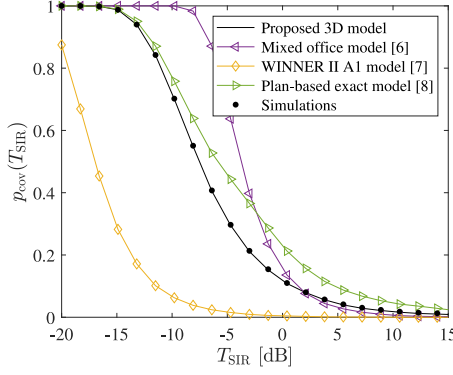


Fig. 5. Coverage rate in a typical office building shown in Fig. 3.

B. Coverage Rate in the Building

Based on the network model above, we further apply the 3-D LOS probability model in the network performance evaluation in terms of the coverage rate. In the computations, we assume that the PLEs $\alpha_L = 1.73$ and $\alpha_N = 3.19$ [6, Table 7.4.1-1], and the density of BSs $\lambda_B = 10^6$ [BSs-km⁻³] in a ultra-dense small cells (UDS) network [12]–[14]. Defining T_{SIR} as the signal-to-interference ratio (SIR) threshold, the coverage rate

$$p_{\text{cov}}(T_{\text{SIR}}) = \Pr \{ \text{SIR} > T_{\text{SIR}} \}. \quad (14)$$

As shown in Fig. 5, the proposed model matches the Monte Carlo simulations well. For the benchmarks, the coverage rate obtained based on the 2-D empirical models shows poor agreements with the simulations. Although the 2-D plan-based exact model may describe the trend of the coverage rate p_{cov} as T_{SIR} increases, it overestimates the coverage rate significantly in this 3-D environment under the parameters assumed. The 2-D models ignore the cross-floor connections/interference, thereby leading to the under- or overestimated network performance. Therefore, the proposed 3-D LOS probability model is critical for the network performance evaluation in practical buildings.

IV. CONCLUSION

In this letter, a 3-D LOS probability model is proposed for the 3-D indoor environments for the first time. According to the exact dimensions of a practical building, both the LOS probability and the coverage rate can be obtained based on the proposed closed-form model accurately. Considering the trend of indoor network densification, the proposed 3-D LOS probability model can be employed to address the indoor blockage effects for practical buildings in the network performance evaluation and optimization schemes.

APPENDIX PROOF OF THEOREM 2

According to the system model, a T–R link is a LOS link if both ends, i.e., the BS and the UE, are in the same cuboid

room. Therefore, deriving the LOS probability for a single room can be seen as deriving the probability that a line segment of length R drops in a cuboid with the dimension $H \times W \times L$ without crossing any face of the cuboid [15]. Let Θ , Φ , and \mathcal{V} represent the ranges of the angle φ , the angle θ and the length of projections (x, y, z) within which the link satisfies the necessary and sufficient condition as a LOS link. Therefore, the LOS probability in a typical room can be derived by the integration of the joint PDF, as

$$\Pr_{\text{LOS}}(R) = \int_{\Phi} \int_{\Theta} \int_{\mathcal{V}} f(x, y, z, \varphi, \theta) dx dy dz d\theta d\varphi \quad (15)$$

where the intervals \mathcal{V} , Θ and Φ are given as shown in (16)–(18) at the bottom of this page, and $f(x, y, z, \varphi, \theta)$ is given in (1). We first integrate the joint PDF with respect to (x, y, z) , i.e.,

$$F(\varphi, \theta) = \iiint_{\mathcal{V}} f(x, y, z, \varphi, \theta) dx dy dz \quad (19)$$

while leave θ and φ as variables. Based on the geometry relationship between the room and the T–R link, (15) can be reformulated as

$$\Pr_{\text{LOS}}(R) = P_0 + P_1 - P_2 \quad (20)$$

where

$$P_0 = \int_{\varphi_1}^{\varphi_3} \int_0^{\frac{\pi}{2}} F(\varphi, \theta) d\theta d\varphi \quad (21)$$

$$P_1 = \int_{\varphi_3}^{\varphi_4} \int_0^{\arcsin\left(\frac{L}{R \sin(\varphi)}\right)} F(\varphi, \theta) d\theta d\varphi \quad (22)$$

$$P_2 = \int_{\varphi_2}^{\varphi_4} \int_0^{\arccos\left(\frac{W}{R \sin(\varphi)}\right)} F(\varphi, \theta) d\theta d\varphi \quad (23)$$

and the angles are defined in (6)–(9). P_0 can be obtained straightforwardly as (3). Solving the integrals with respect to θ , P_1 and P_2 can be rewritten as

$$P_1 = \int_{\varphi_3}^{\varphi_4} \mathcal{H}_1(\varphi) d\varphi, \quad P_2 = \int_{\varphi_2}^{\varphi_4} \mathcal{H}_2(\varphi) d\varphi \quad (24)$$

where $\mathcal{H}_1(\varphi)$ and $\mathcal{H}_2(\varphi)$ are given by (4) and (5), respectively. In order to facilitate the mathematical tractability of the 3-D LOS probability model, we make an approximation of (24) by the Simpson's 1/3 rule [16]. Instead of the integrand $\mathcal{H}_1(\varphi)$ or $\mathcal{H}_2(\varphi)$, a third-order Lagrange interpolating polynomial with the same values at three equally spaced points as the integrand is integrated with respect to φ over Φ . Then (11) can be obtained by inserting the approximation expressions of P_1 and P_2 into (2).

$$\mathcal{V} = \{ [R \cos(\theta) \sin(\varphi) < x < W] \wedge [R \sin(\theta) \sin(\varphi) < y < L] \wedge [R \cos(\varphi) < z < H] \} \quad (16)$$

$$\Theta = \left\{ \arccos \left[\min \left(\frac{W}{R \sin(\varphi)}, 1 \right) \right] < \theta < \arcsin \left[\min \left(\frac{L}{R \sin(\varphi)}, 1 \right) \right] \right\} \quad (17)$$

$$\Phi = \left\{ \arccos \left[\min \left(\frac{H}{R}, 1 \right) \right] < \varphi < \arcsin \left[\min \left(\frac{\sqrt{W^2 + L^2}}{R}, 1 \right) \right] \right\}. \quad (18)$$

REFERENCES

- [1] G. R. Maccartney, T. S. Rappaport, S. Sun and S. Deng, "Indoor office wideband millimeter-wave propagation measurements and channel models at 28 and 73 GHz for ultra-dense 5G wireless networks," *IEEE Access*, vol. 3, pp. 2388–2424, 2015.
- [2] I. Atzeni, J. Arnau and M. Kountouris, "Downlink cellular network analysis with LOS/NLOS propagation and elevated base stations," *IEEE Trans. Commun.*, vol. 17, no. 1, pp. 142–156, Jan. 2018.
- [3] I. Trigui, S. Affes, and B. Liang, "Unified stochastic geometry modeling and analysis of cellular networks in LOS/NLOS and shadowed fading," *IEEE Trans. Commun.*, vol. 65, no. 12, pp. 5470–5486, Dec. 2017.
- [4] M. Ding, P. Wang, D. Lopez-Prez, G. Mao and Z. Lin, "Performance impact of LoS and NLoS transmissions in dense cellular networks," *IEEE Trans. Wireless Commun.*, vol. 15, no. 3, pp. 2365–2380, Mar. 2016.
- [5] J. Yang, M. Ding, G. Mao, Z. Lin, D. Zhang and T. H. Luan, "Optimal base station antenna downtilt in downlink cellular networks," *IEEE Trans. Wireless Commun.*, vol. 18, no. 3, pp. 1779–1791, Mar. 2019.
- [6] 3GPP, "Study on channel model for frequencies from 0.5 to 100 GHz," 3rd Generation Partnership Project (3GPP), Tech. Rep. TR 38.901 V14.1.1 Release 14, Aug. 2017. [Online]. Available: <http://www.3gpp.org/DynaReport/38901.htm>
- [7] P. Kyösti *et al.*, "WINNER II channel models," WINNER II, Tech. Rep., IST-4-027756 D1.1.2 V1.2, 2008.
- [8] H. Zheng, J. Zhang, H. Li, Q. Hong, H. Hu and J. Zhang, "Exact line-of-sight probability for channel modeling in typical indoor environments," *IEEE Antennas Wireless Propag. Lett.*, vol. 17, no. 7, pp. 1359–1362, Jul. 2018.
- [9] W. Yang, J. Zhang and J. Zhang, "Machine learning based indoor line-of-sight probability prediction," in *Proc. Int. Symp. Antennas Propag.*, Xi'an, China, 2019, pp. 1–3.
- [10] A. Zanella, N. Bui, A. Castellani, L. Vangelista and M. Zorzi, "Internet of things for smart cities," *IEEE Internet Things J.*, vol. 1, no. 1, pp. 22–32, Feb. 2014.
- [11] H. Jiang, C. Cai, X. Ma, Y. Yang and J. Liu, "Smart home based on WiFi sensing: A survey," *IEEE Access*, vol. 6, pp. 13317–13325, 2018.
- [12] Z. Pan and Q. Zhu, "Modeling and analysis of coverage in 3-D cellular networks," *IEEE Commun. Lett.*, vol. 19, no. 5, pp. 831–834, May 2015.
- [13] L. Zhang, X. Gu, Z. Liu, L. Zhang and H. Moon, "Modeling and analysis of indoor coverage probability for future 3D dense mobile networks," in *Proc. 20th Int. Symp. Wireless Personal Multimedia Commun.*, 2017, pp. 247–252.
- [14] D. Lopez-Prez, M. Ding, H. Claussen and A. H. Jafari, "Towards 1 Gbps/UE in cellular systems: Understanding ultra-dense small cell deployments," *IEEE Commun. Surveys Tuts.*, vol. 17, no. 4, pp. 2078–2101, Oct.–Dec. 2015.
- [15] R. Fakler, "Buffon's needle problem for a rectangular grid," *Nati. Council Teachers Math.*, vol. 88, no. 3, pp. 205–208, Mar. 1995.
- [16] A. Horwitz, "A version of Simpson's rule for multiple integrals." *J. Comput. Appl. Math.*, vol. 134, pp. 1–11, 2001.