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Theory and Methodology

Interval uneffectiveness distribution for a k-out-of-n multistate reliability system with repair

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Abstract: This paper deals with a parallel load-sharing reliability system with cold standby redundancy and ample repair facilities. That is, we have n identical parallel units, of which at most k units are operating simultaneously. If less than k units are available, the system operates at a proportionally reduced level. For this system, an approximate method is given for the calculation of the probability distribution of that proportion of the system capacity that cannot be used in a given time period. The method is based on an approximation of the k-out-of-n multistate system by a two-state single component. Validation of the approximation using Monte-Carlo simulation shows satisfactory performance. Also, sensitivity results are given, showing in particular a decreasing sensitivity of the measures of performance to the distributional form of the unit lifetimes and repair times as the size of the system increases. Furthermore, it is found that the effect of the distributional form of the unit lifetimes dominates that of the unit repair times.

Keywords: (Un)effectiveness, Markov process, production, k-out-of-n system, phase-type distribution

1. Introduction

In this paper, we consider the k-out-of-n multistate reliability model with repair. That is, we have a system consisting of n identical units, each having a capacity $100/k\%$. Hence k units are required to be available for running the system at full capacity. If more than k units are available, the superfluous units are put on cold standby and cannot fail. If less than k units (say i) are available, the system operates at a proportionally reduced level, i.e. at $100i/k\%$. The model under consideration has ample repair facilities, so all failed units can be under repair simultaneously. We assume perfect switch-over with no start-up failures and no switch-over times. The lifetimes as well as the repair times of the units are independent, identically distributed random variables, both with probability distributions belonging to the important class of phase-type distributions with two phases. Further, the assumption is made that $\mu_R \ll \mu_L$, where μ_L and μ_R denote the mean unit lifetime and the mean unit repair time respectively. It is noted that this assumption is no restriction for most practical situations.

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Here any squared coefficient of variation $c_X^2 > 0.5$ can be reached. By $\lambda_1 > \lambda_2$, the parameter p satisfies $p \leqslant 1$. However, p may be negative, corresponding to the case of $0.5 < c_X^2 < 1$. In case $p \geqslant 0$, the density (3) is the well-known hyperexponential density of order 2 and has $c_X^2 \geqslant 1$.

Many theoretical studies on the k-out-of-n multistate system have been carried out, for which references can be found e.g. in Birolini [1]. In particular, the system long-term uneffectiveness, defined as the long-run fraction of the system capacity which cannot be used due to failure of components, has been a point of concern. However, the stationary interval uneffectiveness distribution is a subject to which relatively little attention has been paid. This distribution represents the probability that the system uneffectiveness will not exceed some pre-specified level in a time interval of given length, when the system has reached statistical equilibrium. This performance measure is of great importance in practice, e.g. when some pre-specified production level has to be attained in a given time period in order for sales contracts to be met.

The main aim of this study is the development of a computationally tractable method for obtaining the stationary interval uneffectiveness distribution for the system as described above. Because both the unit repair times and the unit lifetimes have a phase-type distribution, the long-term uneffectiveness U_∞ can easily be calculated using a continuous Markov chain. However, an exact analysis for the interval uneffectiveness distribution seems very complex and will probably not lead to computationally tractable methods. Therefore we have opted for an approximate approach.

The essence of this approach is the approximation of the k-out-of-n multi-state system by a two-state single component (cf. also Brouwers [2]). The logic behind this is that, for reliable units ($\mu_R \ll \mu_L$), the probability that more than $(n - k + 1)$ units are down simultaneously is negligible for practical purposes. Hence, nearly all of the time, the operational level of the system will fluctuate between 100% and $100(k-1)/k$% of the system capacity. Thus an approximation by a single component with operational levels 100% and 100α% (for some $0 \leqslant \alpha \leqslant 1$ to be specified below) seems to be appropriate. Here we take α as the average operational level of the system when the system is operating below full capacity. This implies that the approximating single component system has the same long-term uneffectiveness as the original k-out-of-n system. The approximate interval uneffectiveness distribution can be computed by a straightforward extension of results in Takács [5] for single items of equipment, once the stationary distributions of the sojourn times in the two states are obtained. We will approximate these sojourn time distributions by computationally tractable distributions. For this, we fit gamma distributions by matching the first two moments. Because in our model the unit lifetimes and repair times have phase-type distributions, the moments of the sojourn time distributions can easily be obtained using the well-known technique of Kolmogoroff's backward differential equations (see e.g. Tijms [6]).

Now our approach can be summarized as follows:

(1) Compute (exactly) the long-term system uneffectiveness U_∞ using a continuous-time Markov chain analysis.

(2) Compute the first two moments of τ_H and τ_L, which are respectively defined as a period of time over which the system operates at the capacity level and a period of time over which the system operates below capacity.

(3) Fit computationally tractable distributions to the first two moments of τ_H and τ_L. For this, gamma distributions are used.

(4) Approximate the k-out-of-n multistate system by an $(\alpha, 1)$ single component, that is, a single unit which can operate at two levels: 100α% and 100% of the capacity. Here α is chosen in such a way that the long-term uneffectiveness of the approximating unit equals U_∞ as computed in step 1.

(5) Compute the (stationary) interval uneffectiveness distribution of the $(\alpha, 1)$ single component using the results from Takács [5].

2. Detailed method description

In this section, the approximation for the stationary interval uneffectiveness distribution of a k-out-of-n multistate system is elaborated further. In the analysis it is crucial that the unit lifetimes and repair times have tractable phase-type distributions. Therefore we first describe the particular class of phase-type distributions we shall use. A random variable X is said to have a second-order

Coxian distribution when it can be represented as (cf. Cox [3] and Yao and Buzacott [8])

$$X = \begin{cases} X_1 & \text{with probability } b, \\ X_1 + X_2 & \text{with probability } (1-b), \end{cases} \qquad (1)$$

where X_1 and X_2 are independent, exponentially distributed random variables with respective means $1/\lambda_1$ and $1/\lambda_2$. In other words, X passes first through an exponential phase X_1 and next, with probability $(1-b)$, through the second exponential phase X_2. It holds that $c_x^2 \geqslant 0.5$, where the coefficient of variation c_X is defined as the ratio of the standard deviation $\sigma(X)$ and the mean $E(X)$. Now two cases can be distinguished:

(i) $\lambda_1 = \lambda_2 = \lambda$. In this case, the density $f(x)$ of X is a mixture of an exponential and an Erlang-2 density with the same scale parameters, i.e.

$$f(x) = b\lambda\, e^{-\lambda x} + (1-b)\lambda^2 x\, e^{-\lambda x}, \quad x \geqslant 0. \qquad (2)$$

For this density it holds that $0.5 \leqslant c_X^2 \leqslant 1$.

(ii) $\lambda_1 \neq \lambda_2$, where it is no restriction to take $\lambda_1 > \lambda_2$. In this case, the density of X is given by

$$f(x) = p\lambda_1\, e^{-\lambda_1 x} + (1-p)\lambda_2\, e^{-\lambda_2 x}, \quad x \geqslant 0, \qquad (3)$$

where p is related to b by $b = p + (1-p)\lambda_2/\lambda_1$.

The fact that both the unit lifetimes and repair times are constructed from two exponentially distributed random variables simplifies the analysis considerably, because use can be made of continuous-time Markov chain tools. A detailed description of the approximate method is now given, via the five steps mentioned in the introduction.

Step 1. Computation of the long-term uneffectiveness U_∞

Let us define the microstate of a system as $I = (i_1, i_2, j_1, j_2, l)$, where

i_m = number of units in the m-th phase of the lifetime, $m = 1, 2$;
j_m = number of units in the m-th phase of the repair time, $m = 1, 2$;
l = number of units put on cold standby.

In fact, a three-dimensional state space (i_1, j_1, j_2) suffices for the calculations. If j_1, j_2 and i_1 are known, then, noting that at most k units are operating simultaneously, it is clear that $i_2 = \min[n - j_1 - j_2 - i_1, k - i_1]$ and further that $l = n - i_1 - i_2 - j_1 - j_2$. However, we shall use the five-dimensional micro-state definition for ease of presentation.

Because the sojourn times in the microstate I are distributed exponentially, we can compute by continuous Markov chain analysis the probabilities:

π_I = the steady-state probability that the system is in state I.

Using the standard reasoning that the rate out of the microstate I equals the rate into that state, we have

$$\sum_{J \in \Omega} r_{JI}\pi_J = \pi_I \sum_{J \in \Omega} r_{IJ}, \quad I \in \Omega, \qquad (4)$$

where r_{IJ} = infinitesimal transition rate from I to J (i.e. $r_{IJ}\Delta t$ gives the probability that the system will move from state I to J in the next short time interval Δt) and the state space Ω is given by

$$\begin{aligned} \Omega = \{ I = (i_1, i_2, j_1, j_2, l) \mid \\ & 0 \leqslant i_1 \leqslant k,\ 0 \leqslant j_1 \leqslant n - i_1, \\ & 0 \leqslant j_2 \leqslant n - i_1 - j_1; \\ & i_2 = \min[n - j_1 - j_2 - i_1, k - i_1], \\ & l = n - i_1 - i_2 - j_1 - j_2 \}. \end{aligned}$$

A specification of the transition rates r_{IJ} can be found in Appendix I. Now the steady-state probabilities π_I, including the normalization constraint $\Sigma\pi_I = 1$, can be computed from the system of linear equations (4). To give an idea of the computational effort, the size of the system (4) varies from 9 for a 1-out-of-2 system via 34 for a 3-out-of-4 system to 83 for a 5-out-of-6 system.

From the steady state probabilities, the long-term uneffectiveness U_∞ can easily be obtained. When it is noted that the operation level of the system in the state $I = (i_1, i_2, j_1, j_2, l)$ equals $100(i_1 + i_2)/k\%$, it is clear that

$$U_\infty = \sum_{I \in \Omega} \frac{k - i_1 - i_2}{k}\pi_I. \qquad (5)$$

Step 2. Computation of the moments of τ_H and τ_L

Recall that τ_H and τ_L are respectively defined as the length of a period in which the system operates at capacity level and the length of a period in which the system operates below capacity. For the distributions of τ_H and τ_L, systems of linear differential equations can be obtained using the powerful technique of Kolmogoroff's backward differential equations. By standard arguments (see e.g. Tijms [6]),

$$Q'_I(t) = \sum_{J \in \Omega_H} r_{IJ} Q_J(t) - Q_I(t) \sum_{J \in \Omega} r_{IJ}, \quad i \in \Omega_H, \tag{6}$$

where

$Q_I(t)$ = $\Pr\{\tau_H > t \mid$ the system is in microstate I at time $0\}$;
$Q_I(t)$ = $dQ_I(t)/dt$;
Ω_H = set of all the microstates in which the system operates at capacity level $= \{I \in \Omega \mid i_1 + i_2 = k\}$.

The specification of r_{IJ} can be found in Table A2 in Appendix II. Now note that the i-th moment of τ_H conditional on the initial state I can be written as

$$m_I^{(i)} = E\left[\tau_H^i \mid \text{the system is in microstate } I \text{ at the initial epoch}\right] = \int_0^\infty i t^{i-1} Q_I(t)\, dt = -\int_0^\infty t^i Q'_I(t)\, dt.$$

Then, by multiplication of both sides of (6) by it^{i-1} and by integration over t, one obtains

$$-im_I^{(i-1)} = \sum_{J \in \Omega_H} r_{IJ} m_J^{(i)} - m_I^{(i)} \sum_{J \in \Omega} r_{IJ}, \quad i \in \Omega_H. \tag{7}$$

Realizing that $m_I^{(0)} = 1$ $(i \in \Omega_H)$, it is clear that the j-th moment $m_I^{(j)}$ can be obtained by solving the system of linear equations (7) successively for $i = 1, 2, \dots, j$.

However, we need the moments of an arbitrary sojourn time at the 100% operational level, independent of the initial situation. Therefore, we take a weighted average of the $m_I^{(i)}$ over all $I \in \Omega_H$ with weights π_I^H defined as

π_I^H = probability that the system will enter Ω_H in state I, given it enters Ω_H.

Let us define Ω_L as the set of all microstates in which the system operates below capacity level, so that $\Omega_L = \{I \in \Omega \mid i_1 + i_2 < k\}$. Then some reflection shows that π_I^H equals the average number of transitions per unit time from Ω_L to the state $I \in \Omega_H$ divided by the average number of transitions per unit time from Ω_L to Ω_H. Hence

$$\pi_I^H = \left\{ \sum_{K \in \Omega_H} \sum_{J \in \Omega_L} \pi_J r_{JK} \right\}^{-1} \sum_{J \in \Omega_L} \pi_J r_{JI}, \quad I \in \Omega_H. \tag{8}$$

A further elaboration of these weighting factors can be found in Appendix III. Now the moments of τ_H can be computed from

$$E\left[\tau_H^i\right] = \sum_{i \in \Omega_H} \pi_I^H m_I^{(i)}. \tag{9}$$

By arguments similar to those above, formulae for the moments of τ_L are obtained. The equivalents of (7), (8), and (9) are, respectively

$$-in_I^{(i-1)} = \sum_{J \in \Omega_L} r_{IJ} n_J^{(i)} - n_I^{(i)} \sum_{J \in \Omega} r_{IJ}, \quad I \in \Omega_L, \tag{10}$$

where $n_I^{(i)} = E[\tau_L^i \mid$ the system is in microstate I at the initial epoch];

$$\pi_I^L = \left\{ \sum_{K \in \Omega_L} \sum_{J \in \Omega_H} \pi_J r_{JK} \right\}^{-1} \sum_{J \in \Omega_H} \pi_J r_{JI}, \quad I \in \Omega_L, \tag{11}$$

where

π_I^L = probability that the system will enter Ω_L in state I, given it enters Ω_L;

$$E\left[\tau_L^i\right] = \sum_{i \in \Omega_L} \pi_I^L n_I^{(i)}. \tag{12}$$

A further elaboration of the transition rates involved in (10) and (11) can be found in Appendix II, Table A3, and in Appendix III respectively.

Step 3. A two-moment approximation for the distributions of τ_L and τ_H

As will be seen in Step 5 subsequently, the formula for the computation of the interval uneffectiveness distribution is rather complex, involving many convolutions. That is, we need the distributions of sums of independent τ_H periods as well as the distributions of sums of independent τ_L periods. Because the sums of independent, identically distributed gamma variables are again gamma distributed, we approximate the distributions of τ_H and τ_L by gamma distributions. A further argument for this is that, in many stochastic systems, only the first two moments of the underlying probability distributions are relevant for the computation of performance measures; see for much empirical evidence Tijms [6]. So we approximate the density $f_S(x)$ of τ_H by

$$\hat{f}_S(x) = \left(\lambda_S^r x^{r-1} \mathrm{e}^{-\lambda_S x}\right)/\Gamma(r) \tag{13}$$

where λ_S and r are chosen in such a way that the first two moments of $f_S(x)$ and $\hat{f}_S(x)$ coincide. After some elementary algebraic manipulations, one obtains

$$r = E^2[\tau_H]/\left\{E\left[\tau_H^2\right] - E^2[\tau_H]\right\},$$
$$\lambda_S = E[\tau_H]/\left\{E\left[\tau_H^2\right] - E^2[\tau_H]\right\}. \tag{14}$$

Similarly, the density $g_S(x)$ of τ_L is approximated by a gamma density $\hat{g}_S(x)$.

Step 4. Computation of the low operational level for the approximating (α, 1) single component

The long-term uneffectiveness of the approximating (α, 1) single component is given by

$$U_\infty^A = (1-\alpha)E[\tau_L]/\{E[\tau_L] + E[\tau_H]\}. \tag{15}$$

Once the long-term uneffectiveness of the k-out-of-n multistate system is computed (Step 1), the low operational level α is easily obtained by matching U_∞ and U_∞^A, yielding

$$\alpha = (1 - U_\infty) - U_\infty E[\tau_H]/E[\tau_L]. \tag{16}$$

This operational level is also the average operational level of the k-out-of-n multistate system whenever the system operates below capacity. It turns out that in general α is somewhat smaller than $(k-1)/k$ due to the (short) time periods in which more than $(n-k+1)$ units are under repair simultaneously and the system operational level falls below $100(k-1)/k\%$ of the capacity.

Step 5. Computation of the interval uneffectiveness distribution

Suppose we have an alternating renewal process, consisting of independent 'on' and 'off' times of a single component. Define $\Psi(x;\, t_0)$ as

$\Psi(x;\, t_0) =$ probability that the cumulative 'on' time of the single component does not exceed x in a given time interval of length t_0, when the process has reached statistical equilibrium.

Assume that the 'on' and 'off' times have distribution functions $F(x)$ and $G(x)$ with means μ_{on} and μ_{off} respectively. Then, Theorem 1 in Takács [5] can easily be extended to:

$$\begin{aligned}\Psi(x;\, t_0) = {} & \frac{\mu_{on}}{\mu_{on}+\mu_{off}} \\ & \times \sum_{i=0}^{\infty} \left\{G^{(i)}(t_0 - x) - G^{(i+1)}(t_0 - x)\right\} \\ & \cdot F^{\mathrm{RES}} * F^{(i)}(x) \\ & + \frac{\mu_{off}}{\mu_{on}+\mu_{off}} \\ & \times \sum_{i=0}^{\infty} \left\{G^{\mathrm{RES}} * G^{(i)}(t_0 - x) \right. \\ & \qquad \left. - G^{\mathrm{RES}} * G^{(i+1)}(t_0 - x)\right\} \\ & \qquad \cdot F^{(i+1)}(x) \\ & + \frac{\mu_{off}}{\mu_{on}+\mu_{off}} \left\{1 - G^{\mathrm{RES}}(t_0 - x)\right\}, \\ x < t_0, & \end{aligned} \tag{17}$$

where

$F^{\mathrm{RES}}(x) = \mu_{on}^{-1}\int_0^x\{1 - F(y)\}\,\mathrm{d}y$ (the stationary residual 'on' time distribution); similarly $G^{\mathrm{RES}}(x)$, with the appropriate substitutions, represents the 'off' time equivalent;

$F^{(i)}(x) =$ the i-fold convolution of $F(x)$, and similarly $G^{(i)}(x)$;

$F^{\mathrm{RES}} * F^{(i)}(x) =$ the convolution of $F^{\mathrm{RES}}(x)$ and $F^{(i)}(x)$, and similarly $G^{\mathrm{RES}} * G^{(i)}(x)$.

Note that the convolution $A * B(x)$ of two probability distribution functions $A(x)$ and $B(x)$, $x \geqslant 0$, is given by

$$A * B(x) = \int_0^x A(x-y)\,\mathrm{d}B(y).$$

Further, it is clear that $\Psi(t_0,\, t_0) = 1$ and so $\Psi(x;\, t_0)$ has a jump of $(\mu_{on} + \mu_{off})^{-1}\mu_{on}\{1 - F^{\mathrm{RES}}(t_0)\}$ at $x = t_0$. Now define for the k-out-of-n multistate system the random variable

$U(t_0) =$ the proportion of system capacity that cannot be used in a given interval of length t_0, assuming that the system has reached statistical equilibrium.

Then, assuming that the low operational level and the high operational level periods are independent, some reflection shows that the approximation by an $(\alpha, 1)$ single component yields for the interval uneffectiveness distribution

$$\Pr\{U(t_0) < x\} \cong 1 - \Psi(t_0 - t_0 x/(1-\alpha);\ t_0) \tag{18}$$

where μ_{on}, μ_{off}, $F(x)$ and $G(x)$ are replaced by $E[\tau_H]$, $E[\tau_L]$, $\hat{F}_S(x)$ and $\hat{G}_S(x)$ respectively.

Although the sojourn times in the high operational level and the low operational level states are not independent, the assumption does not appear to affect the approximation significantly. Further, it is noted that the approximation requires a slight modification, which is discussed in Appendix IV.

3. Validation and sensitivity analysis

In the previous section, we have given a method for approximating the interval uneffectiveness distribution for a k-out-of-n multistate system when the lifetimes and repair times have a Coxian distribution with two phases. However, other probability distributions (e.g. the Weibull distribution for the lifetimes and the log-normal distribution for the repair times) are encountered in practical applications. Nevertheless, we can still apply the methodology of the previous section if we fit approximate phase-type distributions to these distributions by matching the first few moments. Suppose we have a general distribution whose i-th moment is given by ν_i and squared coefficient of variation of c^2. Here it is assumed that $c^2 \geqslant 0.5$ (but see also the end of this section for some comments for the case $c^2 < 0.5$). We distinguish between two cases:

(i) $0.5 \leqslant c^2 \leqslant 1$. Use a mixture of an exponential and an Erlang-2 distribution with the same scale parameter. So we take the density (2) with the same mean ν_i and squared coefficient of variation c^2 as the original distribution, yielding (see Tijms [6]):

$$b = \{1 + c^2\}^{-1}\left\{2c^2 - \sqrt{2(1-c^2)}\right\}, \qquad \lambda = (2-b)/\nu_1. \tag{19}$$

(ii) $c^2 > 1$. Use a hyperexponential distribution of order two, i.e. the density (3) with $0 < p < 1$. We choose the parameters p, λ_1 and λ_2 in such a way that the first three moments of the approximating hyperexponential distribution equal ν_1, ν_2 and ν_3 respectively. Thus

$$\lambda_{1,2} = \left\{\gamma + 1.5\,\beta^2 + 3\nu_1\beta_1^2 \pm \sqrt{\left(\gamma + 1.5\,\beta^2 - 3\nu_1\beta\right)^2 + 18\nu_1^2\beta^3}\right\}^{-1} \times 6\nu_1\beta, \tag{20}$$

$$p = \{\lambda_2 - \lambda_1\}^{-1}\lambda_1(\nu_1\lambda_2 - 1),$$

where $\beta = \nu_2 - 2\nu_1^2$ and $\gamma = \nu_1\nu_3 - 1.5\,\nu_2^2$, see Whitt [7]. This match is only possible if $\gamma \geqslant 0$ (as was found to be the case for the Weibull, log-normal and gamma distributions). In case $\gamma < 0$, it is suggested that p, λ_1 and λ_2 should be chosen such that the first two moments of (3) equal ν_1 and ν_2 and the third moment is as close as possible to ν_3.

To validate our method, we have compared the approximate results to results from Monte-Carlo simulation. It is noted that we have used the Gauss–Seidel method to solve the system of linear equations (4), where we have ordered the states (i_1, j_1, j_2) first to the number of units (i_1) in the first phase of the lifetime, then to the number of units (j_1) in the first phase of the repair time and finally to the number of units (j_2) in the second phase of the repair time. The Gauss–Seidel method is very well suited to solving the linear equations corresponding to a Markov chain (cf. also Tijms [6]). In our model, the method worked in all examples tested, while other methods failed in some instances.

For the validation, we have chosen three system types, namely the 1-out-of-2, the 2-out-of-3 and the 5-out-of-6 multistate systems. For the unit lifetime and repair time distributions, three combinations are considered:

(i) the unit lifetimes and repair times are both exponentially distributed (in the tables this is denoted by M/M).

(ii) a Weibull distribution for the lifetimes with squared coefficient of variation $c_L^2 = 0.75$ and a log-normal distribution for the repair times with $c_R^2 = 2$ (denoted by $W_{0.75}/L_2$).

(iii) the lifetimes and repair times are respectively Weibull-distributed with $c_L^2 = 0.50$ and log-normal-distributed with $c_R^2 = 2$ (denoted by $W_{0.50}/L_2$).

Note that a Weibull-distributed lifetime with

Table 1
Simulated and approximate results for a 1-out-of-2 system

Model		$E[\tau_H]$	$c^2_{\tau_H}$	$E[\tau_L]$	$c^2_{\tau_L}$	$100U_\infty\%$	P_0	P_2	P_5	P_{10}	t_0
M/M	sim	10.0	1.16	0.056	1.00	0.55	0.91	0.94	0.96	0.98	1
							0.40	0.93	1.00	1.00	10
	app	10.0	1.18	0.056	1.00	0.55	0.91	0.93	0.96	0.98	1
							0.40	0.94	1.00	1.00	10
$W_{0.75}/L_2$	sim	13.5	1.14	0.057	1.34	0.42	0.93	0.95	0.97	0.99	1
							0.51	0.95	1.00	1.00	10
	app	12.6	1.19	0.056	1.27	0.44	0.93	0.95	0.97	0.99	1
							0.48	0.95	1.00	1.00	10
$W_{0.5}/L_2$	sim	21.9	1.11	0.059	1.20	0.28	0.96	0.97	0.98	0.99	1
							0.63	0.97	1.00	1.00	10
	app	27.6	1.16	0.061	1.36	0.22	0.96	0.98	0.99	0.99	1
							0.71	0.98	1.00	1.00	10

Table 2
Simulated and approximate results for a 2-out-of-3 system

Model		$E[\tau_H]$	$c^2_{\tau_H}$	$E[\tau_L]$	$c^2_{\tau_L}$	$100U_\infty\%$	P_0	P_2	P_5	P_{10}	t_0
M/M	sim	2.72	1.26	0.057	1.07	1.07	0.95	0.95	0.96	0.97	0.1
							0.71	0.83	0.92	0.98	1
	app	2.75	1.30	0.058	1.05	1.06	0.95	0.95	0.96	0.97	0.1
							0.71	0.83	0.93	0.98	1
$W_{0.75}/L_2$	sim	3.11	1.19	0.058	1.31	0.98	0.95	0.95	0.96	0.97	0.1
							0.73	0.85	0.94	0.98	1
	ap	3.01	1.26	0.058	1.31	0.97	0.95	0.96	0.96	0.97	0.1
							0.73	0.85	0.93	0.98	1
$W_{0.5}/L_2$	sim	3.61	1.09	0.057	1.41	0.85	0.95	0.96	0.96	0.97	0.1
							0.76	0.88	0.95	0.98	1
	app	3.72	1.11	0.057	1.41	0.79	0.96	0.96	0.97	0.98	0.1
							0.76	0.88	0.95	0.99	1

Table 3
Simulated and approximate results for a 5-out-of-6 system

Model		$E[\tau_H]$	$c^2_{\tau_H}$	$E[\tau_L]$	$c^2_{\tau_L}$	$100U_\infty\%$	P_0	P_2	P_5	P_{10}	t_0
M/M	sim	0.56	1.44	0.065	1.18	2.38	0.77	0.81	0.85	0.91	0.1
							0.21	0.58	0.85	0.97	1
	app	0.56	1.46	0.065	1.18	2.38	0.77	0.80	0.84	0.90	0.1
							0.21	0.55	0.85	0.98	1
$W_{0.75}/L_2$	sim	0.61	1.32	0.063	1.50	2.25	0.77	0.80	0.85	0.91	0.1
							0.22	0.63	0.86	0.97	1
	app	0.57	1.38	0.064	1.48	2.32	0.77	0.82	0.86	0.90	0.1
							0.21	0.56	0.85	0.98	1
$W_{0.5}/L_2$	sim	0.62	1.16	0.062	1.47	2.23	0.77	0.80	0.86	0.91	0.1
							0.19	0.62	0.87	0.97	1
	app	0.59	1.19	0.064	1.48	2.22	0.78	0.82	0.86	0.91	0.1
							0.19	0.59	0.86	0.99	1

$c_L^2 \leqslant 1$ has a nondecreasing failure rate. Further, note that for a repair time with $c_R^2 > 1$, the lognormal distribution is often appropriate when the actual repair times can vary from small values to large values. In all cases we have taken the mean unit lifetime as 1 and the mean unit repair time as 1/9, resulting in a unit availability $\mu_L/(\mu_L + \mu_R)$ of 0.9. The length of the stationary time interval t_0 is varied as 1 and 10 times μ_L for the 1-out-of-2 system while $t_0 = 0.1$ or 1 for the other two system types.

In Tables 1–3, a comparison between simulation and approximation is made for the three system types as mentioned. Values for the mean and squared coefficients of variation of both τ_H and τ_L are presented as well as values for $100U_\infty\%$, and some points of the interval uneffectiveness distribution where $P_x = \Pr\{U(t_0) \leqslant x/100\}$. The half lengths of the 95% confidence intervals do not exceed 3% of the simulated values for $E[\tau_H]$ and $E[\tau_L\}$ respectively and 10% of the simulated value for U_∞ and the squared coefficients of variation of τ_H and τ_L. For the simulated probabilities, the half length of the 95% confidence interval never exceeds 0.01 and is about 0.002 for the probabilities very close to 1.

From Tables 1–3 it is concluded that the approximation has a satisfactorily accurate performance. The accuracy of the approximation decreases as c_L^2 deviates more from 1, probably in consequence of a larger deviation between the Weibull distribution and the approximating phase-type distribution. Further, it is clear that the sensitivity of the interval uneffectiveness distribution to the distributional form of the lifetimes and repair times decreases as the size of the system increases.

To study this sensitivity in more detail, we have computed the long-term uneffectiveness U_∞ for the three system types as presented in Tables 1–3, where we have varied the squared coefficients of variation of the (Coxian distributed) unit lifetimes and repair times. A graphical representation of this is shown in Figures 1–3. To compare the sensitivity of U_∞ to c_R^2 and c_L^2 for the three system types, we have chosen the unit availability for each system type such that the long-term uneffectiveness U_∞ for the case of exponential unit lifetimes and repair times ($c_L^2 = c_R^2 = 1$) equals 0.01. This implies unit availabilities of 0.8676, 0.9029 and 0.9358 for the 1-out-of-2, the 2-out-of-3 and the 5-out-of-6 system respectively. From Figures 1–3, it is clear that the sensitivity of U_∞ to c_L^2 and c_R^2 decreases as the size of the system increases. Further, we see that the influence of c_R^2 on U_∞ is much smaller than that of c_L^2. If the unit lifetimes are exponentially distributed, the long-term uneffectiveness is even insensitive to c_R^2 (which is a known insensitivity result for stochastic networks, see e.g. exercise 2.27 in Tijms [6]). So it is concluded that large k-out-of-n multistate systems (e.g. a 5-out-of-6 system) may be analysed assuming an exponential distribution for the unit repair times, while for very large systems an exponential distribution for the unit lifetimes as well may suffice. For smaller systems, more informa-

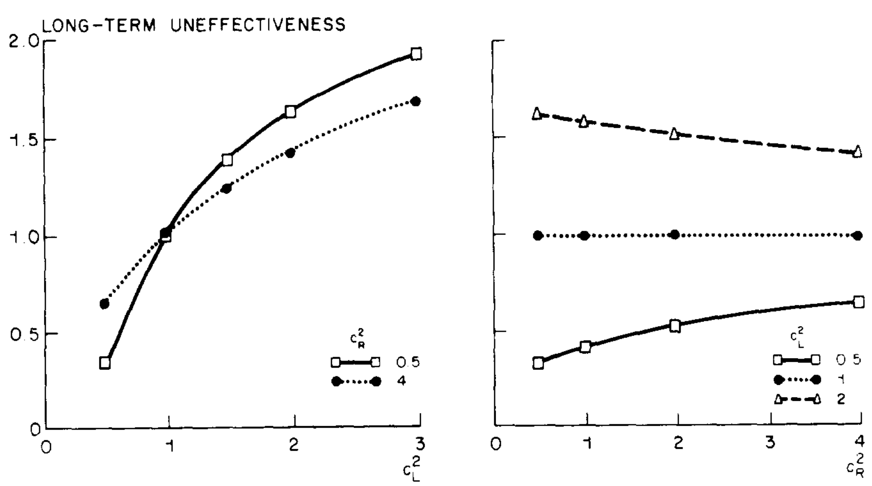

Figure 1. The effect of the distributional form of the unit lifetimes and repair times for a 1-out-of-2 system. Unit availability = 0.8676

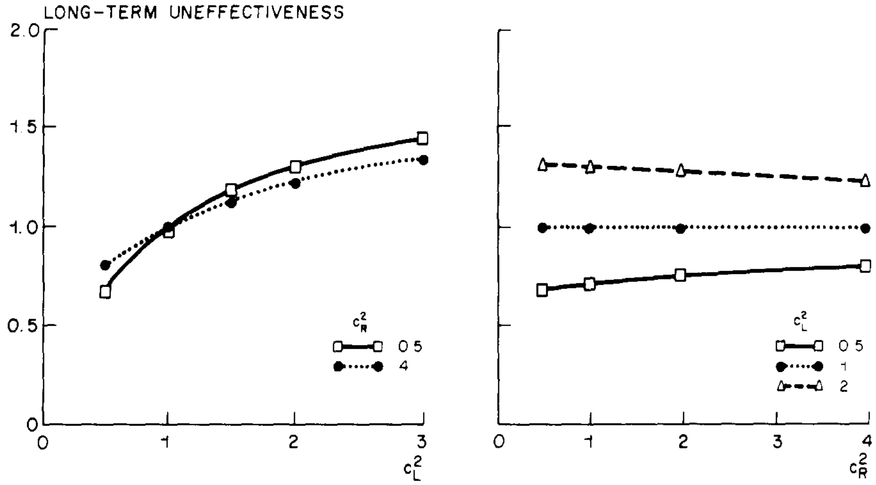

Figure 2. The effect of the distributional form of the unit lifetimes and repair times for a 2-out-of-3 system. Unit availability = 0.9029

tion about the unit lifetime and repair time distributions than just the mean should be used. For the 1-out-of-2 system, an approximation for the interval uneffectiveness distribution using general lifetime and repair time distributions is available (cf. van der Heijden [4]). However, for medium-size *k*-out-of-*n* multistate systems such as the 2-out-of-3 system, the shapes of the unit lifetime and repair time curves have a significant influence on the interval uneffectiveness distribution (and especially on the mean U_∞). For these systems, the approximate method as presented in this paper can be used whenever $c_L^2 \geq 0.5$ and $c_R^2 \geq 0.5$. In case $c_R^2 < 0.5$, we suggest that some extrapolation method should be used, because sensitivity analysis shows that the influence of c_R^2 on the results is only slight. Such an extrapolation method will solve the model for some values of c_R^2 larger than 0.5 and also extrapolate the results to the particular $c_R^2 < 0.5$. When $c_L^2 < 0.5$, some extrapolation method can also be used, although one should then be careful because of greater sensitivity of the results to c_L^2. In this case, the use of a phase-type distribution with more than two phases can be considered as an alternative.

It is remarked that in general a squared coefficient of variation $1/i$ can be reached when a phase-type distribution with i exponential phases is used (see Cox [3] and Whitt [7]). Although an increasing number of phases induces a larger sys-

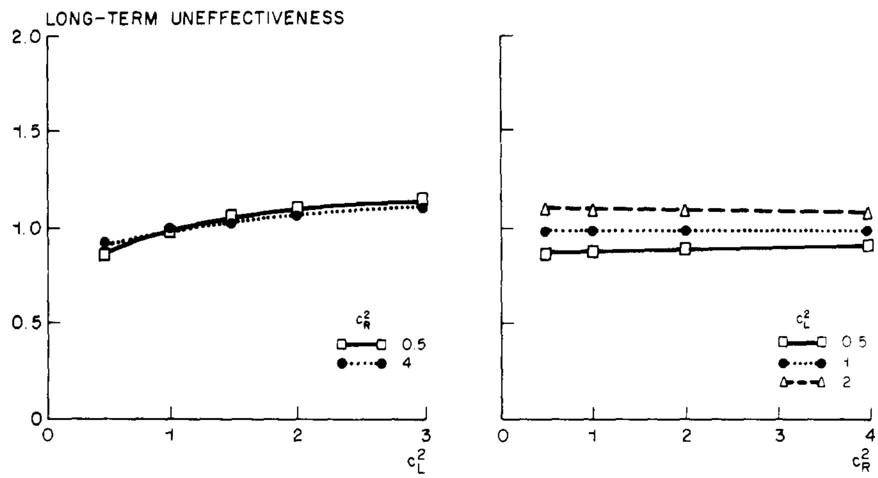

Figure 3. The effect of the distributional form of the unit lifetimes and repair times for a 5-out-of-6 system. Unit availability = 0.9358

tem of linear equations for the computation of U_∞, this is not restrictive because the use of more phases is only required for medium-size systems. For large systems, the unit lifetimes and certainly the repair time distributions can be taken as being exponential, implying relatively small systems of linear equations to be solved.

Appendix I. The transition rates in the continuous Markov chain

For the computation of the steady-state probabilities π_I, the transition rates r_{JI} are required. In Table A1 we give, for an arbitrary microstate $I = (i_1, i_2, j_1, j_2, l)$, all microstates J with $r_{JI} > 0$, together with the conditions under which the particular states J exist. These conditions are useful for implementation in a computer program. Further, note that the transition rate out of a certain state $I = (i_1, i_2, j_1, j_2, l)$, which is also required in (4), simply equals

$$\sum_{J \in \Omega} r_{IJ} = i_1\lambda_1 + i_2\lambda_2 + j_1\mu_1 + j_2\mu_2 \tag{A1}$$

where we use the parameters $(b, \lambda_1, \lambda_2)$ and (q, μ_1, μ_2) for the stochastic two-phase process of the unit lifetime and repair time respectively (see Section 2 for a definition of the parameters).

Appendix II. The transition rates for the computation of $m_I^{(i)}$ and $n_I^{(i)}$

To compute the moments $m_I^{(i)}$, the transition rates r_{IJ} for each pair of microstates $I, J \in \Omega_H$ are required. In Table A2, the transition rates r_{IJ}

Table A2
Transition rates for the computation of $m_I^{(i)}$

$J \in \Omega_H$	Condition	r_{IJ}
$(i_1-1, i_2+1, j_1, j_2, l)$	$(i_1 > 0)$	$(1-b)i_1\lambda_1$
$(i_1, i_2, j_1+1, j_2, l-1)$	$(i_1 > 0)$ $(j_1 + j_2 + 1 \leqslant n-k)$	$bi_1\lambda_1$
$(i_1+1, i_2-1, j_1+1, j_2, l-1)$	$(i_2 > 0)$ $(j_1 + j_2 + 1 \leqslant n-k)$	$i_2\lambda_2$
$(i_1, i_2, j_1-1, j_2+1, l)$	$(j_1 > 0)$	$(1-q)j_1\mu_1$
$(i_1, i_2, j_1-1, j_2, l+1)$	$(j_1 > 0)$	$qj_1\mu_1$
$(i_1, i_2, j_1, j_2-1, l+1)$	$(j_2 > 0)$	$j_2\mu_2$

Table A3
Transition rates for the computation of $n_I^{(i)}$

$J \in \Omega_L$	Condition	r_{IJ}
$(i_1-1, i_2+1, j_1, j_2, l)$	$(i_1 > 0)$	$(1-b)i_1\lambda_1$
$(i_1-1, i_2, j_1+1, j_2, l)$	$(i_1 > 0)$	$bi_1\lambda_1$
$(i_1, i_2-1, j_1+1, j_2, l)$	$(i_2 > 0)$	$i_2\lambda_2$
$(i_1, i_2, j_1-1, j_2+1, l)$	$(j_1 > 0)$	$(1-q)j_1\mu_1$
$(i_1+1, i_2, j_1-1, j_2, l)$	$(j_1 > 0)$ $(i_1 + i_2 + 1 < k)$	$qj_1\mu_1$
$(i_1+1, i_2, j_1, j_2-1, l)$	$(j_2 > 0)$ $(i_1 + i_2 + 1 < k)$	$j_2\mu_2$

for an arbitrary $I = (i_1, i_2, j_1, j_2, l) \in \Omega_H$ are given, the data presentation being similar to that for Table A1. The same is done in Table A3 for the transition rates r_{IJ}, with $I, J \in \Omega_L$, for the computation of $n_I^{(i)}$.

Appendix III. The weighting factors π_I^H and π_I^L

For the computation of π_I^H, the transition rates r_{JI} with $J \in \Omega_L$ are required. The weighting factor π_I^H for some $I = (i_1, i_2, j_1, j_2, l)$ is only positive

Table A1
Specification of the transition rates in the continuous Markov chain

Microstate J	Conditions	r_{JI}
$(i_1+1, i_2-1, j_1, j_2, l)$	$(i_2 > 0)$	$(1-b)(i_1+1)\lambda_1$
$(i_1+1, i_2, j_1-1, j_2, l)$	$(i_1 + i_2 < k) \wedge (j_1 > 0)$	$b(i_1+1)\lambda_1$
$(i_1, i_2, j_1-1, j_2, l+1)$	$(i_1 + i_2 = k) \wedge (j_1 > 0)$	$bi_1\lambda_1$
$(i_1, i_2+1, j_1-1, j_2, l)$	$(i_1 + i_2 < k) \wedge (j_1 > 0)$	$(i_2+1)\lambda_2$
$(i_1-1, i_2+1, j_1-1, j_2, l+1)$	$(i_1 + i_2 = k) \wedge (i_1 > 0) \cup (j_i > 0)$	$(i_2+1)\lambda_2$
$(i_1, i_2, j_1+1, j_2-1, l)$	$(j_2 > 0)$	$(1-q)(j_1+1)\mu_1$
$(i_1-1, i_2, j_1+1, j_2, l)$	$(n - j_1 - j_2 - 1 < k) \wedge (i_1 > 0)$	$q(j_1+1)\mu_1$
$(i_1, i_2, j_1+1, j_2, l-1)$	$(n - j_1 - j_2 - 1 \geqslant k) \wedge (l > 0)$	$q(j_1+1)\mu_1$
$(i_1-1, i_2, j_1, j_2+1, l)$	$(n - j_1 - j_2 - 1 < k) \wedge (i_1 > 0)$	$(j_2+1)\mu_2$
$(i_1, i_2, j_1, j_2+1, l-1)$	$(n - j_1 - j_2 - 1 \geqslant k) \wedge (l > 0)$	$(j_2+1)\mu_2$

if $(i_1 > 0)$ and $(j_1 + j_2 = n - k)$. In this case, state I can be reached from two states $J \in \Omega_L$, namely from $(i_1 - 1, i_2, j_1 + 1, j_2, l)$ with rate $q(j_1 + 1)\mu_1$ and from $(i_1 - 1, i_2, j_1, j_2 + 1, l)$ with rate $(j_2 + 1)\mu_2$.

Similarly, the weighting factor π_I^L for some $I = (i_1, i_2, j_1, j_2, l)$ is only possible if $(j_1 > 0)$ and $(j_1 + j_2 = n - k + 1)$. Then, state I can be reached from two states $J \in \Omega_H$, namely from $(i_1 + 1, i_2, j_1 - 1, j_2, l)$ with rate $b(i_1 + 1)\lambda_1$ and from $(i, i_2 + 1, j_1 - 1, j_2, l)$ with rate $(i_2 + 1)\lambda_2$.

Appendix IV. Approximating the interval uneffectiveness distribution

For the approximation of the interval uneffectiveness distribution, formula (17) has to be evaluated, where $F(x)$ and $G(x)$ are replaced by gamma distribution functions $\hat{F}(x)$ and $\hat{G}(x)$ having the same first two moments as $F(x)$ and $G(x)$. Since the convolution of gamma distributions with the same scale parameters is again a gamma distribution, both convolutions $\hat{F}^{(i)}(x)$ and $\hat{G}^{(i)}(x)$ are easy to evaluate numerically by using widely available codes for computing the incomplete gamma function.

If $\hat{F}^{\text{RES}}(x)$ and $\hat{G}^{\text{RES}}(x)$ denote gamma distribution functions having the same first two moments as $F^{\text{RES}}(x)$ and $G^{\text{RES}}(x)$ respectively, numerical difficulties arise when computing the convolutions $\hat{F}^{\text{RES}} * \hat{F}^{(i)}(x)$ and $\hat{G}^{\text{RES}} * \hat{G}^{(i)}(x)$. The reason is that the latter convolutions are not convolutions of gamma distributions with the same scale parameters and hence are not given by the easily computed gamma functions. Thus, for computational reasons, some further approximation is required.

Step 1

Fit gamma distributions $\tilde{F}^{\text{RES}}(x)$ and $\tilde{G}^{\text{RES}}(x)$ to $F^{\text{RES}}(x)$ and $G^{\text{RES}}(x)$ by matching only the first moment and by taking care that $\tilde{F}^{\text{RES}}(x)$ and $\hat{G}^{\text{RES}}(x)$ have the same scale parameters as $\hat{F}(x)$ and $\hat{G}(x)$ respectively. Then the convolutions $\tilde{F}^{\text{RES}} * \hat{F}^{(i)}(x)$ and $\hat{G}^{\text{RES}} * \hat{G}^{(i)}(x)$ are easily computed gamma distribution functions. In this way, we obtain a 'first-order' approximation $\tilde{\Psi}(x;\, t_0)$ to the interval uneffectiveness distribution. However, using only a first moment fit, we create inaccuracies that are most significant in the 'first-order' approximation $\tilde{U}_0$ and $\tilde{U}_1$ to the boundary points $U_0 = \Pr\{U(t_0) = 0\}$ and $U_1 = \Pr\{U(t_0) = 1 - \alpha\}$. By (17) and (18), these boundary points are given by

$$U_0 = \frac{\mu_{\text{on}}}{\mu_{\text{on}} + \mu_{\text{off}}}\left\{1 - F^{\text{RES}}(t_0)\right\},$$
$$U_1 = \frac{\mu_{\text{off}}}{\mu_{\text{on}} + \mu_{\text{off}}}\left\{1 - G^{\text{RES}}(t_0)\right\}. \quad \text{(A1)}$$

It is important to observe that these boundary points require no computation of convolutions. This observation provides a way to improve the 'first-order' approximation.

Step 2

Fit gamma distribution functions $\hat{F}^{\text{RES}}(x)$ and $\hat{G}^{\text{RES}}(x)$ to $F^{\text{RES}}(x)$ and $G^{\text{RES}}(x)$ by matching the first two moments and use these gamma distributions to compute improved approximations $\hat{U}_0$ and $\hat{U}_1$ from (A1).

Step 3

Correct the 'first order' approximation $\tilde{\Psi}(x;\, t_0)$ in such a way that the boundary probabilities agree with $\hat{U}_0$ and $\hat{U}_1$. Thus the corrected approximation becomes

$$\hat{\Psi}(x;\, t_0) = \tilde{\Psi}(x;\, t_0) \times \left\{ \frac{x(1 - \hat{U}_0)}{t_0(1 - \tilde{U}_0)} + \frac{(t_0 - x)}{t_0} P* \right\}, \quad \text{(A2)}$$

where

$$P* = \begin{cases} 1 & \text{if } \hat{U}_1 < 0.001 \text{ and } \tilde{U}_1 < 0.001, \\ \hat{U}_1/\tilde{U}_1 & \text{else.} \end{cases}$$

The latter correction is to avoid non-monotonic behaviour of $\hat{\Psi}$.

Note that the moments of $F^{\text{RES}}(x)$ and $G^{\text{RES}}(x)$, as required in Step 1 and Step 2, can easily be obtained from the moments of $F(x)$ and $G(x)$. Denoting by ν_i and ν_i^{RES} the i-th moment of $F(x)$ and $F^{\text{RES}}(x)$ respectively, some elementary algebraic manipulations show that

$$\nu_i^{\text{RES}} = \frac{\nu_{i+1}}{(i+1)\nu_1}. \quad \text{(A3)}$$

A similar relation applies between the moments of $G(x)$ and $G^{\text{RES}}(x)$.

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