



# Optimal linear–quadratic control of asymptotically stabilizable systems using approximations

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## ABSTRACT

In this paper we study approximations to the infinite-horizon quadratic optimal control problem for linear systems that may be only asymptotically stabilizable. For linear systems, this issue only arises with infinite-dimensional systems. We provide sufficient conditions which guarantee when approximations to the optimal feedback result in the cost converging to the optimal cost. One technique for approximate solution of the optimal control problem is to use Newton–Kleinman iterations for the associated Riccati equation. Some new results in this direction are provided. Several important classes of systems, lightly damped second-order systems and a platoon-type system, are shown to be optimizable. Also, finding an initial stabilizing control for the Newton–Kleinman iteration can be non-trivial. The initial iterate for these classes is described.

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## 1. Introduction

The linear–quadratic (LQ) optimal-control problem is one of the oldest control problems studied for state–space linear systems, and its solution can be found in many text books; for instance, in [1], where the problem is studied for systems on an infinite-dimensional state–space. The solution of this problem is very similar in theory to the finite-dimensional case. However, the calculation of the corresponding optimal feedback, which is through solution to an operator-valued Algebraic Riccati Equation (ARE), can almost never be done exactly. Hence, approximations are needed. A fairly complete theory exists for systems that are exponentially stabilizable; see for instance the book [2].

However, some infinite-dimensional systems are at best asymptotically stable but not exponentially stabilizable. This is one of the challenges of controlling infinite-dimensional systems. Lightly damped waves and some platoon systems are important classes of these systems. In [3] approximations to asymptotically stabilizable systems are studied. They showed that, under standard assumptions on the approximation scheme, the optimal cost

of the approximating systems converges to the true optimal cost, and also that the approximating optimal control converges to the true optimal feedback. The question of what happens when the approximate optimal feedback is applied to the original system is not answered. It is conjectured in [3] that the approximating control might not yield close to optimal cost, but an example or proof is missing.

In this paper we provide a counter-example where even though the approximation scheme satisfies the usual assumptions, and the approximating optimal controls converge, these approximating controls do not even stabilize the original system and also the cost does not converge. Thus, we show that the conjecture in [3] holds. Additional assumptions are needed for approximation of asymptotically stabilizable systems. We provide sufficient conditions under which the approximating controls do provide convergent and finite cost. We show that if a sequence of feedbacks converges to the optimal feedback and these feedbacks give uniformly bounded cost, then these feedbacks can be used to obtain a cost arbitrarily close to the optimal cost.

Newton–Kleinman iterations are one of the techniques for constructing a sequence of approximating stabilizing feedback laws. The initial iterate in this scheme has to be chosen so that the controlled system has finite cost. We show how this problem can be solved for two classes of systems: lightly damped second-order systems and a platoon-type system, and also that these classes of systems are well-posed.

This paper is structured as follows: After introducing the class of systems to be considered and notation in Section 2, we show

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in Section 3 that a natural approximation scheme may lead to feedback operators that do not even stabilize the original infinite-dimensional system. We then provide sufficient conditions for the approximate control to provide a convergent cost. In Section 4 the optimal feedback construction via Newton–Kleinman algorithm for asymptotically stabilizable infinite-dimensional systems is treated. We apply this approach to nontrivial asymptotically-stabilizable systems that are not exponentially stabilizable: lightly damped waves and vibrations (Section 4.1) and a vehicular platoon model (Section 4.2).

## 2. Asymptotically stabilizable systems

Consider a linear infinite-dimensional system  $\Sigma(A, B, C)$  on separable Hilbert spaces  $Z, U$ , and  $Y$

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0, \quad (1)$$

$$y(t) = Cz(t), \quad (2)$$

where  $B$  and  $C$  are bounded operators, that is,  $B \in \mathcal{L}(U, Z)$ ,  $C \in \mathcal{L}(Z, Y)$  and  $A$  with domain  $\mathcal{D}(A) \subset Z$  generates a  $C_0$ -semigroup on  $Z$ . Associated to this we consider the cost functional

$$J(z_0, u) = \int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 dt. \quad (3)$$

**Definition 1.** A  $C_0$ -semigroup  $T(t)$  on the Hilbert space  $Z$  is *asymptotically stable* if for all  $z \in Z$ ,  $T(t)z \rightarrow 0$  as  $t \rightarrow \infty$ .

**Definition 2.** If for every initial condition  $z_0 \in Z$  there exists an input  $u \in L_2([0, \infty); U)$  such that  $J(z_0, u) < \infty$ , then the system  $\Sigma(A, B, C)$  is *optimizable*.

Note that in [3] optimizable is called *output stabilizable* while the term *finite-cost-condition* is used in [4, Theorem 9.2.5].

If the system is optimizable, then the well-known associated (control) algebraic Riccati equation (ARE)

$$A^*Xz_0 + XAz_0 - XBB^*Xz_0 + C^*Cz_0 = 0, \quad z_0 \in \mathcal{D}(A), \quad (4)$$

has a minimal self-adjoint non-negative solution  $X$ . Furthermore, this solution defines the optimal cost and control:

$$\langle z_0, Xz_0 \rangle = \min_u J(z_0, u) = J(z_0, -B^*Xz),$$

where  $z$  is the solution of (1) with  $u(t) = -B^*Xz(t)$ , see e.g. [1, Thm. 6.2.4].

Often minimizing the cost is not the only requirement on the control input, but it should also stabilize the system as well. Here by stability we mean asymptotic stability. Sufficient conditions on the system  $\Sigma(A, B, C)$  such that the closed loop operator  $A - BB^*X$  generates an asymptotically stable semigroup are given in [3] and [5].

## 3. Approximation of LQ-optimal control

Calculating the solution of the ARE is normally done via an approximation scheme, i.e., a method that yields a sequence of approximations of  $X$ . Most schemes for approximation of partial differential equations are open loop. That means that, given the initial condition, an accurate reproduction of the trajectory is found under a known forcing term. An approximation scheme that is satisfactory in this respect may yield a controller that is not close to optimal, or worse, that does not even stabilize the system, when used in controller design. For examples of this, we refer to [6,7], and [2]. Stronger conditions than those sufficient for open-loop simulation are needed when a scheme is used for controller design. Sufficient conditions for an approximation of

an exponentially stabilizable system to be successfully used in controller synthesis are now well-established; see the book [2] for an overview.

In [3], it was shown that a specified set of conditions for approximation of an asymptotically stabilizable system is sufficient for convergence of solutions to the approximating AREs to that of the infinite-dimensional ARE. The following simple example shows that, as conjectured in [3, pg. 1923], these conditions are not sufficient to guarantee that the approximated optimal control stabilizes the original system for large approximation order.

**Example 3.** Take  $A = \text{diag}_{n \in \mathbb{Z}}(in)$  on  $Z = \ell^2(\mathbb{Z})$ , i.e.,  $A(\sum_{n \in \mathbb{Z}} \alpha_n e_n) = \sum_{n \in \mathbb{Z}} in \alpha_n e_n$ , where  $\{e_n, n \in \mathbb{Z}\}$  is the standard basis of  $Z$ . Also define

$$b = (\dots, b_{-2}; b_{-1}; b_0; b_1; \dots), \quad Bu = b \cdot u,$$

with  $b \in Z$  and  $u \in U = \mathbb{C}$ . Define  $C = B^*$ , and so  $Y = \mathbb{C}$ . We assume that  $b_n \neq 0$  for all  $n$ , and so  $\Sigma(A, B, C)$  is approximately controllable and observable [4, Theorem 6.3.6]. By inspection it is easy to see that the identity operator on  $Z$  is a non-negative solution of the ARE (4). It is also the unique non-negative solution. Defining  $F = B^*$ , the feedback  $u = -Fz$  yields an asymptotically stable system and also a finite cost.

As approximating systems, we use the familiar Galerkin approximation with the eigenfunctions as a basis,

$$A_N = \text{diag}_{|n| \leq N}(in), \quad B_N = (b_{-N}; \dots; b_{-1}; b_0; b_1; \dots; b_N), \quad C_N = B_N^*.$$

The unique non-negative solution of the ARE of the system  $\Sigma(A_N, B_N, C_N)$  equals  $I_{2N+1}$ , the identity on  $\mathbb{C}^{2N+1}$ . The approximating optimal feedback is

$$F_N = -B_N^* \pi_N,$$

where  $\pi_N : Z \mapsto \mathbb{C}^{2N+1}$  is the (natural) projection operator. This example and approximation scheme satisfies all the conditions in Section 5 of [3]. The approximating Riccati operators,  $I_{2N+1}$ , converge strongly to the Riccati operator of the full system and the feedback control operators  $F_n$  converge uniformly since the input space is  $\mathbb{C}$ . The approximating feedback  $F_N$  yields the closed loop system operator  $A - BB_N^* \pi_N$ . The closed loop system is not asymptotically stable. Furthermore, it is not hard to show that the approximate feedback does not give finite cost either.

In the above example we showed that even with a natural choice of approximation scheme, the approximate optimal feedback may not be suitable for the original system. In particular, it does not give a finite cost. In the following theorem we show that if the approximate feedback does provide a finite cost, then it also stabilizes the original system.

**Theorem 4.** Consider the system  $\Sigma(A, B, C)$ , (1)–(2) with cost criterion (3). Assume that it is optimizable and let  $X$  be the minimal non-negative solution of (4). Denote the optimal feedback,  $-B^*X$ , by  $F_{\text{opt}}$  and let  $z_{\text{opt}}$  denote the optimal state trajectory, that is, the solution of  $\dot{z}(t) = (A + BF_{\text{opt}})z(t)$ ,  $z(0) = z_0$ .

If  $A + BF_{\text{opt}}$  generates an asymptotically stable semigroup on  $Z$ , and there exists a sequence  $F_n \in \mathcal{L}(Z, U)$  such for all  $z_0 \in Z$

1.  $\lim_{n \rightarrow \infty} F_n z_0 = F_{\text{opt}} z_0$  and
2. letting  $z_n(t)$  denote the solution of  $\dot{z}(t) = (A + BF_n)z(t)$ ,  $z(0) = z_0$ , the cost with feedbacks  $F_n$  are uniformly bounded, i.e.,

$$\sup_n J(z_0, F_n z_n) < \infty; \quad (5)$$

then for all  $z_0 \in Z$

$$\lim_{n \rightarrow \infty} J(z_0, F_n z_n) = \langle z_0, Xz_0 \rangle = J(z_0, F_{\text{opt}} z_{\text{opt}}).$$

Furthermore,  $F_n z_n$  converges in  $L_2((0, \infty); U)$  to  $F_{\text{opt}} z_{\text{opt}}$  and  $Cz_n$  converges in  $L_2((0, \infty); Y)$  to  $Cz_{\text{opt}}$ .

**Proof.** Step 1: Let  $T_{F_n}(t)$  denote the  $C_0$ -semigroup generated by  $A + BF_n$ , and so  $z_n(t) = T_{F_n}(t)z_0$ . From the fact that

$$J(z_0, F_n z_n) = \int_0^\infty \|CT_{F_n}(t)z_0\|^2 + \|F_n T_{F_n}(t)z_0\|^2 dt,$$

we see that condition (5) is a uniform bounded in norm condition. The Uniform Boundedness Theorem implies that there exists an  $M > 0$  such that for all  $n \in \mathbb{N}$

$$J(z_0, F_n z_n) \leq M \|z_0\|^2. \quad (6)$$

Step 2: Using the ARE (4), we find for its (optimal) solution

$$\begin{aligned} X(A + BF_n)z_0 + (A + BF_n)^* X z_0 &= X A z_0 + A^* X z_0 - X B F_n z_0 - F_n^* B^* X z_0 \\ &= -C^* C z_0 - F_n^* F_n z_0 + (F_{opt} - F_n)^* (F_{opt} - F_n) z_0. \end{aligned} \quad (7)$$

From this we obtain for  $z_0 \in D(A)$

$$\begin{aligned} \frac{d}{dt} \langle T_{F_n}(t)z_0, X T_{F_n}(t)z_0 \rangle &= \|(F_{opt} - F_n)T_{F_n}(t)z_0\|^2 - \|CT_{F_n}(t)z_0\|^2 \\ &\quad - \|F_n T_{F_n}(t)z_0\|^2. \end{aligned}$$

Thus for every  $t > 0$

$$\begin{aligned} \langle z_0, X z_0 \rangle - \langle T_{F_n}(t)z_0, X T_{F_n}(t)z_0 \rangle &= \\ &= - \int_0^t \|(F_{opt} - F_n)T_{F_n}(\tau)z_0\|^2 d\tau \\ &\quad + \int_0^t [\|CT_{F_n}(\tau)z_0\|^2 + \|F_n T_{F_n}(\tau)z_0\|^2] d\tau. \end{aligned} \quad (8)$$

Since all components in the above equality are continuous in  $z_0$ , we have that this equality holds for all  $z_0 \in Z$ .

Step 3: Now we fix an  $z_0 \in Z$  and  $\varepsilon > 0$ . Let  $T_{F_{opt}}(t)$  denote the  $C_0$ -semigroup generated by  $A + BF_{opt}$ . By assumption,  $T_{F_{opt}}(t)$  is asymptotically stable and so there exists a  $t_1 > 0$  such that  $\|T_{F_{opt}}(t)z_0\| \leq \varepsilon$  for all  $t > t_1$ .

Since  $F_n$  converges strongly, we have that

$$\sup_n \|F_n z\| \leq M_1 \|z\| \quad (9)$$

for some  $M_1 > 0$ . So in particular, the function  $h_n(t)$  defined as  $h_n(t) = \|(F_{opt} - F_n)T_{F_{opt}}(t)z_0\|$  is uniformly bounded on  $[0, t_1]$ . Furthermore, the strong convergence of  $F_n$  to  $F_{opt}$  implies that this function converges pointwise to zero. So by the Lebesgue Dominated Convergence Theorem we have that

$$\lim_{n \rightarrow \infty} \int_0^{t_1} \|(F_{opt} - F_n)T_{F_{opt}}(t)z_0\|^2 dt = 0. \quad (10)$$

Step 4: Next we show that  $\int_0^{t_1} \|(F_{opt} - F_n)T_{F_n}(t)z_0\| dt$  converges to zero. From the equality

$$T_{F_{opt}}(t)z_0 = T_{F_n}(t)z_0 + \int_0^t T_{F_{opt}}(t - \tau)B[F_{opt} - F_n]T_{F_n}(\tau)z_0 d\tau$$

we find that

$$\begin{aligned} (F_{opt} - F_n)T_{F_{opt}}(t)z_0 &= (F_{opt} - F_n)T_{F_n}(t)z_0 + \\ &\quad \int_0^t (F_{opt} - F_n)T_{F_{opt}}(t - \tau)B[F_{opt} - F_n]T_{F_n}(\tau)z_0 d\tau. \end{aligned}$$

Thus

$$\begin{aligned} q_n(t) &:= \|(F_{opt} - F_n)T_{F_n}(t)z_0\| \\ &\leq \|(F_{opt} - F_n)T_{F_{opt}}(t)z_0\| + \\ &\quad \int_0^t \|(F_{opt} - F_n)T_{F_{opt}}(t - \tau)B\| \|(F_{opt} - F_n)T_{F_n}(\tau)z_0\| d\tau \end{aligned} \quad (11)$$

$$\begin{aligned} &\leq \|(F_{opt} - F_n)T_{F_{opt}}(t)z_0\| + \int_0^t 2M_1 \|B\| M_2 q_n(\tau) d\tau \\ &= h_n(t) + \int_0^t 2M_1 \|B\| M_2 q_n(\tau) d\tau, \end{aligned} \quad (12)$$

where  $M_2 = \sup_{t \geq 0} \|T_{F_{opt}}(t)\|$ . Next we define,  $M_3 = 2M_1 \|B\| M_2$ , and  $Q_n(t) = \int_0^t q_n(\tau) d\tau$ . Using (12) we see that

$$\frac{d}{dt} (e^{-M_3 t} Q_n(t)) \leq e^{-M_3 t} h_n(t),$$

which directly implies that

$$e^{-M_3 t} Q_n(t) = e^{-M_3 t} Q_n(t) - Q_n(0) \leq \int_0^t e^{-M_3 \tau} h_n(\tau) d\tau.$$

Thus for  $q_n$  there holds

$$\int_0^{t_1} q_n(\tau) d\tau \leq \int_0^{t_1} e^{M_3(t_1 - \tau)} h_n(\tau) d\tau. \quad (13)$$

We remark that the left-hand side of this inequality is the  $L_1((0, t_1); Z)$ -norm of  $(F_{opt} - F_n)T_{F_n}(t)z_0$ . Using (10) we conclude that this  $L_1$ -norm converges to zero as  $n$  converges to infinity.

Step 5: Next we show that the  $L_2((0, t_1); Z)$ -norm of  $(F_{opt} - F_n)T_{F_n}(t)z_0$  converges to zero as  $n$  converges to infinity.

The assumption that  $\sup_n J(z_0, F_n z_n) < \infty$  implies that

$$\int_0^{t_1} \|F_n T_{F_n}(t)z_0\|^2 dt$$

is uniformly bounded. This implies that the solution of

$$\dot{z}(t) = Az(t) + Bu_n(t), \quad z(0) = z_0$$

with  $u_n(t) = F_n T_{F_n}(t)z_0$  is uniformly bounded on the time interval  $[0, t_1]$ . Note that the solution of this abstract differential equation is precisely  $T_{F_n}(t)z_0$ . Hence there exists an  $M_4$  such that for all  $n$  and  $t \in [0, t_1]$

$$\|T_{F_n}(t)z_0\| \leq M_4. \quad (14)$$

Combining this estimate with the uniform boundedness of  $F_n$  implies that the function  $q_n(t) = \|(F_{opt} - F_n)T_{F_n}(t)z_0\|$  is uniformly bounded on  $[0, t_1]$  by the constant  $2M_1 M_4$ . So

$$\begin{aligned} \int_0^{t_1} \|(F_{opt} - F_n)T_{F_n}(t)z_0\|^2 dt &= \int_0^{t_1} q_n(t)^2 dt \\ &\leq 2M_1 M_4 \int_0^{t_1} q_n(t) dt \rightarrow 0 \quad n \rightarrow \infty, \end{aligned} \quad (15)$$

where we have used the result from the previous step.

Step 6: Using the equality

$$T_{F_{opt}}(t)z_0 = T_{F_n}(t)z_0 + \int_0^t T_{F_{opt}}(t - \tau)B[F_{opt} - F_n]T_{F_n}(\tau)z_0 d\tau,$$

and (15) we find that  $T_{F_n}(t)z_0 \rightarrow T_{F_{opt}}(t)z_0$  for  $t \in [0, t_1]$  as  $n \rightarrow \infty$ .

Thus, by the above there exists an  $N$  such that for  $n \geq N$ ,

- $\|T_{F_n}(t_1)z_0 - T_{F_{opt}}(t_1)z_0\| \leq \varepsilon$ ;
- $\langle T_{F_n}(t_1)z_0, X T_{F_n}(t_1)z_0 \rangle \leq \varepsilon^2 + \langle T_{F_{opt}}(t_1)z_0, X T_{F_{opt}}(t_1)z_0 \rangle \leq \varepsilon^2 + \|X\| \varepsilon^2$  and
- $\int_0^{t_1} \|(F_{opt} - F_n)T_{F_n}(\tau)z_0\|^2 d\tau \leq \varepsilon^2$ .

Defining  $z_n^{t_1}(t) = T_{F_n}(t_1 + t)z_0$ ,  $t \geq 0$ , and using the semigroup property of  $T_{F_n}$  to split the cost into two parts

$$J(z_0, F_n z_n) = \int_0^{t_1} \|CT_{F_n}(\tau)z_0\|^2 + \|F_n T_{F_n}(\tau)z_0\|^2 d\tau + J(T_{F_n}(t_1)z_0, F_n z_n^{t_1}).$$

Combining this with (8) gives

$$\begin{aligned} \langle z_0, Xz_0 \rangle + \int_0^{t_1} \|(F_{opt} - F_n)T_{F_n}(\tau)z_0\|^2 d\tau + J(T_{F_n}(t_1)z_0, F_n z_n^{t_1}) \\ = J(z_0, F_n z_n) + \langle T_{F_n}(t_1)z_0, XT_{F_n}(t_1)z_0 \rangle. \end{aligned}$$

By (6) we have that  $J(T_{F_n}(t_1)z_0, F_n z_n^{t_1}) \leq M \|T_{F_n}(t_1)z_0\|^2$ . By construction, this last expression can be bounded by a constant times  $\varepsilon^2$ . So we see that the difference between the optimal cost,  $\langle z_0, Xz_0 \rangle$ , and the approximate cost,  $J(z_0, F_n z_n(\cdot))$ , is bounded by a constant times  $\varepsilon^2$  for all  $n$  sufficiently large. We can do this for any  $\varepsilon > 0$ , and so

$$J(z_0, F_n z_n) \rightarrow \langle z_0, Xz_0 \rangle = J(z_0, F_{opt} z).$$

*Step 7:* Next we show that  $F_n T_{F_n}(t)z_0 \rightarrow F_{opt} T_{opt}(t)z_0$  in  $L_2((0, t_1); U)$ . We write

$$\begin{aligned} F_n T_{F_n}(t)z_0 - F_{opt} T_{opt}(t)z_0 &= (F_n - F_{opt})T_{F_n}(t)z_0 - \\ &F_{opt} (T_{F_n}(t)z_0 - T_{opt}(t)z_0). \end{aligned}$$

By (15) the first term on the right-hand side converges to zero in  $L_2((0, t_1); U)$ . By (14) we have that the function  $\|F_{opt} (T_{F_n}(t)z_0 - T_{opt}(t)z_0)\|$  is uniformly bounded on  $[0, t_1]$ . By Step 6, we know that this function converges pointwise to zero, and thus the Lebesgue Dominated Convergence Theorem gives that the last term on the right-hand side converges to zero in  $L_2((0, t_1); U)$ .

*Step 8:* In the previous steps we have shown that  $T_{F_n}(t)z_0 \rightarrow T_{opt}(t)z_0$  for  $t \in [0, t_1]$  and that  $F_n T_{F_n}(t)z_0 \rightarrow F_{opt} T_{opt}(t)z_0$  in  $L_2((0, t_1); U)$ . This immediately implies that  $CT_{F_n}(t)z_0 \rightarrow CT_{opt}(t)z_0$  in  $L_2((0, t_1); Y)$ .

*Step 9:* It remains to show that  $F_n T_{F_n}(t)z_0 \rightarrow F_{opt} T_{opt}(t)z_0$  in  $L_2((0, \infty); U)$  and  $CT_{F_n}(t)z_0 \rightarrow CT_{opt}(t)z_0$  in  $L_2((0, \infty); Y)$ . Now

$$\begin{aligned} \int_0^\infty \|F_n T_{F_n}(t)z_0 - F_{opt} T_{opt}(t)z_0\|^2 dt + \int_0^\infty \|CT_{F_n}(t)z_0 - CT_{opt}(t)z_0\|^2 dt \\ = \int_0^{t_1} \|F_n T_{F_n}(t)z_0 - F_{opt} T_{opt}(t)z_0\|^2 dt + \\ \int_{t_1}^\infty \|F_n T_{F_n}(t)z_0 - F_{opt} T_{opt}(t)z_0\|^2 dt + \\ \int_0^{t_1} \|CT_{F_n}(t)z_0 - CT_{opt}(t)z_0\|^2 dt + \\ \int_{t_1}^\infty \|CT_{F_n}(t)z_0 - CT_{opt}(t)z_0\|^2 dt \\ \leq \int_0^{t_1} \|F_n T_{F_n}(t)z_0 - F_{opt} T_{opt}(t)z_0\|^2 dt + \\ \int_0^{t_1} \|CT_{F_n}(t)z_0 - CT_{opt}(t)z_0\|^2 dt + \\ 2 \int_{t_1}^\infty \|F_n T_{F_n}(t)z_0\|^2 + \|CT_{F_n}(t)z_0\|^2 dt + \\ 2 \int_{t_1}^\infty \|F_{opt} T_{opt}(t)z_0\|^2 + \|CT_{opt}(t)z_0\|^2 dt \\ \leq \int_0^{t_1} \|F_n T_{F_n}(t)z_0 - F_{opt} T_{opt}(t)z_0\|^2 dt + \\ 2J(z_n(t_1), F_n z_n^{t_1}) + 2\langle z_{opt}(t_1), Xz_{opt}(t_1) \rangle. \end{aligned}$$

By the asymptotic stability of  $T_{F_{opt}}(t)$ , Step 6 and inequality (6) we find that the last two terms can be made arbitrarily small. Combining this with the first part, we see that  $F_n T_{F_n}(t)z_0 \rightarrow F_{opt} T_{opt}(t)z_0$  in  $L_2((0, \infty); U)$  and  $CT_{F_n}(t)z_0 \rightarrow CT_{opt}(t)z_0$  in  $L_2((0, \infty); Y)$   $\square$

#### 4. LQ optimal feedback using the Newton–Kleinman method

The Newton–Kleinman method [8] is a well-known approach to approximating the solution to the Riccati equation (4), particularly for large-scale systems. The following extension to asymptotically stabilizable infinite-dimensional systems can be found in [9].

**Theorem 5.** Consider an infinite-dimensional system  $\Sigma(A, B, C)$ . If there exists  $F \in \mathcal{L}(Z, U)$  such that

AS1.  $J(z_0, Fz) < \infty$  and

AS2.  $A + BF$  generates an asymptotically stable semigroup,  $T_{BF}(t)$ ,

then

a. the following Lyapunov equation has a nonnegative solution  $X_0 \in \mathcal{L}(Z)$  for  $z \in \mathbf{D}(A)$ :

$$(A + BF)^* X_0 z + X_0 (A + BF)z = -F^* Fz - C^* Cz,$$

and  $A - BB^* X_0$  generates an asymptotically stable semigroup;

b. there exists a nonincreasing sequence  $(X_n)_{n \in \mathbb{N}}$ , of bounded nonnegative operators such that for  $z \in \mathbf{D}(A)$

$$\begin{aligned} (A - BB^* X_n)^* X_{n+1} z + X_{n+1} (A - BB^* X_n) z \\ = -C^* Cz - X_n BB^* X_n z, \end{aligned} \quad (16)$$

and  $A_{n+1} := A - BB^* X_n$ ,  $n \geq 0$ , generates an asymptotically stable semigroup;

c. the sequence  $(X_n)_{n \in \mathbb{N}}$ , has the strong limit  $X_{\max} \geq 0$  which satisfies the algebraic Riccati equation (4);

d.  $X_{\max}$  is the maximal solution to the Riccati equation (4) and to the Riccati inequality

$$\langle Xz, Az \rangle + \langle Az, Xz \rangle - \langle B^* Xz, B^* Xz \rangle + \langle Cz, Cz \rangle \geq 0, \quad z \in \mathbf{D}(A).$$

Suppose that in addition to the stabilizability assumptions of Theorem 5 a system satisfies a detectability assumption; that is, there is  $L \in \mathcal{L}(Z, Y)$  such that  $A - LC$  generates an asymptotically semigroup  $S_{LC}$  and there is  $\beta > 0$  such that for all  $z_0 \in Z$ ,

$$\int_0^\infty \left\| \begin{bmatrix} B^* \\ L^* \end{bmatrix} S_{LC}^*(t) \right\|^2 dt \leq \beta \|z_0\|^2.$$

Then the solution to Eq. (4) is unique, and the Newton–Kleinman iterations provide a sequence of approximating feedbacks satisfying condition 2 of Theorem 4 [9].

**Theorem 6.** With the notation of Theorem 5, for all  $n \in \mathbb{N}$  there hold

$$J(z_0, -B^* X_n z) = \langle z_0, X_{n+1} z_0 \rangle \leq M \|z_0\|^2$$

and

$$\|X_{n+1} - X_{\max}\| = \sup_{\|z_0\|=1} \int_0^\infty \|B^* (X_n - X_{\max}) T_{-BB^* X_n}(t) z_0\|^2 dt.$$

**Proof.** Let  $z_0 \in \mathbf{D}(A)$  and let  $z(t)$  be the solution of  $\dot{z}(t) = A_{n+1} z(t) = (A - BB^* X_n) z(t)$ ,  $z(0) = z_0$ , then Eq. (16) gives for every  $t_1 > 0$

$$\int_0^{t_1} \|y(t)\|^2 + \|B^* X_n z(t)\|^2 dt = \langle z_0, X_{n+1} z_0 \rangle - \langle z(t_1), X_{n+1} z(t_1) \rangle.$$

Using the fact that  $X_{n+1}$  is non-negative and that  $A_{n+1}$  generates an asymptotically stable semigroup gives the desired result. Since the sequence  $X_n$  converges, the costs are uniformly bounded.

Since the maximal solution of the ARE is the limit of the Newton–Kleinman iterates,  $X_{\max}$  satisfies

$$A^* X_{\max} z_0 + X_{\max} A z_0 = -C^* C z_0 - X_{\max} BB^* X_{\max} z_0.$$

This can be reformulated as

$$(A - BB^*X_n)^*X_{\max}z_0 + X_{\max}(A - BB^*X_n)z_0 = -C^*Cz - X_{\max}BB^*X_{\max}z_0 - X_nBB^*X_{\max}z_0 - X_{\max}BB^*X_nz_0. \quad (17)$$

From [Theorem 5](#), part b, the Newton–Kleinman iterates  $X_n \in \mathcal{L}(Z)$  also satisfy

$$(A - BB^*X_n)^*X_{n+1}z_0 + X_{n+1}(A - BB^*X_n)z_0 = -C^*Cz_0 - X_nBB^*X_nz_0. \quad (18)$$

Subtracting (17) from (18) gives

$$(A - BB^*X_n)^*(X_{n+1} - X_{\max})z_0 + (X_{n+1} - X_{\max})(A - BB^*X_n)z_0 = -(X_n - X_{\max})BB^*(X_n - X_{\max})z_0.$$

Hence

$$\langle (X_{n+1} - X_{\max})z_0, z_0 \rangle = \int_0^\infty \|B^*(X_n - X_{\max})T_{-BB^*X_n}(t)z_0\|^2 dt.$$

Since this holds for every  $z_0 \in \mathbf{D}(A)$  and since  $\mathbf{D}(A)$  is dense in  $Z$ , the result follows.  $\square$

A practical difficulty related to the Newton–Kleinman method is to find an initially stabilizing feedback  $F$  for step (a). In the following subsections, two nontrivial classes of asymptotically stabilizable systems that are not exponentially stabilizable will be considered: lightly damped second-order systems and a platoon type system. It will be shown that these systems are optimizable, and an explicit representation of a stabilizing feedback will be provided. We also show that these classes of systems are well-posed.

#### 4.1. Lightly damped second-order systems

Models for lightly damped wave and vibrations are an important class of asymptotically stabilizable systems. Consider second-order systems of the form

$$\ddot{w}(t) + A_0 w(t) + D\dot{w}(t) = B_0 u(t). \quad (19)$$

It will be assumed that the stiffness operator  $A_0$ , damping operator  $D$ , and control operator  $B_0$  satisfy the following assumptions:

**(A1)** The operator  $A_0 : \mathbf{D}(A_0) \subset H \rightarrow H$  is a self-adjoint, positive-definite linear operator on the Hilbert space  $H$  such that zero is in the resolvent set of  $A_0$ . Here  $\mathbf{D}(A_0)$  denotes the domain of  $A_0$ . Since  $A_0$  is self-adjoint and positive definite,  $A_0^\alpha$  is well-defined for  $\alpha \geq 0$ . Define  $H_{\frac{1}{2}} = \mathbf{D}(A_0^{\frac{1}{2}})$  equipped with the norm induced by the inner product

$$\langle x, y \rangle_{H_{\frac{1}{2}}} = \langle A_0^{\frac{1}{2}} x, A_0^{\frac{1}{2}} y \rangle_H, \quad x, y \in H_{\frac{1}{2}}$$

and  $H_{-\frac{1}{2}} = H_{\frac{1}{2}}^*$ . Here the duality is taken with respect to the pivot space  $H$ , that is, equivalently  $H_{-\frac{1}{2}}$  is the completion of  $H$  with respect to the norm  $\|z\|_{H_{-\frac{1}{2}}} = \|A_0^{-\frac{1}{2}} z\|_H$ . Thus  $A_0$  extends (restricts) to  $A_0 : H_{\frac{1}{2}} \rightarrow H_{-\frac{1}{2}}$ . We use the same notation  $A_0$  to denote this extension (restriction).

We denote the inner product on  $H$  by  $\langle \cdot, \cdot \rangle_H$  or  $\langle \cdot, \cdot \rangle$ , and the duality pairing on  $H_{-\frac{1}{2}} \times H_{\frac{1}{2}}$  by  $\langle \cdot, \cdot \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}$ . Note that for  $(z', z) \in H \times H_{\frac{1}{2}}$ ,

$$\langle z', z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle z', z \rangle_H.$$

**(A2 i)** The control operator  $B_0$  is a linear, bounded operator from  $\mathbb{C}^m$  to  $H$ .

**(A2 ii)** The operator  $D : H_{\frac{1}{2}} \rightarrow H_{-\frac{1}{2}}$  is a bounded operator such that  $A_0^{-1/2} D A_0^{-1/2}$  is a bounded self-adjoint non-negative operator

on  $H$ . This implies in particular

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \quad z \in H_{\frac{1}{2}}.$$

The operator  $D$  determines the stability of the uncontrolled system.

These assumptions are the same as in [10] except the stronger assumption that  $B_0$  is bounded into  $H$  is made here to be consistent with the rest of this paper.

A first-order form of the system (19) on the state-space  $Z = H_{\frac{1}{2}} \times H$  is defined by

$$\dot{z}(t) = Az(t) + Bu(t) \quad (20)$$

where  $A : \mathbf{D}(A) \subset Z \rightarrow Z$ ,  $B : U \rightarrow Z \times H$ ,  $U = \mathbb{C}^m$ , and

$$A = \begin{bmatrix} 0I & -D \\ -A_0 & -D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$

$$\mathbf{D}(A) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 z + Dw \in H \right\}.$$

It is well-known that the assumptions guarantee that  $A$  generates a contraction semigroup on  $Z$ . If  $D = 0$  then of course the system is not asymptotically stable. At the other extreme, if  $D$  is coercive with respect to  $H$  then  $A$  generates an exponentially stable semigroup. With intermediate values of damping,  $A$  may be asymptotically stable. In particular, if  $\langle D\phi, \phi \rangle \neq 0$  for all eigenfunctions of  $A_0$ , the system is asymptotically stable.

For  $C_0 \in \mathcal{L}(H, \mathbb{C}^m)$ , define velocity measurements or cost  $C_v : Z \rightarrow Y$ , where  $Y = \mathbb{C}^m$ , by

$$y_v(t) = C_v z(t) \quad (21)$$

where  $C_v = \begin{bmatrix} 0 & C_0 \end{bmatrix}$ .

The following assumption on the damping will be made. This is a weaker assumption on the system than considered in [10].

**(A3)** There exist  $\alpha_b > 0$  and  $\alpha_c > 0$  such that

$$\langle Dv, v \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq \alpha_b \|B_0^* v\|^2 + \alpha_c \|C_0 v\|^2, \quad v \in H_{\frac{1}{2}}.$$

**Theorem 7.** *If assumptions (A1)–(A3) are satisfied, then the control system (20) with cost (3) defined by output (21) is optimizable with  $u = 0$ . Furthermore, the system is state-output and input-output stable.*

**Proof.** The proof of this proposition uses the approach in [10,11]. First, from [10, Prop. 4.1] for  $u \in H^2(0, \infty; \mathbb{C}^m)$  and  $w(0), \dot{w}(0) \in H_{\frac{1}{2}}$  there is a solution  $w \in C^1([0, \infty); H_{\frac{1}{2}}) \cap C^2((0, \infty); H)$  of (19). Also  $A$  generates a contraction semigroup on  $Z$  and

$$\frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} \right\|^2 = -\langle D\dot{w}(t), \dot{w}(t) \rangle + \text{Re} \langle B_0 u(t), \dot{w}(t) \rangle.$$

Using (A3) and Young's inequality leads to

$$\frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} \right\|^2 \leq (-\alpha_b + \frac{\varepsilon}{2}) \|B_0^* \dot{w}(t)\|^2 - \alpha_c \|C_0 \dot{w}(t)\|^2 + \frac{1}{2\varepsilon} \|u(t)\|^2.$$

Choosing  $\varepsilon < 2\alpha_b$ , there are constants  $c_1, c_2 > 0$  such that

$$\frac{d}{dt} \left\| \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} \right\|^2 \leq c_1 \|u(t)\|^2 - c_2 \|B_0^* \dot{w}(t)\|^2 - 2\alpha_c \|C_0 \dot{w}(t)\|^2. \quad (22)$$

Integrating this inequality over time, rearranging, and writing the state  $z(t) = \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix}$ ,

$$\begin{aligned} \|z(T)\|^2 + 2\alpha_c \int_0^T \|C_0 \dot{w}(t)\|^2 dt + c_2 \int_0^T \|B_0^* \dot{w}(t)\|^2 dt \\ \leq c_1 \int_0^T \|u(t)\|^2 dt + \|z(0)\|^2. \end{aligned}$$

Thus, since  $y(t) = C_0 \dot{w}(t)$ ,

$$2\alpha_c \int_0^T \|y(t)\|^2 dt \leq c_1 \int_0^\infty \|u(t)\|^2 dt + \|z(0)\|^2.$$

which implies that the system is optimizable with zero input. Also, the system is state-output and input-output stable.  $\square$

A similar result holds for position measurements. The following proposition will be useful.

**Proposition 8** ([11, Prop. 5.3]). For every  $s \in \rho(A)$ ,

1.  $(sI - A)^{-1}$  is a bounded and invertible map from  $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$  to  $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ .

2. The operator  $s^2I + Ds + A_0 \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$  has a bounded inverse  $V(s) \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}})$ ,

$$V(s) = (s^2I + Ds + A_0)^{-1}.$$

3. On  $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ , for every non-zero  $s \in \rho(A)$ ,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} [I - V(s)A_0] & V(s) \\ -V(s)A_0 & sV(s) \end{bmatrix}.$$

**Theorem 9.** Suppose assumptions (A1)–(A3) hold and the cost is defined through  $C_p : H_{\frac{1}{2}} \times H \rightarrow \mathbb{C}^m$  as

$$C_p = \begin{bmatrix} C_0 & 0 \end{bmatrix},$$

so

$$y_p(t) = C_p z(t). \quad (23)$$

The control system with this cost is optimizable, state-output and input-output stable.

**Proof.** As for the previous situation with velocity in the cost, the operator  $A$  generates a  $C_0$ -semigroup. Letting  $G_p(s)$  indicate the transfer function of this system, and  $G_v(s)$  that of the system with velocity measurements  $\begin{bmatrix} 0 & C_0 \end{bmatrix}$ ,

$$G_p(s) = \frac{1}{s} G_v(s).$$

**Theorem 7** implies that  $G_v \in \mathcal{H}_\infty(U, Y)$ . Since

$$G_v(s) = sC_0(s^2I + Ds + A_0)^{-1}B_0$$

and  $0 \in \rho(A_0)$ ,

$$G_p(s) = C_0(s^2I + Ds + A_0)^{-1}B_0$$

is also in  $\mathcal{H}_\infty(U, Y)$ . Thus, the map from  $u$  to  $y$  is bounded from  $\mathcal{H}_2(U)$  to  $\mathcal{H}_2(Y)$ .

Similarly, with zero control, and initial condition  $z_0 = \begin{bmatrix} w_0 & v_0 \end{bmatrix}$ , the velocity measurement  $y_v$  has Laplace transform

$$\hat{y}_v(s) = -C_0V(s)A_0w_0 + sC_0V(s)v_0$$

where  $w_0 \in H_{\frac{1}{2}}$ ,  $v_0 \in H$ . From **Theorem 7**,  $y_v \in L^2((0, \infty); Y)$  and so  $\hat{y}_v \in \mathcal{H}_2(Y)$ . For position measurements, for  $w_0 \in H_{\frac{1}{2}}$ ,  $v_0 \in H$ ,

$$\begin{aligned} \hat{y}_p(s) &= C_0 \left( \frac{1}{s} (I - V(s)A_0)w_0 + V(s)v_0 \right) \\ &= C_0sV(s)w_0 + C_0V(s)Dw_0 + C_0V(s)v_0. \end{aligned}$$

The signal  $\hat{y}_p(s)$  is analytic for  $s \in \rho(A)$  and hence for all  $\text{Re}(s) > 0$ . Since  $sC_0V(s)v_0 \in \mathcal{H}_2(Y)$  for all  $v_0 \in H$ ,  $C_0sV(s)w_0 \in \mathcal{H}_2(Y)$  for all  $w_0 \in H_{\frac{1}{2}} \subset H$  and also  $C_0V(s)v_0 \in \mathcal{H}_2(Y)$  for all  $v_0 \in H$ . Similarly, defining  $\tilde{w}_0 = A_0^{-1}Dw_0$ ,  $C_0V(s)Dw_0 = C_0V(s)A_0w_0 \in \mathcal{H}_2(Y)$ . It follows that  $\hat{y}_p \in \mathcal{H}_2(Y)$  and so  $y_p \in L^2((0, \infty); Y)$ . The system is state-output stable and also optimizable.  $\square$

The above results on optimizability imply that, for  $F_0 = 0$ , the conditions in **Theorem 5** are satisfied. However, if  $B_0^* \phi \neq 0$  for all eigenfunctions of  $A_0$  then  $F_0 = \begin{bmatrix} 0 & B_0^* \end{bmatrix}$  yields an asymptotically stable system. (See the discussion and list of references in [10] for details.) This choice of  $F_0$  is probably a better choice of initial iterate in **Theorem 5**. Also, in this situation, the system is asymptotically stabilizable and so the feedback obtained from solving the ARE leads to an asymptotically stable system [3].

The following generalization applies to systems where the damping is very weak but the system is stabilizable.

**Theorem 10.** Suppose assumptions (A1)–(A2) are satisfied,  $B_0^* \phi \neq 0$  for all eigenfunctions  $\phi$  of  $A_0$ , and the control system (20) has cost defined by  $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$  or  $\begin{bmatrix} C_0 & 0 \end{bmatrix}$  where there is  $\beta > 0$  such that for all  $v \in H$ ,  $\|B_0^* w\| \geq \beta \|C_0 w\|$ . Then the system is optimizable. If  $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$  where  $C_0 \phi \neq 0$  for any eigenfunction  $\phi$  of  $A_0$ , then  $A - BB^*X$  where  $X$  solves the associated ARE (4) generates an asymptotically stable semigroup.

**Proof.** Defining the control  $u(t) = -B_0^* v(t) = B_0^* \dot{w}(t)$ , the resulting system has damping satisfying (A3), with  $\alpha_b = \frac{1}{2}$ ,  $\alpha_c = \frac{\beta}{2}$ . With either choice of cost, the resulting system is state-output stable and defining  $F = \begin{bmatrix} 0 & -B_0^* \end{bmatrix}$ ,  $A + BF$  generates an asymptotically stable semigroup and the system is optimizable. In fact, also the system with cost defined through  $\begin{bmatrix} C \\ F \end{bmatrix}$  is optimizable and state-output stable. The remainder of the result follows from [3, Thm. 2.3]. The cost is minimized by  $u(t) = -B^*X$  where  $X$  solves the associated ARE (4). If also  $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$  where  $C_0 \phi \neq 0$  for any eigenfunction  $\phi$  of  $A_0$ , then  $A - BB^*X$  generates an asymptotically stable semigroup.  $\square$

#### 4.2. Platoons

The example considered in this section is motivated by vehicular platooning [12], and belongs to the class of *spatially invariant systems* described by

$$\dot{z}_r(t) = \sum_{l=-\infty}^{\infty} a_l z_{r-l}(t) + \sum_{l=-\infty}^{\infty} b_l u_{r-l}(t), \quad (24)$$

$$y_r(t) = \sum_{l=-\infty}^{\infty} c_l z_{r-l}(t), \quad r \in \mathbb{Z}, \quad t \geq 0, \quad (25)$$

where  $\mathbb{Z}$  is the set of integer numbers,  $a_l, b_l, c_l$  are matrices and  $z_r(t)$ ,  $u_r(t)$  and  $y_r(t)$  are the state, the input and the output vectors, respectively, at time  $t \geq 0$  and spatial point  $r \in \mathbb{Z}$ . The system (24)–(25) can be formulated as a standard state linear system  $\Sigma(A, B, C)$  [13]

$$\dot{z}(t) = (Az)(t) + (Bu)(t), \quad (26)$$

$$y(t) = (Cz)(t), \quad t \geq 0,$$

where  $Z, U$ , and  $Y$  are the state-space, the input space, and the output space, respectively. The spaces  $Z, U$ , and  $Y$  contain  $\ell_2$ -sequences of vectors, that is, are of the form  $\ell_2(\mathbb{C}^n) = \{z \mid z = (z_r)_{r=-\infty}^{\infty}, z_r \in \mathbb{C}^n, \sum_{r=-\infty}^{\infty} \|z_r\|^2 < \infty\}$  for some  $n$ . The operators  $A, B, C$  are convolution operators; for example,

$$(Az)_r = \sum_{l=-\infty}^{\infty} a_l z_{r-l} = \sum_{l=-\infty}^{\infty} a_{r-l} z_l.$$

Take Fourier transforms  $\check{z} = \mathfrak{F}z$ , of the system equations (26), to obtain, letting  $\check{z} = \mathfrak{F}z$  etc.

$$\dot{\check{z}}(t) = \mathfrak{F}\dot{z}(t) = \check{A}\check{z}(t) + \check{B}\check{u}(t), \quad (27)$$

$$\check{y}(t) = \mathfrak{F}y(t) = \check{C}\check{z}(t), \quad t \geq 0.$$

The transformed operators  $\check{A} = \mathfrak{F}A\mathfrak{F}^{-1}$ ,  $\check{B} = \mathfrak{F}B\mathfrak{F}^{-1}$  and  $\check{C} = \mathfrak{F}C\mathfrak{F}^{-1}$  are multiplicative operators; for example

$$((\check{A})(e^{j\theta})) = \check{A}(e^{j\theta})\check{z}(e^{j\theta}) = \left( \sum_{l=-\infty}^{\infty} a_l e^{j\theta l} \right) \check{z}(e^{j\theta}), \quad \theta \in [0, 2\pi].$$

If only finitely many of the coefficients are nonzero,  $\check{A}$ ,  $\check{B}$ ,  $\check{C}$  define bounded operators on  $L_2(\partial\mathbb{D}; \mathbb{C})$ , where  $\partial\mathbb{D}$  is the unit circle [9]. Note that  $L_2(\partial\mathbb{D}; \mathbb{C})$  and  $\ell_2(\mathbb{C})$  are isometrically isomorphic, the systems  $\Sigma(A, B, C)$  and  $\Sigma(\check{A}, \check{B}, \check{C})$  are also isometrically isomorphic, therefore have the same system theoretic properties (see [1, Exercise 2.5]). For  $\theta \in [0, 2\pi]$  the system (27) can be written as

$$\begin{aligned} \dot{\check{z}}(e^{j\theta}, t) &= \check{A}(e^{j\theta})\check{z}(e^{j\theta}, t) + \check{B}(e^{j\theta})\check{u}(e^{j\theta}, t) \\ \check{y}(e^{j\theta}, t) &= \check{C}(e^{j\theta})\check{z}(e^{j\theta}, t), \quad t \geq 0, \end{aligned} \quad (28)$$

The following conditions for asymptotic stability can be used.

**Theorem 11 ([13]).**

The closed loop system is asymptotically stable if and only if

1.  $\text{Re}(\lambda(\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v)) \leq 0$  for all  $\theta \in [0, 2\pi]$ ,
2.  $\text{Re}(\lambda(\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v)) = 0$  for at most a countable number of  $\theta \in [0, 2\pi]$ ,
3.  $\sup_{\theta \in [0, 2\pi], t \geq 0} \|e^{(\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v)t}\| < \infty$ .

As an example, consider

$$\check{A}(e^{j\theta}) = \begin{pmatrix} 0 & \overline{e^{j\theta}} - 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \check{B}(e^{j\theta}) = \check{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and  $\check{C}(e^{j\theta}) = \text{diag}(1 - \overline{e^{j\theta}}, 0, 0)$ . This system is non-exponentially stabilizable [12], but it is asymptotically stabilizable. If  $\nu$  is negative and very small (take for example  $\nu = -0.05$ ), the constant matrix

$$P_v = \begin{pmatrix} 3\nu^2 & 2\nu & \nu \\ 2\nu & 2 & 1 \\ \nu & 1 & 1 \end{pmatrix}$$

satisfies

$$\begin{aligned} (\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v)^*P_v + P_v(\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v) \\ + P_v\check{B}\check{B}^*P_v + \check{C}^*(e^{j\theta})\check{C}(e^{j\theta}) \leq 0. \end{aligned}$$

Hence it follows immediately that the system  $\Sigma(\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v, \check{B}, \check{C}(e^{j\theta}))$  is optimizable so condition AS1 from Theorem 5 is satisfied.

Take  $\check{F} = -\check{B}^*P_v$ . Then the closed-loop system has

$$\det(\lambda I - \check{A}(e^{j\theta}) - \check{B}\check{F}) = \lambda^3 + 2\lambda^2 + \lambda + \nu(\overline{e^{j\theta}} - 1).$$

For negative  $\nu$  with values close to zero (take for example  $\nu = -0.05$ ), the closed-loop system satisfies the first two conditions in Theorem 11. The third condition in Theorem 11 is satisfied if the following inequality holds [12]

$$\min_{\theta} |\lambda_k((\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v)) - \lambda_l((\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v))| > 0 \quad (29)$$

for  $k \neq l$ ,  $k, l = 1, 2, 3$ . It can be easily checked that (29) is satisfied for our system. Hence the closed-loop system  $\Sigma(\check{A}(e^{j\theta}) - \check{B}\check{B}^*P_v, \check{B}, \check{C}(e^{j\theta}))$  is asymptotically stable, so condition AS2 in Theorem 5 is satisfied.

Since the conditions AS1 and AS2 in Theorem 5 are satisfied, there exists a nonnegative solution to the Riccati equation

$$\check{A}^*(e^{j\theta})\check{X}(e^{j\theta}) + \check{X}(e^{j\theta})\check{A}(e^{j\theta}) - \check{X}(e^{j\theta})\check{B}\check{B}^*\check{X}(e^{j\theta}) + \check{C}^*(e^{j\theta})\check{C}(e^{j\theta}) = 0.$$

For each fixed  $\theta$ , the Lyapunov equation

$$(\check{A}(e^{j\theta}) + \check{B}\check{F})^*\check{X}_0(e^{j\theta}) + \check{X}_0(e^{j\theta})(\check{A}(e^{j\theta}) + \check{B}\check{F}) = -\check{F}^*\check{F} - \check{C}^*(e^{j\theta})\check{C}(e^{j\theta}) \quad (30)$$

has a unique solution  $\check{X}_0(e^{j\theta})$ . One can obtain an explicit  $\check{X}_0(e^{j\theta})$  as follows. Consider

$$\check{X}_0(e^{j\theta}) = \begin{pmatrix} \alpha & p_{12} & p_{13} \\ \overline{p_{12}} & \beta & p_{23} \\ \overline{p_{13}} & \overline{p_{23}} & \gamma \end{pmatrix}.$$

Then the Lyapunov equation (30) leads to the following system

$$\nu(p_{13} + \overline{p_{13}}) = \nu^2 + |e^{j\theta} - 1|^2 = \nu^2 + 4\sin^2(\theta/2) \quad (31)$$

$$\alpha(\overline{e^{j\theta}} - 1) - p_{13} - \nu\overline{p_{23}} + \nu = 0 \quad (32)$$

$$p_{12} - 2p_{13} - \nu\gamma + \nu = 0 \quad (33)$$

$$\overline{p_{12}}(\overline{e^{j\theta}} - 1) + p_{12}(e^{j\theta} - 1) - p_{23} - \overline{p_{23}} + 1 = 0 \quad (34)$$

$$\beta - 2p_{23} + p_{13}(e^{j\theta} - 1) - \gamma + 1 = 0 \quad (35)$$

$$p_{23} + \overline{p_{23}} - 4\gamma + 1 = 0 \quad (36)$$

Take  $p_{13} = a + jb$ ,  $p_{12} = c + jd$  and  $p_{23} = e + jf$ . Note also that in (32), (33) and (35) one has equations corresponding to the real and imaginary parts. Then one has a system with 9 equations and 9 unknowns. For each fixed  $\theta$ , this linear system can be solved explicitly to find its guaranteed unique solution  $\check{X}_0(e^{j\theta})$ . Once  $\check{X}_0(e^{j\theta})$  is available, one can continue the procedure to construct a sequence of feedbacks  $\check{F}_n(e^{j\theta}) = -\check{B}^*\check{X}_n(e^{j\theta})$  which satisfy the conditions from Theorem 4.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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