Cycles Containing All Vertices of Maximum Degree

H. J. Broersma
J. van den Heuvel
FACULTY OF APPLIED MATHEMATICS
UNIVERSITY OF TWENTE
ENSCHEDE, THE NETHERLANDS
H. A. Jung
FACHBEREICH MATHEMATIK
TECHNISCHE UNIVERSITÄT BERLIN
BERLIN, GERMANY
H. J. Veldman
FACULTY OF APPLIED MATHEMATICS
UNIVERSITY OF TWENTE
ENSCHEDE, THE NETHERLANDS

ABSTRACT

For a graph G and an integer k, denote by V_k the set $\{v \in V(G) \mid d(v) \geq k\}$. Veldman proved that if G is a 2-connected graph of order n with $n \leq 3k - 2$ and $|V_k| \leq k$, then G has a cycle containing all vertices of V_k. It is shown that the upper bound k on $|V_k|$ is close to best possible in general. For the special case $k = \Delta(G)$, it is conjectured that the condition $|V_k| \leq k$ can be omitted. Using a variation of Woodall's Hopping Lemma, the conjecture is proved under the additional condition that $n \leq 2\Delta(G) + \delta(G) + 1$. This result is an almost-generalization of Jackson's Theorem that every 2-connected k-regular graph of order n with $n \leq 3k$ is hamiltonian. An alternative proof of an extension of Jackson's Theorem is also presented. © 1993 John Wiley & Sons, Inc.

1. RESULTS

We use Bondy and Murty [3] for terminology and notation not defined here, and consider finite simple graphs only. In particular, we denote

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by δ the minimum degree of a graph under consideration and by Δ its maximum degree. For a graph G and an integer k, we denote by V_k the set $\{v \in V(G) \mid d(v) \geq k\}$.

The following result occurs in Veldman [7].

Theorem 1 ([7]). Let G be a 2-connected graph of order n and k an integer with $n \leq 3k - 2$. If $|V_k| \leq k$, then G has a cycle containing all vertices of V_k.

It was shown in [7] that the upper bound $3k - 2$ imposed on n in Theorem 1 cannot be relaxed. The question what would be the best possible upper bound to impose on $|V_k|$, however, was left open. Here we first show that, in fact, the upper bound k on $|V_k|$ is close to best possible in general. To this end we exhibit, for integers n and k with $3 \leq k \leq \frac{1}{2}(n + 5) - \sqrt{n + 2}$, a 2-connected graph of order n in which no cycle contains all vertices of V_k, while $|V_k| = k + O(\sqrt{k}) = k(1 + o(1))$ $(k \to \infty)$. We give the example only for the case where $2k - 2$ is a square. Let n, k, and l be integers such that $k \geq 3$, $l = -2 + \sqrt{2k - 2}$, and $n \geq 2k + 2l + 1$ (or equivalently, $k \leq \frac{1}{2}(n + 5) - \sqrt{n + 2}$). Then it is possible to construct a 2-connected bipartite graph G_0' with bipartition (A, B) such that $|A| = k + l$, $|B| = k + l + 1$, A has $l + 3$ vertices of degree $k + l + 1$, and $k - 3$ vertices of degree $k - 1$, and all vertices of B have degree k. (For example, remove from the complete bipartite graph with bipartition (A, B) the edges of $\frac{1}{2}l + 2$ disjoint copies of $K_{l,l+2}$, where the smaller class of the bipartition of each $K_{l,l+2}$ is contained in A.) Let G_0 be a subdivision of G_0' with n vertices. Clearly, no cycle of G_0 contains all vertices of V_k, while $|V_k| = k + 2l + 4 = k + 2\sqrt{2k - 2}$.

The next result is due to Shi Ronghua, and also follows from a more general result of Bollobás and Brightwell. It shows that the condition $|V_k| \leq k$ in Theorem 1 can be left out completely in the special case where $n \leq 2k$.

Theorem 2 (Bollobás and Brightwell [1], Shi Ronghua [6]). In every 2-connected graph of order n there exists a cycle containing all vertices of degree at least $\frac{1}{2}n$.

Here we focus on the special case where $k = \Delta$. We conjecture that in this case, too, the condition $|V_k| \leq k$ in Theorem 1 can be omitted.

Conjecture 3. If G is a 2-connected graph of order n with $n \leq 3\Delta - 2$, then G has a cycle containing all vertices of degree Δ.

Our main result is that Conjecture 3 holds under the additional condition that $n \leq 2\Delta + \delta + 1$.

Theorem 4. If G is a 2-connected graph of order n with $n \leq 3\Delta - 2$ and $n \leq 2\Delta + \delta + 1$, then G has a cycle containing all vertices of degree Δ.

Theorem 4 is best possible in the sense that the condition $n \leq 3\Delta - 2$ cannot be relaxed. To see this, construct a graph G_1 from $3K_{p-1}$ ($p \geq 5$) by adding two vertices v_1, v_2 and joining them to vertices of $3K_{p-1}$ in such a way that $d(v_1) = d(v_2) = p$, each copy of K_{p-1} contains at least one vertex adjacent to both v_1 and v_2, and G_1 is 2-connected. With $n = |V(G_1)|$ we have $n = 3p - 1 = 3\Delta - 1$, and since $\delta = \Delta - 2$, $n = 2\Delta + \delta + 1$, while no cycle of G_1 contains all vertices of degree Δ. Obviously, the graph G_1 (and every 2-connected spanning subgraph of G_1 such that each copy of K_{p-1} contains a vertex of degree p) also shows that Conjecture 3, if true, is best possible.

An immediate consequence of Theorem 4 is the following.

Corollary 5. If G is a 2-connected graph of order n with $n \leq 2\Delta + \delta - 2$, then G has a cycle containing all vertices of degree Δ.

Corollary 5 in turn implies the following.

Corollary 6. Every 2-connected k-regular graph on at most $3k - 2$ vertices is hamiltonian.

Thus, Theorem 4 and Corollary 5 are almost-generalizations of the theorem in Jackson [5] stating that every 2-connected k-regular graph on at most $3k$ vertices is hamiltonian.

In fact, by combining Theorem 4 with Jackson's Theorem we obtain an improvement of Corollary 5.

Theorem 7. If G is a 2-connected graph of order n with $n \leq 2\Delta + \delta - 1$, then G has a cycle containing all vertices of degree Δ.

It is conceivable that the condition $n \leq 2\Delta + \delta - 1$ in Theorem 7 can be relaxed to $n \leq 2\Delta + \delta$, which would yield a full generalization of Jackson's Theorem. The graph G_1 shows that a further relaxation to $n \leq 2\Delta + \delta + 1$ is not possible.

The proof of Theorem 4 is given in Section 4. It uses Lemma 9 in Section 3, a variation of the Hopping Lemma in Woodall [9]. In Section 5, Lemma 9 is used to give an alternative proof of (an extension of) Jackson's Theorem. Section 2 contains preliminary definitions.

2. TERMINOLOGY AND NOTATION

Let G be a graph, C a cycle of G, and A and B subsets of $V(G)$. The cycle C is called *A-longest* if $|A \cap V(C)| \geq |A \cap V(C')|$ for every cycle

C' of G, and *A-dominating* if every vertex in $A - V(C)$ has all its neighbors on C. We call C *(A,B)-extendable* if there exists an *(A,B)-extension* of C, i.e., a cycle C' such that $V(C') \subseteq B$ and $A \cap V(C)$ is properly contained in $A \cap V(C')$. The cycle C is *A-extendable* if C is $(A, V(G))$-extendable, and *extendable* if C is $V(G)$-extendable. By $\omega(G)$ we denote the number of components of G. By $\varepsilon(A,B)$ we denote the number of edges of G with one end in A and the other in B, the edges with both ends in $A \cap B$ being counted twice.

We denote by $\vec{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. If $A \subseteq V(C)$, then $A^+ = \{a^+ \mid a \in A\}$ and $A^- = \{a^- \mid a \in A\}$. Similar notation is used for paths. When more than one cycle or path is under consideration, we sometimes write $u^{+C}, u^{-C} \ldots$ instead of just u^+, $u^- \ldots$ in order to avoid ambiguity.

3. A VARIATION OF WOODALL'S HOPPING LEMMA

Woodall's Hopping Lemma reads as follows.

Lemma 8 (Woodall [9]). Let $\vec{C}$ be a cycle of length m in a graph G. Assume G contains no cycle of length $m + 1$ and no cycle C' of length m with $\omega(G - V(C')) < \omega(G - V(C))$, and a is an isolated vertex of $G - V(C)$. Set $Y_0 = \emptyset$ and, for $i \geq 1$,

$$X_i = N\big(Y_{i-1} \cup \{a\}\big),$$
$$Y_i = \big(X_i \cap V(C)\big)^+ \cap \big(X_i \cap V(C)\big)^-.$$

Set $X = \cup_{i=1}^{\infty} X_i$ and $Y = \cup_{i=1}^{\infty} Y_i$. Then

(a) $X \subseteq V(C)$;
(b) if $x_1, x_2 \in X$, then $x_1^+ \neq x_2$;
(c) $X \cap Y = \emptyset$.

For our variation of Lemma 8, Lemma 9 below, we adapt the assumptions and definitions in Lemma 8 as follows.

Let G be a graph, A a subset of $V(G)$ and $\vec{C}$ a cycle in G with $A \not\subseteq V(C)$. Let a be a vertex in $A - V(C)$. Set $D = V(C) \cup \{a\}$ and $R = V(G) - D$.

Assume C is not (A,D)-extendable. Set $Y_0 = \emptyset$ and, for $i \geq 1$,

$$X_i = N(Y_{i-1} \cup \{a\}) \cap V(C),$$
$$Y_i = X_i^+ \cap X_i^-.$$

Then $N(a) \cap V(C) = X_1 \subseteq X_2 \subseteq \cdots$ and $\emptyset = Y_0 \subseteq Y_1 \subseteq \cdots$. Set $X = \bigcup_{i=1}^{\infty} X_i$ and $Y = \bigcup_{i=1}^{\infty} Y_i$. Note that

(1) $Y \subseteq V(C)$ and $N(Y) \cap V(C) \subseteq X$.

The *height* $h(x)$ of $x \in X$ is defined by

$$h(x) = \min\{i \mid x \in X_i\}.$$

A path $P = x_1 \vec{P} x_2$ is called a *hopping path* if each of the following conditions is satisfied:

(2) $x_1, x_2 \in X$;
(3) $V(P) \subseteq V(C)$;
(4) $A \cap V(C) \subseteq V(P)$;
(5) if $i < \max\{h(x_1), h(x_2)\}$ and $y \in Y_i \cap (V(P) - \{x_1, x_2\})$, then $y^{-P}, y^{+P} \in X_i$.

The *height* $h(P)$ of a hopping path $P = x_1 \vec{P} x_2$ is defined by

$$h(P) = \max\{h(x_1), h(x_2)\}.$$

Lemma 9.

(a) There exists no hopping path.
(b) If $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $x_1^+ \neq x_2$ and $x_1^+ \vec{C} x_2^- \cap A \neq \emptyset$.
(c) $X \cap Y = \emptyset$.
(d) $Y \subseteq A$.
(e) Y is an independent set.

Proof. By induction on i we prove the following assertion for all $i \geq 1$: $A(i)$: there exists no hopping path of height i.
If there exists a hopping path $P = x_1 \vec{P} x_2$ of height 1, then $x_1, x_2 \in X_1 = N(a) \cap V(C)$ and $x_1 \vec{P} x_2 a x_1$ is an (A,D)-extension of C, contradicting our assumptions and proving $A(1)$. Now fix $i \geq 1$ and assume

(6) $A(j)$ holds for all $j \leq i$.

In particular, if $x_1, x_2 \in X_i$ and $x_1 \neq x_2$, then $x_2 \vec{C} x_1$ is not a hopping path, whence

(7) if $x_1, x_2 \in X_i$ and $x_1 \neq x_2$, then $x_1^+ \neq x_2$ and $x_1^+ \vec{C} x_2^- \cap A \neq \emptyset$.

Suppose $x \in X_i \cap Y_i$. Then by definition of Y_i, $x^+ \in X_i$. This contradicts (7) and so

(8) $X_i \cap Y_i = \emptyset$.

If $y \in Y_i$, then $y^+, y^- \in X_i$. Now by (7), $y \in A$. We thus have

(9) $Y_i \subseteq A$.

To prove $A(i+1)$, suppose there exists a hopping path $P = x_1\vec{P}x_2$ of height $i+1$. We assume $h(x_1) \geq h(x_2)$ and distinguish two cases.

Case 1. $h(x_2) < h(x_1) = i + 1$.

Then $x_1 \in X_{i+1} - X_i$ and hence there exists a vertex $y \in Y_i - Y_{i-1}$ with $x_1 y \in E(G)$. By (1), (9), and (4), $y \in V(P)$. We have $y \neq x_2$; otherwise $y \in X_i \cap Y_i$, contradicting (8). By (5), $y^{-P}, y^{+P} \in X_i$. Set $Q = y^{-P}\overleftarrow{P}x_1 y\vec{P}x_2$. Since $V(Q) = V(P)$, the path Q satisfies (2)–(4). We have $x_1 \notin Y_{i-1}$ (otherwise $y \in X_i \cap Y_i$, contradicting (8)) and $y \notin Y_{i-1}$. Furthermore, $\{v^{-P}, v^{+P}\} = \{v^{-Q}, v^{+Q}\}$ for each internal vertex v of Q with $v \neq x_1, y$. It follows that Q satisfies (5) also, whence Q is a hopping path of height at most i, contradicting (6).

Case 2. $h(x_1) = h(x_2) = i + 1$.

In this case there exist vertices $y_1, y_2 \in Y_i - Y_{i-1}$ with $x_1 y_1, x_2 y_2 \in E(G)$. By (1), (9), and (4), $y_1, y_2 \in V(P)$. We have $y_1 \neq x_2$; otherwise $x_2^{+C} \in X_i$ and $x_2^{+C}\vec{C}x_2$ is a hopping path of the type excluded in Case 1. Similarly, $y_2 \neq x_1$. As in Case 1, we obtain a contradiction by constructing a hopping path of height at most i: $y_1^{-P}\overleftarrow{P}x_1 y_1\vec{P}y_2 x_2\overleftarrow{P}y_2^{+P}$ if $y_1 \in x_1\vec{P}y_2$, $y_2^{-P}\overleftarrow{P}x_1 y_1\overleftarrow{P}y_2 x_2\vec{P}y_1^{+P}$ if $y_2 \in x_1\vec{P}y_1$.

Since thus $A(i)$ holds for all $i \geq 1$, (7), (8), and (9) also hold for all $i \geq 1$. (a)–(d) follow. Finally, (e) is an immediate consequence of (1) and (c). ▮

Set $U' = X^+ - Y$ and $W' = X^- - Y$. Let $p_1, \ldots, p_t$ be the vertices of U', occurring on $\vec{C}$ in the order of their indices. For $i = 1, \ldots, t$, let q_i be the unique vertex of W' in the component of $C - X$ containing p_i. By Lemma 9(b), $p_i\vec{C}q_i \cap A \neq \varnothing$ $(1 \leq i \leq t)$. Let p_i' and q_i' be the first and last vertex of A on $p_i\vec{C}q_i$, respectively. Define the vertices u_i and w_i by

$$(u_i, w_i) = \begin{cases} (p_i', q_i'), & \text{if } p_i' \neq q_i'; \\ (p_i', q_i), & \text{if } p_i' = q_i' \neq q_i; \\ (p_i, q_i), & \text{if } p_i' = q_i' = q_i. \end{cases}$$

Now set $U = \{u_1, \ldots, u_t\}$ and $W = \{w_1, \ldots, w_t\}$.

Corollary 10.

(a) For $i \neq j$, u_i is not adjacent to any vertex in $p_j\vec{C}u_j$. In particular, U is an independent set.

(b) For $i \neq j$, w_i is not adjacent to any vertex in $w_j\vec{C}q_j$. In particular, W is an independent set.
(c) If $v \in q_j^{++}\vec{C}p_i^{--}$ and $u_iv \in E(G)$, then $w_jv^- \notin E(G)$ and $w_jv^+ \notin E(G)$ $(1 \leq i \leq t, 1 \leq j \leq t, i \neq j+1 \pmod{t}$; possibly $i = j)$.
(d) If $v \in u_i^{++}\vec{C}p_j^-$ and $u_iv \in E(G)$, then $u_jv^- \notin E(G)$ $(i \neq j)$.
(e) If $v \in q_i^{++}\vec{C}w_j^-$ and $w_iv \in E(G)$, then $w_jv^- \notin E(G)$ $(i \neq j)$.

Proof.

(a) Suppose $u_iv \in E(G)$ for some $v \in p_j\vec{C}u_j$. Then, since $u_i, v \notin Y$, $p_i^-\overleftarrow{C}vu_i\overleftarrow{C}p_j^-$ is a hopping path, contradicting Lemma 9(a).
(b) The proof of (b) is similar to that of (a).
(c) Suppose $v \in q_j^{++}\vec{C}p_i^{--}$ and $u_iv, w_jv^- \in E(G)$. By (1), $v, v^- \notin Y$. Now the path $p_i^-\overleftarrow{C}vu_i\vec{C}w_jv^-\overleftarrow{C}q_j^+$ is a hopping path, a contradiction. The rest of (c) is proved similarly.
(d) Suppose $v \in u_i^{++}\vec{C}p_j^-$ and $u_iv, u_jv^- \in E(G)$ $(i \neq j)$. Then, since $u_i, u_j, v, v^- \notin Y$, $p_i^-\overleftarrow{C}u_jv^-\overleftarrow{C}u_iv\vec{C}p_j^-$ is a hopping path.
(e) The proof of (e) is similar to that of (d). ∎

In our next consequence of Lemma 9 we make a stronger assumption about C.

Corollary 11. Assume C is not A-extendable.

(a) If x_1, x_2, v_1, v_2 are distinct vertices such that $x_1, x_2 \in X$, $v_1 \in x_1\vec{C}x_2$, $v_2 \in x_2\vec{C}x_1$, $x_1\vec{C}v_1 \cap A \subseteq \{x_1, v_1\}$ and $x_2\vec{C}v_2 \cap A \subseteq \{x_2, v_2\}$, then there exists no (v_1, v_2)-path with all internal vertices in R.
(b) No pair of vertices in $(Y \times Y) \cup (Y \times U) \cup (Y \times W) \cup (U \times U) \cup (W \times W)$ is joined by a path with all internal vertices in R.

Proof.

(a) Suppose v_1 and v_2 are joined by a path P with all internal vertices in R. Set $G' = G + v_1v_2$. We show that

(10) C is not (A, D)-extendable in G'.

Suppose there exists an (A, D)-extension C' of C in G'. If $v_1v_2 \notin E(C')$, then C' is a cycle in G, too, whence C is A-extendable in G, contradicting the hypothesis of the corollary. Now assume $v_1v_2 \in E(C')$. By definition, C' contains no vertex of R. Hence the subgraph of G induced by $(E(C') - \{v_1v_2\}) \cup E(P)$ is a cycle. This cycle is an A-extension of C in G, again a contradiction. (10) follows.

We now derive a contradiction from (10) by applying Lemma 9 to G'. Define X' and Y' for G' in the same way as X and Y were defined for G. Since $x_1, x_2 \in X \subseteq X'$, $x_1\vec{C}v_1 \cap A \subseteq \{x_1, v_1\}$

and $x_2\overrightarrow{C}v_2 \cap A \subseteq \{x_2, v_2\}$, we have $v_1, v_2 \notin X'$ by Lemma 9(b). Since $N_{G'}(Y') \cap V(C) \subseteq X'$, it follows that $v_1, v_2 \notin Y'$. But then $x_1\overleftarrow{C}v_2v_1\overrightarrow{C}x_2$ is a hopping path, contradicting Lemma 9(a).

(b) This is an immediate consequence of (a) (applied to $\overleftarrow{C}$ instead of $\overrightarrow{C}$ if W is involved) and the definitions of Y, U, W. ■

Comparing Lemma 8 with Lemma 9 for $A = V(G)$, we observe that Lemma 9, in contrast to Lemma 8, can be applied to arbitrary nonextendable cycles. On the other hand, whereas $N(Y) = X \subseteq V(C)$ under the hypothesis of Lemma 8, nonextendability of C only assures that no vertex of Y is joined to a vertex of $Y \cup U \cup W$ by a path with all internal vertices in R. However, in our applications this weaker conclusion will turn out to be almost equally useful.

We finally note that obviously Lemma 9 and Corollaries 10 and 11 remain valid if $\{a\}$ is replaced by a subset S of $V(G) - V(C)$ such that every pair of vertices in $N(S) \cap V(C)$ is joined by a path with all internal vertices in S and at least one internal vertex in A.

4. PROOF OF THEOREM 4

In the proof of Theorem 4 we use the following result (from which Theorem 1 is easily deduced).

Theorem 12 (Veldman [8]). Let G be a 2-connected graph of order n and k an integer with $n \leq 3k - 2$. Then G contains a V_k-longest cycle which is V_k-dominating.

Proof of Theorem 4. Let G be a 2-connected graph with vertex set V such that $|V| = n \leq 3\Delta - 2$ and $n \leq 2\Delta + \delta + 1$. Set $A = V_\Delta$. By Theorem 12, G contains an A-longest cycle C, which is A-dominating. In particular, C is not A-extendable. We assume $A \not\subseteq V(C)$ and derive a contradiction. Let a be a vertex in $A - V(C)$. Define $R, X, Y, p_1, \ldots, p_t, q_1, \ldots, q_t, u_1, \ldots, u_t, w_1, \ldots, w_t, U, W$ as in Section 3. For $1 \leq i \leq t$, set $S_i = p_i\overrightarrow{C}q_i$ and $s_i = |S_i|$. Set $Z = U \cup W$, $B = Y \cup Z \cup \{a\}$, $S = \cup_{i=1}^{t} S_i$, $r = |R|$, $s = |S|$, $x = |X|$ and $y = |Y|$. Note that $V(C) = X \cup Y \cup S$, $n = x + y + s + r + 1$ and $x = y + t$.

We derive a lower bound for $\varepsilon(B, X)$. As in Jackson [5, Lemma 2] we deduce from Corollary 10(a), (b), (c) that

(11) $\varepsilon(\{u_j, w_j\}, S_i) \leq s_i - 1 \qquad (j \neq i).$

Also,

(12) $\varepsilon(\{u_i, w_i\}, S_i) \leq 2(s_i - 1) \qquad (1 \leq i \leq t).$

Therefore, since $\varepsilon(Y \cup \{a\}, S) = 0$ by (1),

$$(13)\ \varepsilon(B,S) = \varepsilon(Z,S) = \sum_{i=1}^{t}\sum_{j=1}^{t} \varepsilon(\{u_j, w_j\}, S_i) \le \sum_{i=1}^{t}(t+1)(s_i - 1) = (t+1)(s-t).$$

By (1) and Lemma 9(c),

$$(14)\ \varepsilon(B, Y \cup \{a\}) = 0.$$

Since C is A-dominating, $\varepsilon(\{a\}, R) = 0$. If $v \in R$, then, by Corollary 11, v is adjacent to at most one vertex of U and at most one vertex of W, and if v is adjacent to a vertex of Y, then, again by Corollary 11, v is not adjacent to any other vertex of B. We conclude that

$$\varepsilon(B,R) = \varepsilon(Z \cup Y, R) \le \begin{cases} 1 \cdot r, & \text{if } t = 0 \\ 2 \cdot r, & \text{if } t \ge 1 \end{cases},$$

whence

$$(15)\ \varepsilon(B,R) \le (t+1)r,$$

where equality is possible only if $r = 0$ or $t \in \{0,1\}$. By the way u_i and w_i were chosen, at least one of the vertices u_i and w_i belongs to A, so that $\max\{d(u_i), d(w_i)\} = \Delta$; furthermore, $\min\{d(u_i), d(w_i)\} \ge \delta$ $(1 \le i \le t)$. Since $Y \subseteq A$ by Lemma 9(d), and $t = x - y$, we obtain

$$(16)\ \varepsilon(B,V) \ge (t + y + 1)\Delta + t\delta = (x+1)\Delta + t\delta.$$

From (13)–(16),

$$(17)\ \varepsilon(B,X) = \varepsilon(B,V) - \varepsilon(B,S) - \varepsilon(B, Y \cup \{a\}) - \varepsilon(B,R) \ge (x+1)\Delta + t\delta - (t+1)(s-t) - (t+1)r.$$

On the other hand,

$$(18)\ \varepsilon(B,X) \le x\Delta.$$

Combining (17) and (18), we obtain

$$(19)\ (t+1)(s-t) \ge \Delta + t\delta - (t+1)r.$$

Using $y = x - t$ and $x \ge d(a) = \Delta$, we deduce from (19) that

$$(20)\ (t+1)n = (t+1)(x+y+s+r+1) = (t+1)(s-t) + (t+1)(2x+r+1) \ge \Delta + t\delta - (t+1)r + (t+1)(2\Delta + r + 1),$$

whence

$$(21)\ (t+1)(2\Delta + \delta - n + 1) + \Delta - \delta \le 0.$$

By the hypothesis of the theorem, $2\Delta + \delta - n + 1 \geq 0$. Thus by (21), $2\Delta + \delta - n + 1 = 0$ and $\Delta - \delta = 0$. But then $n = 2\Delta + \delta + 1 = 3\Delta + 1$, contradicting $n \leq 3\Delta - 2$. ∎

We note that if it were true that, under the hypothesis of Theorem 12, *every* V_k-longest cycle is V_k-dominating, then we could have concluded that $N(Y) \subseteq V(C)$ (as in Lemma 8). However, as shown in Veldman [8], Theorem 12 cannot be strengthened in this direction (for general k). Moreover, even if $N(Y) \subseteq V(C)$, we do not obtain a better estimate for $\varepsilon(B, R)$, which is the reason that for the proof of Theorem 4 the conclusions of Lemma 9 are just as useful as those of Lemma 8.

5. AN ALTERNATIVE PROOF OF JACKSON'S THEOREM

Using the results of Section 3, we now give a relatively short proof of the following extension of Jackson's Theorem, which occurs in Zhu, Liu, and Yu [10] and Bondy and Kouider [2].

Theorem 13 [2,10]. Every 2-connected k-regular graph on at most $3k + 1$ vertices is hamiltonian, except for the Petersen graph.

Before we prove Theorem 13, we introduce some additional terminology. A cycle C of a graph G is a D_λ-*cycle* if all components of $G - V(C)$ have order less than λ. A D_1-cycle is a Hamilton cycle, a D_2-cycle is often called a dominating cycle. Two subgraphs H_1 and H_2 of a graph are *remote* if they are disjoint and $\varepsilon(V(H_1), V(H_2)) = 0$.

Proof of Theorem 13. Let G be a nonhamiltonian 2-connected k-regular graph with vertex set V, where $|V| = n \leq 3k + 1$. We distinguish two cases.

Case 1. G contains a D_2-cycle.

Let $\vec{C}$ be a nonextendable D_2-cycle and let a be a vertex in $V - V(C)$. On replacing Δ and δ by k and A by V we may copy the proof of Theorem 4 up to (21), which becomes

(22) $(t + 1)(3k + 1 - n) \leq 0$.

Since $n \leq 3k + 1$, equality holds in (22) and hence throughout (11)–(20). In particular, we have

(23) $n = 3k + 1$,

(24) $x = k$,

(25) $\varepsilon(\{u_j, w_j\}, S_i) = s_i - 1 \qquad (j \neq i)$,

(26) $\varepsilon(\{u_i, w_i\}, S_i) = 2(s_i - 1) \qquad (1 \leq i \leq t)$.

Since (18) holds with equality,

(27) $N(X) \subseteq B$,

and hence

(28) $\varepsilon(Z, X) = \varepsilon(S, X)$.

Suppose $r > 0$. Since (15) holds with equality, $t \in \{0, 1\}$. If $t = 0$, then a vertex v of R is adjacent to no vertex of X (by (27)), no vertex of S (since $S = \emptyset$), no vertex of $V - V(C)$ (since C is a D_2-cycle) and at most one vertex of Y (by Corollary 11); thus $d(v) \leq 1$, a contradiction. If $t = 1$, then (15) holds with equality only if $\varepsilon(Y, R) = 0$. But then $N(Y \cup \{a\}) = X$ and hence

$$\begin{aligned} xk = \varepsilon(X, V) &\geq \varepsilon(X, Y \cup \{a, p_1, q_1\}) \\ &\geq (y + 1)k + 2 = xk + 2, \end{aligned}$$

a contradiction. We conclude that

(29) $r = 0$.

As in Bondy and Kouider [2, Case 1], from (23)–(29) and Corollary 10 it can be deduced that G is the Petersen graph.

Case 2. G contains no D_2-cycle.

Set $\lambda = \min\{s \mid G \text{ contains a } D_{s+1}\text{-cycle}\}$. By assumption, $\lambda \geq 2$. Let $\vec{C}$ be a $D_{\lambda+1}$-cycle such that $G - V(C)$ has as few components of order λ as possible, H_0 a component of $G - V(C)$ of order λ and $v_1, \ldots, v_p$ the neighbors of H_0, occurring on $\vec{C}$ in the order of their indices. Clearly,

(30) $p \geq \max\{2, k - \lambda + 1\}$.

By the proof of Veldman [7, Theorem 2] G contains, for $1 \leq i \leq p$, a connected subgraph H_i of order λ with $v_i^+ \in V(H_i)$ and $V(H_i) \cap V(C) \subseteq v_i^+ \vec{C} v_{i+1}^-$ (indices mod p), such that $H_0, H_1, \ldots, H_p$ are pairwise remote. Since $p \geq 2$ by (30) and $n \leq 3k + 1$ by hypothesis,

$$3k + 1 \geq n \geq \lambda + p(\lambda + 1) \geq 3\lambda + 2,$$

implying that

(31) $\lambda \leq k - 1$.

Set $A = \cup_{i=0}^{p} V(H_i)$ and $B = V - A$. G is k-regular and $H_0, H_1, \ldots, H_p$ are pairwise remote. Furthermore, $k - \lambda + 1$ is nonnegative by (31). Thus by (30),

(32) $\varepsilon(A, B) \geq |A|(k - \lambda + 1) = \lambda(p + 1)(k - \lambda + 1)$
$\geq \lambda(k - \lambda + 2)(k - \lambda + 1)$.

On the other hand,

$$(33)\ \varepsilon(A,B) \leq |B|k = (n - \lambda(p + 1))k \\ \leq (3k + 1 - \lambda(k - \lambda + 2))k.$$

Combining (32) and (33) we obtain

$$(34)\ (3k + 1)k \geq \lambda(k - \lambda + 2)(2k - \lambda + 1).$$

Set $f(x) = x(k - x + 2)(2k - x + 1)$. On the interval $(0, k + 1)$ the function f has exactly one local extremum, which is a local maximum. Thus by (31) and (34),

$$(3k + 1)k \geq \min\{f(2), f(k - 1)\} \\ = \min\{2k(2k - 1), 3(k - 1)(k + 2)\}.$$

It follows that $k = 3$ and, by (31), $\lambda = 2$, so that $H_0 \cong K_2$. Since (34) holds with equality, (32) holds with equality, implying that $p = 2$. But then v_1 and v_2 have degree at least 4, a contradiction. ∎

Comparing our proof of Theorem 13 with the proof in Bondy and Kouider [2], we conclude that our Case 1 is only slightly more involved than [2, Case 1]. On the other hand, by its stronger hypothesis, our Case 2 is significantly simpler than [2, Case 2], resulting in a net gain with respect to the total length of the proof (even if the relevant part of the proof of Veldman [7, Theorem 2] is included).

We finally note that for $k \geq 4$, Case 2 can be excluded by the following result from Fraisse [4].

Theorem 14 ([4]). Every 2-connected k-regular graph with fewer than $3(k + 1 - (2/k))$ vertices has a D_2-cycle.

We preferred to settle Case 2 directly in order to allow a comparison of our proof of Theorem 13 with the proof in [2]. Moreover, to our knowledge, Fraisse's paper [4] has never been published.

Note. Jackson recently informed us that he has proved Conjecture 3 [B. Jackson, Cycles through vertices of large maximum degree. Preprint (1992)].

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